On Categorical Models of GoI
Lecture 2

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August 25, 2009

In this lecture

- We shall discuss constructions based on a GoI Situation.
- I shall follow the papers: Haghverdi (MSCS 2000), Abramsky, Haghverdi & Scott (MSCS 2002).

Abramsky's Program:

GoI Situation

↓ $\mathcal{G}$

Weak Linear Categories

↓ $\mathcal{G}(\mathbb{C})(I, V)$

Linear Combinatory Algebra

↓ standardisation

Combinatory Algebra

↓ quotienting

λ-algebra

GoI construction (Abramsky), Int construction (JSV)

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- Objects: (A^+, A^-) where A^+ and A^- are objects of $\mathbb{C}$.

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- Identity: $1_{(A^+,A^-)} = s_{A^+,A^-}$.

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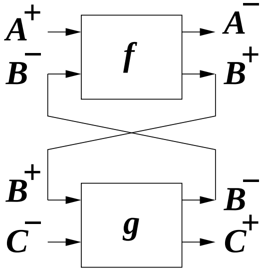
- Identity: $1_{(A^+, A^-)} = s_{A^+, A^-}$.
- Composition: Composition is given by symmetric feedback. Given $f : (A^+, A^-) \longrightarrow (B^+, B^-)$ and $g : (B^+, B^-) \longrightarrow (C^+, C^-)$, $gf : (A^+, A^-) \longrightarrow (C^+, C^-)$ is given by:

$$gf = Tr^{B^- \otimes B^+}_{A^+ \otimes C^-, A^- \otimes C^+}(\beta(f \otimes g)\alpha)$$

where $\alpha = (1_{A^+} \otimes 1_{B^-} \otimes s_{C^-, B^+})(1_{A^+} \otimes s_{C^-, B^-} \otimes 1_{B^+})$ and
$\beta =$
$(1_{A^-} \otimes 1_{C^+} \otimes s_{B^+, B^-})(1_{A^-} \otimes s_{B^+, C^+} \otimes 1_{B^-})(1_{A^-} \otimes 1_{B^+} \otimes s_{B^-, C^+})$.

In pictures



Monoidal structure

- Tensor: $(A^+, A^-) \otimes (B^+, B^-) = (A^+ \otimes B^+, A^- \otimes B^-)$ and for $f : (A^+, A^-) \longrightarrow (B^+, B^-)$ and $g : (C^+, C^-) \longrightarrow (D^+, D^-)$,

$$f \otimes g = (1_{A^-} \otimes s_{B^+,C^-} \otimes 1_{D^+})(f \otimes g)(1_{A^+} \otimes s_{C^+,B^-} \otimes 1_{D^-})$$

- Unit: (I, I).

Proposition

Let $\mathbb{C}$ be a traced symmetric monoidal category , $\mathcal{G}(\mathbb{C})$ defined as above is a compact closed category. Moreover, $F : \mathbb{C} \longrightarrow \mathcal{G}(\mathbb{C})$ with $F(A) = (A, I)$ and $F(f) = f$ is a full and faithful embedding.

This says that any traced symmetric monoidal category $\mathbb{C}$ arises as a monoidal subcategory of a compact closed cateorgy, namely $\mathcal{G}(\mathbb{C})$.

Proof.

Sketch

- For (A^+, A^-) and (B^+, B^-) in $\mathcal{G}(\mathbb{C})$, we define $s_{(A^+,A^-),(B^+,B^-)} =_{def} (1_{A^-} \otimes s_{B^+,B^-} \otimes 1_{A^+})(s_{B^+,A^-} \otimes s_{A^+,B^-})(1_{B^+} \otimes s_{A^+,A^-} \otimes 1_{B^-})(s_{A^+,B^+} \otimes s_{B^-,A^-})$.

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- The dual of (A^+, A^-) is given by $(A^+, A^-)^* = (A^-, A^+)$

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- unit, $\eta : (I, I) \longrightarrow (A^+, A^-) \otimes (A^+, A^-)^* =_{def} s_{A^-,A^+}$

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- The internal homs, $(A^+, A^-) \multimap (B^+, B^-) = (B^+ \otimes A^-, B^- \otimes A^+)$.

□

Useful facts

- Let $A^+ \cong B^+$ and $A^- \cong B^-$ in $\mathbb{C}$, then $(A^+, A^-) \cong (B^+, B^-)$ in $\mathcal{G}(\mathbb{C})$.
- If $A^+ \lhd B^+ \, (f_1, g_1)$ and $A^- \lhd B^- \, (f_2, g_2)$ in $\mathbb{C}$, then $(A^+, A^-) \lhd (B^+, B^-) \; (s_{B^+, A^-}(f_1 \otimes g_2), s_{A^+, B^-}(g_1 \otimes f_2))$ in $\mathcal{G}(\mathbb{C})$.

Weak Linear Category (WLC)

Definition

A *Weak Linear Category (WLC)* $(\mathbb{C}, !)$ consists of the following data:

- A symmetric monoidal closed category $\mathbb{C}$,
- A symmetric monoidal functor $! : \mathbb{C} \longrightarrow \mathbb{C}$ (officially, $! = (!, \varphi, \varphi_I)$),
- The following monoidal pointwise natural transformations:
  1. $\mathtt{der} :! \Rightarrow Id$
  2. $\delta :! \Rightarrow !!$
  3. $\mathtt{con} :! \Rightarrow ! \otimes !$
  4. $\mathtt{weak} :! \Rightarrow \mathcal{K}_I$. Here $\mathcal{K}_I$ is the constant I functor.

Important remark

- Pointwise naturality: $\alpha : F \Rightarrow G$: For all $f : I \longrightarrow A$,

$$\begin{array}{ccc} FI & \xrightarrow{\alpha_I} & GI \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FA & \xrightarrow{\alpha_A} & GA \end{array}$$

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- In the GoI models we discuss the monoidal transformations $\mathtt{der}, \delta, \mathtt{con}, \mathtt{weak}$ exist but are merely pointwise natural
- Pointwise naturality suffices for the construction of linear combinatory algebras
- We do not require $(!, \mathtt{der}, \delta)$ to form a comonad,
- We do not require $(!A, \mathtt{con}_A, \mathtt{weak}_A)$ to form a comonoid.

Reflexive object

Definition

A *reflexive* object in a WLC $(\mathbb{C}, !)$ is an object V in $\mathbb{C}$ with the following retracts:

- $V \multimap V \lhd V$
- $!V \lhd V$
- $I \lhd V$

Another remark

Since CCCs are SMCCs, all the usual domain theoretic constructions of reflexive objects in CCCs also yield reflexive objects in the WLC-sense, as follows:

Proposition

Let $\mathbb{C}$ be a CCC and V be a reflexive object in $\mathbb{C}$, i.e., $V^V \lhd V$. Then $(\mathbb{C}, Id)$ is a WLC and V is a reflexive object in the WLC-sense.

Proof.

Any CCC is an SMCC. *Id* is a symmetric monoidal functor from $\mathbb{C}$ to itself and the following are monoidal natural transformations:

1. $\mathtt{der}_A = 1_A$
2. $\delta_A = 1_A$
3. $\mathtt{con}_A = \langle 1_A, 1_A \rangle$
4. $\mathtt{weak}_A = f : A \longrightarrow T$; the unique map to the terminal object.

It can be easily shown that $V^V \lhd V$ implies $T \lhd V$. Therefore $V \multimap V = V^V \lhd V$, $!V = Id(V) = V \lhd V$ and $I = T \lhd V$ and hence V is a reflexive object in the WLC-sense. □

Linear Combinatory Algebra (LCA)

Definition

A *Linear Combinatory Algebra* $(A, \cdot, !)$ consists of the following data:

- An applicative structure $(A, \cdot)$
- A unary operator $! : A \to A$
- Distinguished elements $B, C, I, K, W, D, \delta, F$ of A,

satisfying the following identities (we associate $\cdot$ to the left and write xy for $x \cdot y$, $x!y = x \cdot (!(y))$, etc.) for all variables x, y, z ranging over A.

1. $Bxyz = x(yz)$ Composition, Cut

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5. $Wx!y = x!y!y$ Contraction
6. $D!x = x$ Dereliction
7. $\delta!x = !!x$ Comultiplication

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6. $D\mathbf{!}x = x$ Dereliction
7. $\delta\mathbf{!}x = \mathbf{!!}x$ Comultiplication
8. $F\mathbf{!}x\mathbf{!}y = \mathbf{!}(xy)$ Monoidal Functoriality

Remarks

- The notion of LCA corresponds to a Hilbert style axiomatization of the $\{!, \multimap\}$ fragment of int. linear logic.
- The *principal types of the combinators correspond to the axiom schemes which they name.*

  1. $B : (\beta \multimap \gamma) \multimap (\alpha \multimap \beta) \multimap \alpha \multimap \gamma$
  2. $C : (\alpha \multimap \beta \multimap \gamma) \multimap (\beta \multimap \alpha \multimap \gamma)$
  3. $I : \alpha \multimap \alpha$
  4. $K : \alpha \multimap \mathbf{!}\beta \multimap \alpha$
  5. $W : (\mathbf{!}\alpha \multimap \mathbf{!}\alpha \multimap \beta) \multimap \mathbf{!}\alpha \multimap \beta$
  6. $D : \mathbf{!}\alpha \multimap \alpha$
  7. $\delta : \mathbf{!}\alpha \multimap \mathbf{!!}\alpha$
  8. $F : \mathbf{!}(\alpha \multimap \beta) \multimap \mathbf{!}\alpha \multimap \mathbf{!}\beta$

From WLC to LCA

Theorem
Let $(\mathbb{C}, !)$ be a WLC and V be a reflexive object in $\mathbb{C}$ with retracts $V \multimap V \lhd V\,(r, s)$ and $!V \lhd V\,(p, q)$. Then $(\mathbb{C}(I, V), \bullet, \mathbf{!})$ with $\bullet$ and $\mathbf{!}$ defined below is a linear combinatory algebra.

Proof.
Sketch

- Given $f, g \in \mathbb{C}(I, V)$, $f \bullet g = ev(sf \otimes g)$
- Given $f \in \mathbb{C}(I, V)$, $\mathbf{!}f = p!f\varphi_I$ where $\varphi_I : I \longrightarrow !I$ and $! = (!, \varphi, \varphi_I)$.

□

Standard Combinatory Algebra (SCA)

Definition

A *Standard Combinatory Algebra* consists of a pair $(A, \cdot_s)$ where A is a nonempty set and $\cdot_s$ is a binary operation on A and B_s, C_s, I_s, K_s, and W_s are distinguished elements of A satisfying the following identities for all x, y, z variables ranging over A:

1. $B_s \cdot_s x \cdot_s y \cdot_s z = x \cdot_s (y \cdot_s z)$
2. $C_s \cdot_s x \cdot_s y \cdot_s z = (x \cdot_s z) \cdot_s y$
3. $I_s \cdot_s x = x$
4. $K_s \cdot_s x \cdot_s y = x$
5. $W_s \cdot_s x \cdot_s y = x \cdot_s y \cdot_s y$

Remarks

- S_s can be defined from B_s, C_s, I_s and W_s.

From LCA to CA

Let $(A, \cdot, !)$ be a linear combinatory algebra. We define a binary operation $\cdot_s$ on A as follows: for $\alpha, \beta \in A$, $\alpha \cdot_s \beta =_{def} \alpha \cdot ! \beta$. We define $D' = C(BBI)(BDI)$. Note that

$$D' x ! y = xy.$$

Consider the following elements of A.

1. $B_s =_{def} C \cdot (B \cdot (B \cdot B \cdot B) \cdot (D' \cdot I)) \cdot (C \cdot ((B \cdot B) \cdot F) \cdot \delta)$
2. $C_s =_{def} D' \cdot C$
3. $I_s =_{def} D' \cdot I$
4. $K_s =_{def} D' \cdot K$
5. $W_s =_{def} D' \cdot W$

Theorem

Let $(A, \cdot, !)$ be a linear combinatory algebra. Then $(A, \cdot_s)$ with $\cdot_s$ and the elements B_s, C_s, I_s, K_s, W_s as defined above is a combinatory algebra.

In the case of WLCs coming from CCCs, the associated linear combinatory algebra agrees with the (standard) combinatory algebra structure, since

$$x \cdot_s y = x \cdot !y = x.y \ .$$

General GoI Construction

A *GoI Situation* is a triple $(\mathbb{C}, T, U)$ where:

- $\mathbb{C}$ is a traced symmetric monoidal category
- $T : \mathbb{C} \longrightarrow \mathbb{C}$ is a traced symmetric monoidal functor with the following retractions (which are monoidal natural transformations):
    1. $TT \lhd T\,(e, e')$ (Comultiplication)
    2. $Id \lhd T\,(d, d')$ (Dereliction)
    3. $T \otimes T \lhd T\,(c, c')$ (Contraction)
    4. $\mathcal{K}_I \lhd T\,(w, w')$ (Weakening), where $\mathcal{K}_I$ is the constant I functor.
- U is an object of $\mathbb{C}$, called *a reflexive object*, with retractions:
    1. $U \otimes U \lhd U\,(j, k)$
    2. $I \lhd U$
    3. $TU \lhd U\,(u, v)$

- $\mathcal{G}(\mathbb{C})$ with the distinguished objects $I = (I, I)$ and $V = (U, U)$.
- Note that by definition (since we are in the strict case) $\mathcal{G}(\mathbb{C})(I, V) = \mathbb{C}(U, U)$.
- We can define an endofunctor $! : \mathcal{G}(\mathbb{C}) \longrightarrow \mathcal{G}(\mathbb{C})$ as follows: $!(A^+, A^-) = (TA^+, TA^-)$ and given $f : (A^+, A^-) \longrightarrow (B^+, B^-)$, $!f \ =_{def} TA^+ \otimes TB^- \xrightarrow{\cong} T(A^+ \otimes B^-) \xrightarrow{Tf} T(A^- \otimes B^+) \xrightarrow{\cong} TA^- \otimes TB^+$.

Proposition

Let $(\mathbb{C}, T, U)$ *be a GoI Situation. Then:*

(i) $(\mathcal{G}(\mathbb{C}), !)$ *is a WLC with reflexive object* $V = (U, U)$,

(ii) $(\mathcal{G}(\mathbb{C})(I, V), \cdot, \mathbf{!})$ *is an LCA, where for any* $f, g \in \mathcal{G}(\mathbb{C})(I, V) = \mathbb{C}(U, U)$,
$f \cdot g = Tr^{U}_{U,U}((1_U \otimes g)(kfj))$, *and*
$\mathbf{!} f = u(Tf)v$.

Proof.

Sketch

Note that $\mathcal{G}(\mathbb{C})$ is a compact closed category and hence it is symmetric monoidal closed.

It can be easily shown that ! is a symmetric monoidal functor.

Next we define the following maps:

- $\mathtt{der}_{(A^+,A^-)}: \ !(A^+, A^-) \longrightarrow (A^+, A^-) =_{def} s_{A^+,TA^-}(d'_{A^+} \otimes d_{A^-})$ where $A \lhd TA\,(d_A, d'_A)$,

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- $\delta_{(A^+,A^-)} :\ !(A^+, A^-) \longrightarrow !!(A^+, A^-) =_{def} s_{T^2A^+,TA^-}(e'_{A^+} \otimes e_{A^-})$ where $T^2A \lhd TA\,(e_A, e'_A)$,

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- $\mathtt{con}_{(A^+,A^-)}: \ !(A^+, A^-) \longrightarrow !(A^+, A^-) \otimes !(A^+, A^-) =_{def} s_{TA^+\otimes TA^+,TA^-}(c'_{A^+} \otimes c_{A^-})$ where $TA \otimes TA \lhd TA\,(c_A, c'_A)$,

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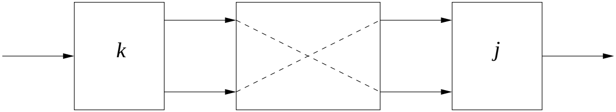
- $\delta_{(A^+,A^-)}: \,!(A^+,A^-) \longrightarrow !!(A^+,A^-) =_{def} s_{T^2A^+,TA^-}(e'_{A^+} \otimes e_{A^-})$ where $T^2A \lhd TA\,(e_A, e'_A)$,
- $\mathtt{con}_{(A^+,A^-)}: \,!(A^+,A^-) \longrightarrow !(A^+,A^-) \otimes !(A^+,A^-) =_{def} s_{TA^+\otimes TA^+,TA^-}(c'_{A^+} \otimes c_{A^-})$ where $TA \otimes TA \lhd TA\,(c_A, c'_A)$,
- $\mathtt{weak}_{(A^+,A^-)}: \,!(A^+,A^-) \longrightarrow (I,I) =_{def} s_{I,TA^-}(w'_{A^+} \otimes w_{A^-})$ where $I \lhd TA\,(w_A, w'_A)$.

□

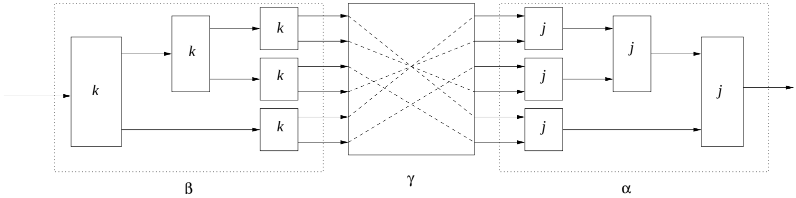
The combinators

$I =_{def} js_{U,U}k,$



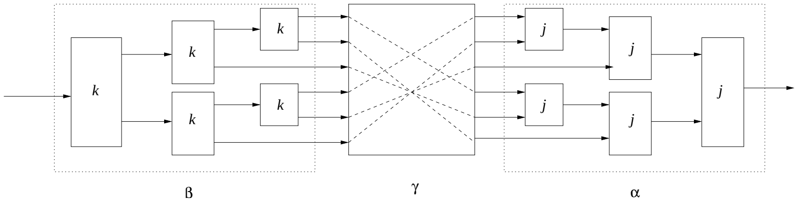
$B =_{def} \alpha\gamma\beta$, where

1. $\alpha = j(j \otimes 1_U)(j \otimes j \otimes j)$
2. $\beta = (k \otimes k \otimes k)(k \otimes 1_U)k$
3. γ see figure below.



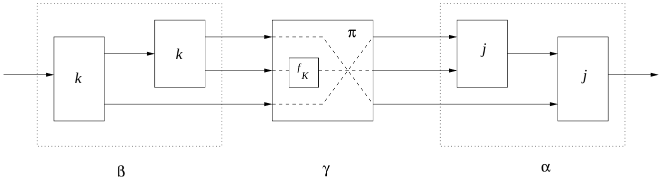
$C =_{def} \alpha\gamma\beta$, where

1. $\alpha = j(j \otimes j)(j \otimes 1_U \otimes j \otimes 1_U)$
2. $\beta = (k \otimes 1_U \otimes k \otimes 1_U)(k \otimes k)k$
3. γ see figure below.



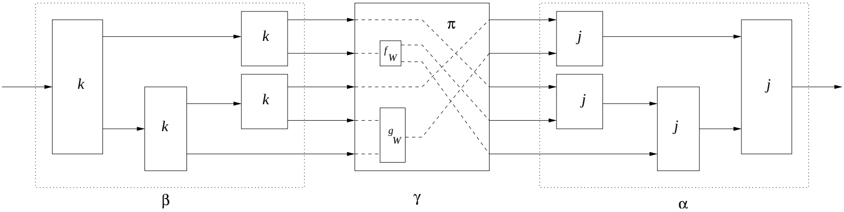
$K =_{def} \alpha\gamma\beta$, where

1. $\alpha = j(j \otimes 1)$
2. $\beta = (k \otimes 1)k$
3. $\gamma = \pi(1_U \otimes f_K \otimes 1_U)$, where $f_K = uw_U w'_U v$ and π as in figure below.



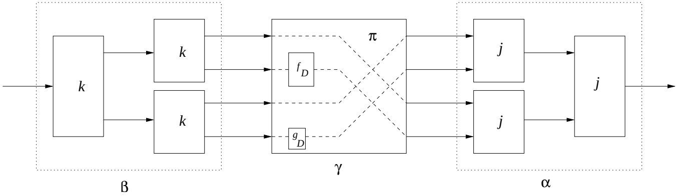
$W =_{def} \alpha\gamma\beta$, where

1. $\alpha = j(1_U \otimes j)(j \otimes j \otimes 1_U)$
2. $\beta = (k \otimes k \otimes 1_U)(1_U \otimes k)k$
3. $\gamma = \pi(1_U \otimes g_W \otimes 1_U \otimes f_W)(1_U \otimes 1_U \otimes 1_U \otimes \sigma)$, where $g_W = (u \otimes u)c'_U v$, $f_W = uc_U(v \otimes v)$, and π is the permutation in the figure below.



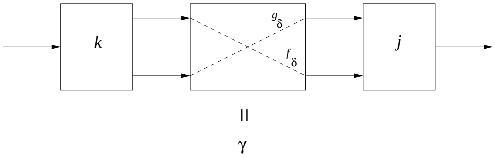
$D =_{def} \alpha\gamma\beta$, where

1. $\alpha = j(j \otimes j)$
2. $\beta = (k \otimes k)k$
3. $\gamma = \pi(1_U \otimes g_D \otimes 1_U \otimes f_D)$, where $f_D = ud_U$, $g_D = d'_U v$ and π as in the figure below.



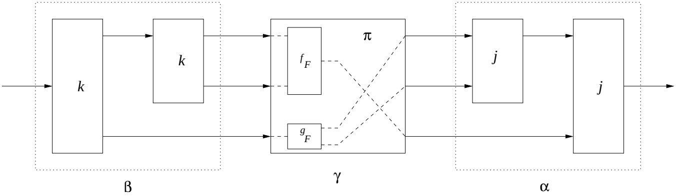
$\delta =_{def} \alpha\gamma\beta$, where

1. $\alpha = j$
2. $\beta = k$
3. $\gamma = \sigma_{U,U}(f_\delta \otimes g_\delta)$, where $f_\delta = ue_U T(v)v$ and $g_\delta = uT(u)e'_U v$.



$F =_{def} \alpha\gamma\beta$, where

1. $\alpha = j(j \otimes 1_U)$
2. $\beta = (k \otimes 1_U)k$
3. $\gamma = \pi(f_F \otimes g_F)$, where $f_F = uT(j)\psi_{U,U}(v \otimes v)$, $g_F = (u \otimes u)\psi^{-1}_{U,U}T(k)v$, π is the permutation given in the figure below, and $T = (T, \psi, \psi_I)$.



Combinators in UDCs: Particle-style Semantics

Haghverdi, (MSCS 2000)

- $\mathbb{C}$ a traced UDC, $(\mathbb{C}, T, U)$ a GoI Situation.
- $(\mathbb{C}(U,U), \bullet, \mathbf{!})$, $TU \lhd U(u,v)$, $U \otimes U \lhd U(j,k)$.
- $\alpha \bullet \beta = Tr^{U}_{U,U}((1_U \otimes \beta)(k\alpha j))$
- $\mathbf{!}\alpha = uT(\alpha)v$.

In matricial form we have:

$$\alpha \cdot \beta = Tr^{U}_{U,U}\left(\begin{bmatrix} 1 & 0 \\ 0 & \beta \end{bmatrix}\begin{bmatrix} k_1\alpha j_1 & k_1\alpha j_2 \\ k_2\alpha j_1 & k_2\alpha j_2 \end{bmatrix}\right)$$

$$j = \begin{bmatrix} j_1 & j_2 \end{bmatrix} \qquad k = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

Combinators

- $I = j_2 k_1 + j_1 k_2$
- $K = j_2 k_1^2 + j_1^2 k_2$.
- $W = j_2 j_1^2 k_1^2 + j_1 j_2 f_{W1} k_2^2 + j_1 j_2 f_{W2} k_2 k_1 k_2 + j_2^2 g_{W1} k_2 k_1 + j_2 j_1 j_2 g_{W2} k_2 k_1 + j_1^2 k_1^2 k_2$.
  Where $f_{W1} = uc_U(v \otimes v)\iota_1$, $f_{W2} = uc_U(v \otimes v)\iota_2$, $g_{W1} = \rho_1(u \otimes u)c'_U v$, and $g_{W2} = \rho_2(u \otimes u)c'_U v$.

- $B = j_2 j_1 k_1^3 + j_1^3 k_1 k_2 + j_2^2 k_1 k_2 k_1 + j_1 j_2 j_1 k_2^2 + j_1^2 j_2 k_2^2 k_1 + j_1 j_2^2 k_2 k_1^2$
- $C = j_2 j_1^2 k_1^3 + j_1^3 k_1^2 k_2 + j_1 j_2 k_2 k_1 k_2 + j_2 j_1 j_2 k_2 k_1 + j_1^2 j_2 k_2^2 + j_2^2 k_2 k_1^2$
- $D = j_2 j_1 k_1^2 + j_1^2 k_1 k_2 + j_1 j_2 f_D k_2^2 + j_2^2 g_D k_2 k_1,$
- $\delta = j_2 f_\delta k_1 + j_1 g_\delta k_2,$
- $F = j_2 f_{F1} k_1^2 + j_1^2 g_{F1} k_2 + j_1 j_2 g_{F2} k_2 + j_2 f_{F2} k_2 k_1$
  Here $f_D = u l_U, g_D = l'_U v, f_\delta = u e_U T(v) v, g_\delta = u T(u) e'_U v,$
  $f_{F1} = u T(j) \varphi(v \otimes v) \iota_1$, $g_{F1} = \rho_1 (u \otimes u) \varphi^{-1} T(k) v$,
  $f_{F2} = u T(j) \varphi(v \otimes v) \iota_2$, $g_{F2} = \rho_2 (u \otimes u) \varphi^{-1} T(k) v$, and φ is the component of the monoidal functor T.