Construction of an invariant for integral homology spheres via kauffman bracket skein algebra and its application

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1. Kauffman bracket skein algebra
2. Filtration
3. Torelli group
4. An invariant for $\mathbb{Z}HS^3$
5. Casson invariant

§1 Kauffman bracket skein algebra

Σ : cpt conn. ori. surface,

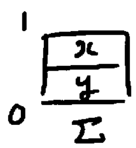
$I = [0,1]$,

$\mathcal{L}(\Sigma) = \{$ unoriented framed links in $\Sigma \times I \}$,

$\mathcal{S}(\Sigma) = \mathbb{Q}[A^{\pm}]\mathcal{L}(\Sigma) / (\text{skein rel. triv. knot rel.})$.

skein rel.  $= A$)($+ A^{-1}$ ,

triv. rel.  $= (-A^2 - A^{-2})$ ,

$x, y \in \mathcal{S}(\Sigma)$, $xy =$ , $[x, y] = \frac{1}{-A + A^{-1}}(xy - yx)$.

§2 Filtration

$$\varepsilon : \mathcal{S}(\Sigma) \to \mathbb{Q} \qquad ; \text{ well-defined.}$$

$$A \mapsto -1$$

$$L \mapsto (-2)^{|L|} \quad |L| \leftarrow \#\text{ comp. of } L.$$

$\gamma \in \pi_1(\Sigma)$,

$L_\gamma \in \ell(\Sigma)$,

the homotopy class of L_γ is the conj. class of $\gamma \in \pi_1(\Sigma) = \pi_1(\Sigma \times I)$.

$w(L_\gamma)$: the self linking number of L_γ,

$$\varpi(L_\gamma) = L_\gamma + 2 - 3(A - A^{-1})\,w(L_\gamma) \in (\ker\varepsilon)/(\ker\varepsilon)^2.$$

$$\varpi : \mathbb{Q}\pi_1(\Sigma) \to \ker\varepsilon/(\ker\varepsilon)^2,$$

$$\lambda : H \wedge H \wedge H \to \ker\varepsilon/(\ker\varepsilon)^2$$

$$[\alpha]\wedge[\beta]\wedge[\gamma] \mapsto \varpi((\alpha-1)(\beta-1)(\gamma-1)).$$

<u>Thm</u> (T) λ is well-defined and injective.

$$\delta : \ker\varepsilon \to (\ker\varepsilon/(\ker\varepsilon)^2)/\mathrm{Im}\,\lambda$$

$F^0\mathcal{S}(\Sigma) = \mathcal{S}(\Sigma)$, $F^1\mathcal{S}(\Sigma) = F^2\mathcal{S}(\Sigma) = \ker\varepsilon$,

$F^3\mathcal{S}(\Sigma) = \ker\delta$, $F^{n+2}\mathcal{S}(\Sigma) = \ker\varepsilon\, F^n\mathcal{S}(\Sigma)$ $(n \geq 2)$

<u>Prop</u> $F^n\mathcal{S}(\Sigma)F^m\mathcal{S}(\Sigma) \subset F^{n+m}\mathcal{S}(\Sigma)$,

$[F^n\mathcal{S}(\Sigma), F^m\mathcal{S}(\Sigma)] \subset F^{n+m-2}\mathcal{S}(\Sigma)$.

$$\lambda_2 : \mathbb{Q} \oplus S^2H \to F^2/F^3,$$
$$q \mapsto q(A+1)$$
$$[\alpha][\beta] \mapsto \varpi((\alpha-1)(\beta-1)).$$

$$\lambda_3 : \Lambda^3 H \to F^3/F^4$$
$$\overset{\shortparallel}{\lambda}$$

$$\lambda_4 : \mathbb{Q} \oplus S^2H \oplus S^2(S^2H) \to F^4/F^5,$$
$$q \qquad q(A+1)$$
$$[\alpha][\beta] \qquad \varpi((\alpha-1)(\beta-1))(A+1)$$
$$([\alpha_1][\beta_1])([\alpha_2][\beta_2]) \qquad \frac{1}{2}\Big(\varpi((\alpha_1-1)(\beta_1-1))\,\varpi((\alpha_2-1)(\beta_2-1))$$
$$+\varpi((\alpha_2-1)(\beta_2-1))\,\varpi((\alpha_1-1)(\beta_1-1))\Big).$$

§3 Torelli group

$$\widehat{\mathcal{S}(\Sigma)} = \varprojlim_{i\to\infty} \mathcal{S}(\Sigma)/F^i\mathcal{S}(\Sigma).$$

<u>Thm</u> (T) $\partial\Sigma \neq \emptyset \Rightarrow \mathcal{S}(\Sigma) \hookrightarrow \widehat{\mathcal{S}(\Sigma)}$.

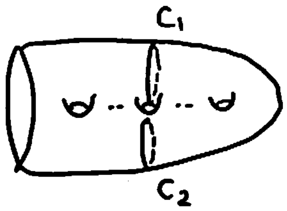
C : scc in Σ,

$$L(C) = \frac{-A+A^{-1}}{4\log(-A)}\left(\operatorname{arccosh}\left(-\frac{C}{2}\right)\right)^2 - (-A+A^{-1})\log(-A).$$

$\Sigma = \Sigma_{g,1}$,

$\mathcal{M}(\Sigma)$: mapping class group,

$\mathcal{I}(\Sigma)$: Torelli group.



Thm (T)

$\zeta : \mathcal{I}(\Sigma_{g,1}) \longrightarrow (F^3 \widehat{\mathcal{S}}(\Sigma), \mathrm{bch})$ is well-defined and injective.

$$t_{c_1} t_{c_2}^{-1} \mapsto L(c_1) - L(c_2)$$

Rem

- $C \subset \Sigma_{g,1}$: null-homologous SCC.
  $\Rightarrow \zeta(t_c) = L(c)$.

- $\mathcal{I}(\Sigma_{g,1}) \xrightarrow{\zeta} F^3 \widehat{\mathcal{S}}(\Sigma) \xrightarrow{\lambda_3^{-1}} H \wedge H \wedge H$

  coincides with the 1st Johnson homo.

- $\mathcal{K}(\Sigma_{g,1})$: Johnson kernel.

$$\begin{tikzcd}
\mathcal{K}(\Sigma_{g,1}) \arrow[r] \arrow[d, "\text{2nd Johnson}"'] & F^4 \mathcal{S}(\Sigma_{g,1}) \arrow[r, two heads] & F^4/F^5 \arrow[r, "\lambda_4^{-1}", "\cong"'] & S^2(S^2H) \oplus S^2H \oplus \mathbb{Q} \arrow[d] \\
S^2(\Lambda^2 H) \arrow[rrr] & & & S^2(S^2H)
\end{tikzcd}$$

$$S^2H \to \underbrace{S^2H}_{0\text{ map}}, \qquad \mathbb{Q} \to \underbrace{\mathbb{Q}}_{6\times \text{Casson core}}$$

$$(a_1 \wedge b_1)(a_2 \wedge b_2) \mapsto \frac{1}{2}\left((a_1 a_2)(b_1 b_2) - (a_1 b_2)(b_1 a_1) \right)$$

§4 An invariant for $\mathbb{Z}HS^3$

$S^3 = H_g^+ \cup_\iota H_g^-$, $\iota : \partial H_g^+ \to \partial H_g^-$: diffeo.

$\Sigma_{g,1} \subset \partial H_g^+$,

$\xi \in \mathcal{M}(\Sigma)$, $M(\xi) = H_g^+ \cup_{\iota\circ\xi} H_g^-$.

$\mathcal{H}(3)$: the set of $\mathbb{Z}HS^3$.

$e : \Sigma_{g,1} \times I \to S^3$: ori. pres. embedding

s.t. $e|_{\Sigma_{g,1}\times\{0\}} : \Sigma_{g,1}\times\{0\} \to \Sigma_{g,1} \hookrightarrow \partial H_g^+ \hookrightarrow S^3$

$(t, 0) \mapsto t$

e induces $e : \widehat{\mathcal{S}(\Sigma_{g,1})} \to \mathbb{Q}[[A+1]]$.

($\mathbb{Q}[[A+1]]$ - module homo.)

<u>Thm</u> (T)

The map $Z : \mathcal{I}(\Sigma_{g,1}) \to \mathbb{Q}[[A+1]]$ induces

$\xi \mapsto \sum_{i=0}^{\infty} \frac{1}{(-A+A^{-1})^i\, i!}\, e(\zeta(\xi)^i)$

$z : \mathcal{H}(3) \to \mathbb{Q}[[A+1]]$.

$M(\xi) \mapsto Z(\xi)$

<u>Question</u>

$z(M)|_{A^4=q}$: Ohtsuki series ?

<u>Rem</u> The coeff. of $(A+1)$ in $z(M)$ is

(-24) times the Casson inv. for M.

§5 Casson invariant

$\lambda_3' : H \wedge H \wedge H \to F^3 / F^5$

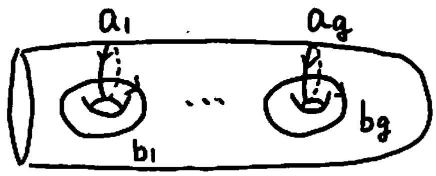
- $$\begin{tikzcd} H \wedge H \wedge H \arrow[r, "\lambda_3'"] \arrow[dr, "\lambda_3"'] & F^3/F^5 \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] \\ & F^3/F^4 \end{tikzcd}$$

- $\lambda_3'(H \wedge H \wedge H) \subset \mathrm{span}\,(\zeta(\mathcal{I}(\Sigma_{g,1})))$

- ${}^\forall v \in H \wedge H \wedge H$, $e(\lambda_3'(v)) = 0 \mod (A+1)^3$.

<u>Prop</u> There exists such λ_3' (not unique).

<u>Prop</u> $\zeta(\mathcal{I}(\Sigma_{g,1})) \subset \lambda_3'(H \wedge H \wedge H) \oplus \lambda_4(\mathbb{Q} \oplus S^2(S^2 H))$.



Thm (T)

$\xi \in \mathcal{I}(\Sigma_{g,1})$.

If
$$\zeta(\xi) = \sum_{\substack{i<j<k \\ \varepsilon_i, \varepsilon_j, \varepsilon_k}} q(i,j,k,\varepsilon_i,\varepsilon_j,\varepsilon_k)\lambda_3'\big((b_i+\varepsilon_i a_i) \wedge (b_j+\varepsilon_j a_j) \wedge (b_k+\varepsilon_k a_k)\big)$$
$$+ \sum_{i<j} q'(i,j)\lambda_4\big((a_i b_j)(a_i b_j)\big)$$
$$+ \sum_i q''(i)\lambda_4\big((a_i a_i)(b_i b_i)\big)$$
$$+ \sum_i q'''(i)\lambda_4\big((a_i b_i)^2\big) + q''''(A+1)^2 + X,$$

where $X \in$ span {other basis},

the Casson inv. for $M(\xi)$ is

$$\sum \varepsilon_i \varepsilon_j \varepsilon_k \big(q(i,j,k)\, \varepsilon_i \varepsilon_j \varepsilon_k\big)^2 \qquad (*_1)$$
$$+ \frac{1}{2}\sum_{i<j} q'(i,j) + \frac{3}{4}\sum q''(i) + \sum q'''(i) + \frac{1}{48} q''''. \qquad (*_2)$$

(Outline of proof)

$$z(M(\xi)) = 1 + \underbrace{\frac{1}{-A+A^{-1}} e(\zeta(\xi))}_{\substack{\shortparallel \\ -24(A+1)\, *_2}} + \underbrace{\frac{1}{2(-A+A^{-1})^2} e(\zeta(\xi)^2)}_{\substack{\shortparallel \\ -24(A+1)\, *_1}} \mod (A+1)^2.$$