From asynchronous games to coherence spaces

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Workshop on

Geometry of Interaction, Traced Monoidal Categories, Implicit Complexity

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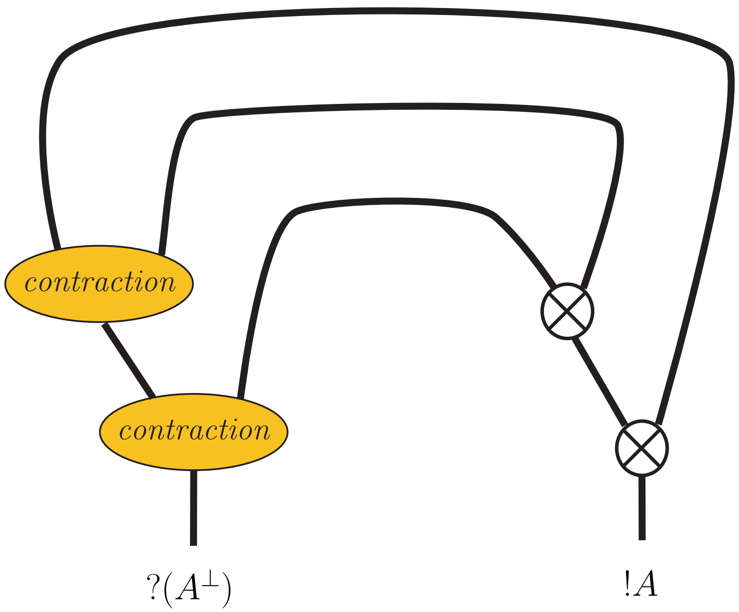
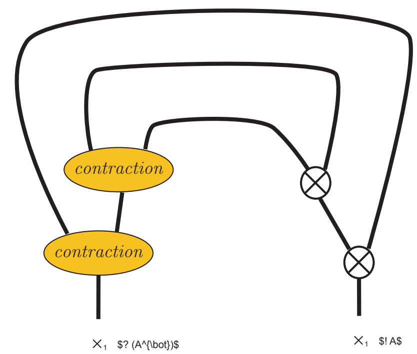
An anomaly of the Geometry of Interaction





Very much studied in the field of game semantics

Game semantics

Every proof of formula A initiates a dialogue where

Proponent tries to convince Opponent

Opponent tries to refute Proponent

An interactive approach to logic and programming languages

Four basic operations on logical games

the negation	$\neg\, A$
the sum	$A \oplus B$
the tensor	$A \otimes B$
the exponential	$!\, A$

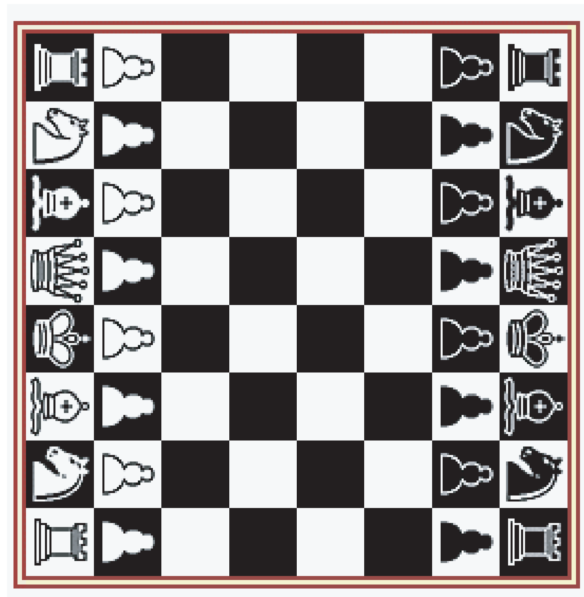
Algebraic structure similar to linear algebra !

Negation

Proponent
Program

plays the game

A



Opponent
Environment

plays the game

$\neg\ A$

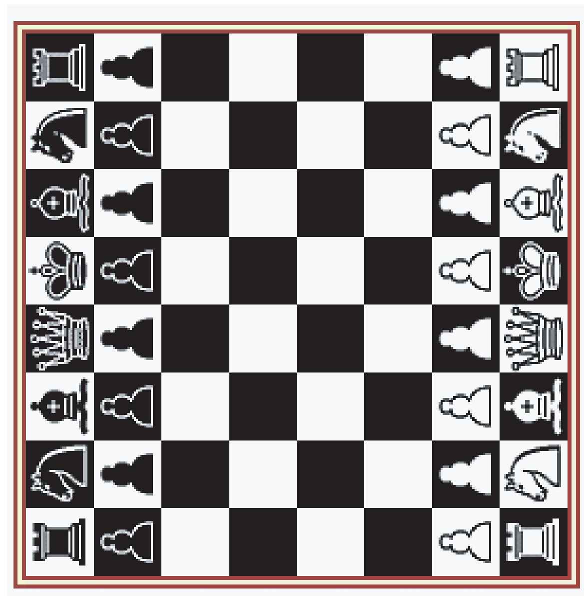
Negation permutes the rôles of Proponent and Opponent

Negation

Opponent
Environment

plays the game

$\neg A$



Proponent
Program

plays the game

A

Negation permutes the rôles of Opponent and Proponent

Sum

$\oplus$

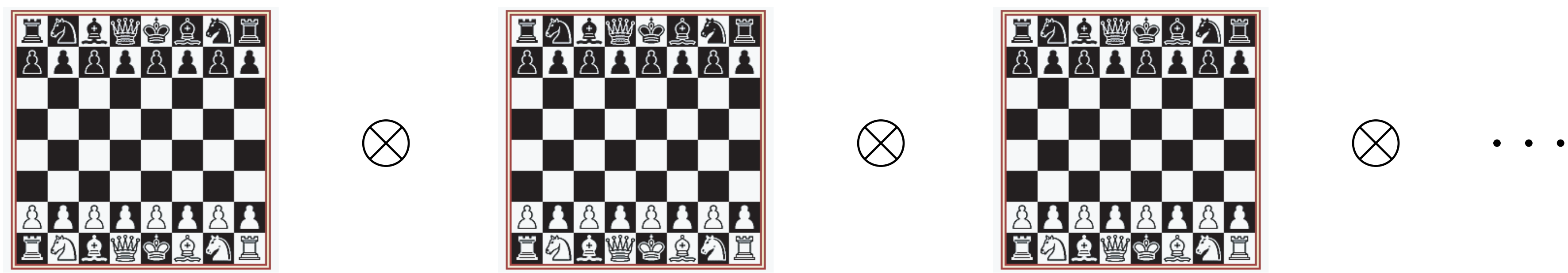
Proponent selects one component

Tensor product

$$\otimes$$

Opponent plays the two games in parallel

Exponentials



Opponent opens as many copies as necessary to beat Proponent

Policy of the talk

In order to clarify game semantics, compare it to relational semantics...

Key idea: the strategy σ associated to a proof π should contain its clique.

Part I

Additives in sequential games

Sequential strategies at the leaves

Sequential games

A proof π

alternating
sequences of moves

A proof π

Sequential games

A sequential game (M, P, λ) consists of

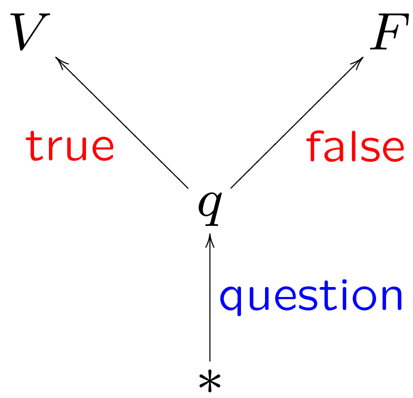
M	a set of **moves**,
$P \subseteq M^*$	a set of **plays**,
$\lambda : M \to \{-1, +1\}$	a **polarity** function on moves

such that every play is **alternating** and **starts by Opponent.**

Alternatively, a sequential game is an alternating decision tree.

Sequential games

The boolean game $\mathbb{B}$:



Player in red
Opponent in blue

Strategies

A **strategy** σ is a set of **alternating plays** of **even-length**

$$s \quad = \quad m_1 \cdots m_{2k}$$

such that:

— σ contains **the empty play,**

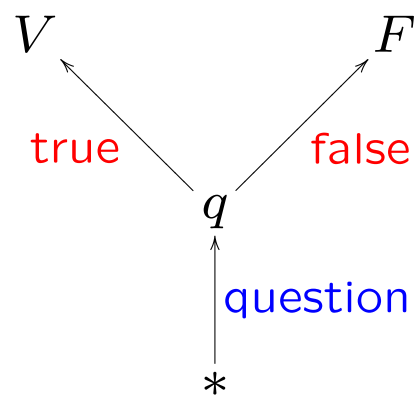
— σ is **closed by even-length prefix:**

$$\forall s, \forall m, n \in M, \qquad s \cdot m \cdot n \in \sigma \;\Rightarrow\; s \in \sigma$$

— σ is **deterministic:**

$\forall s \in \sigma, \forall m, n_1, n_2 \in M, \qquad s \cdot m \cdot n_1 \in \sigma$ and $s \cdot m \cdot n_2 \in \sigma \;\Rightarrow\; n_1 = n_2$.

Three strategies on the boolean game $\mathbb{B}$



Player in red
Opponent in blue

Total strategies

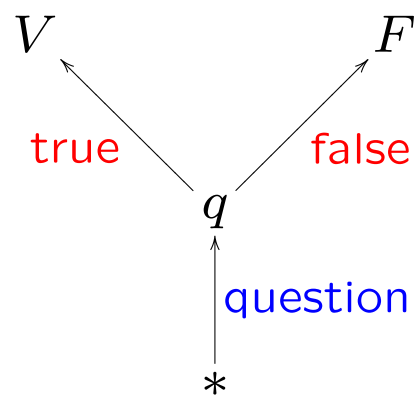
A strategy σ is **total** when

— for every play s of the strategy σ,

— for every Opponent move m such that $s \cdot m$ is a play,

there exists a Proponent move n such that $s \cdot m \cdot n$ is a play of σ.

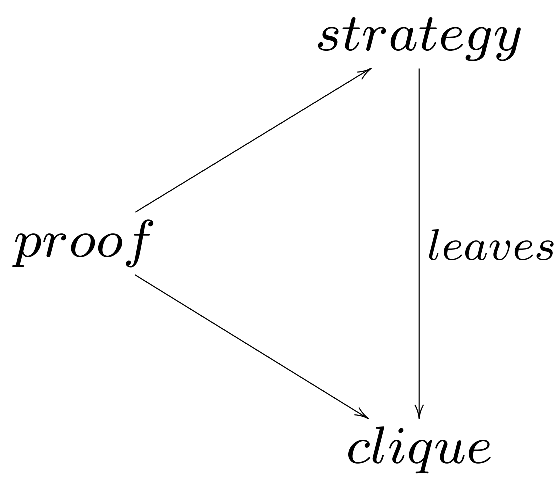
Two total strategies on the boolean game $\mathbb{B}$



Player in red
Opponent in blue

From sequential games to coherence spaces

The diagram commutes



for every proof of a purely additive formula.

From sequential games to coherence spaces

Let $\mathscr{G}$ denote the category

— with families of sequential games as objects,
— with families of sequential strategies as morphisms.

Proposition. The category $\mathscr{G}$ is the **free category** with sums, equipped with a contravariant functor

$$\neg \quad : \quad \mathscr{G} \quad \longrightarrow \quad \mathscr{G}^{op}$$

and a bijection

$$\varphi_{x,y} \quad : \quad \mathscr{G}(x, \neg y) \quad \cong \quad \mathscr{G}(y, \neg x)$$

natural in x and y.

A theorem for free

There exists a functor

$$leaves \quad : \quad \mathscr{G} \quad \longrightarrow \quad \mathbf{Coh}$$

which preserves the sum, and transports the non-involutive negation of the category $\mathscr{G}$ into the involutive negation of the category $\mathbf{Coh}$.

This functor collapses the dynamic semantics into a static one

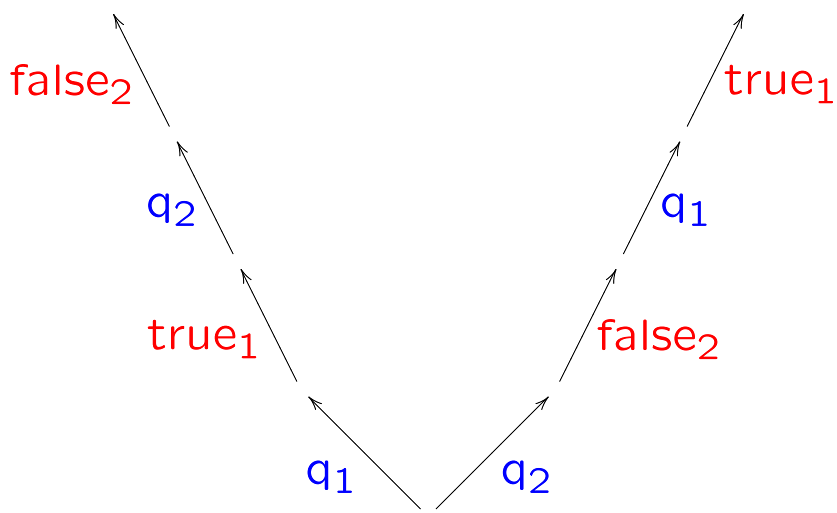
Part II (a)

Multiplicatives in asynchronous games

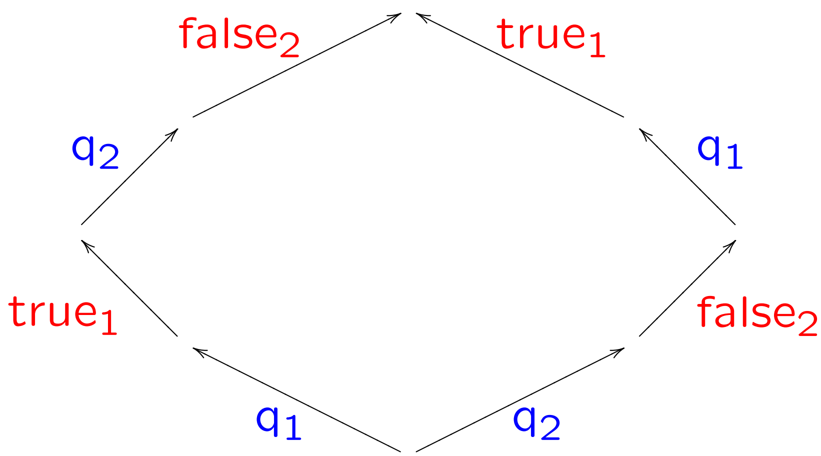
From trajectories to positions

Sequential games: an interleaving semantics

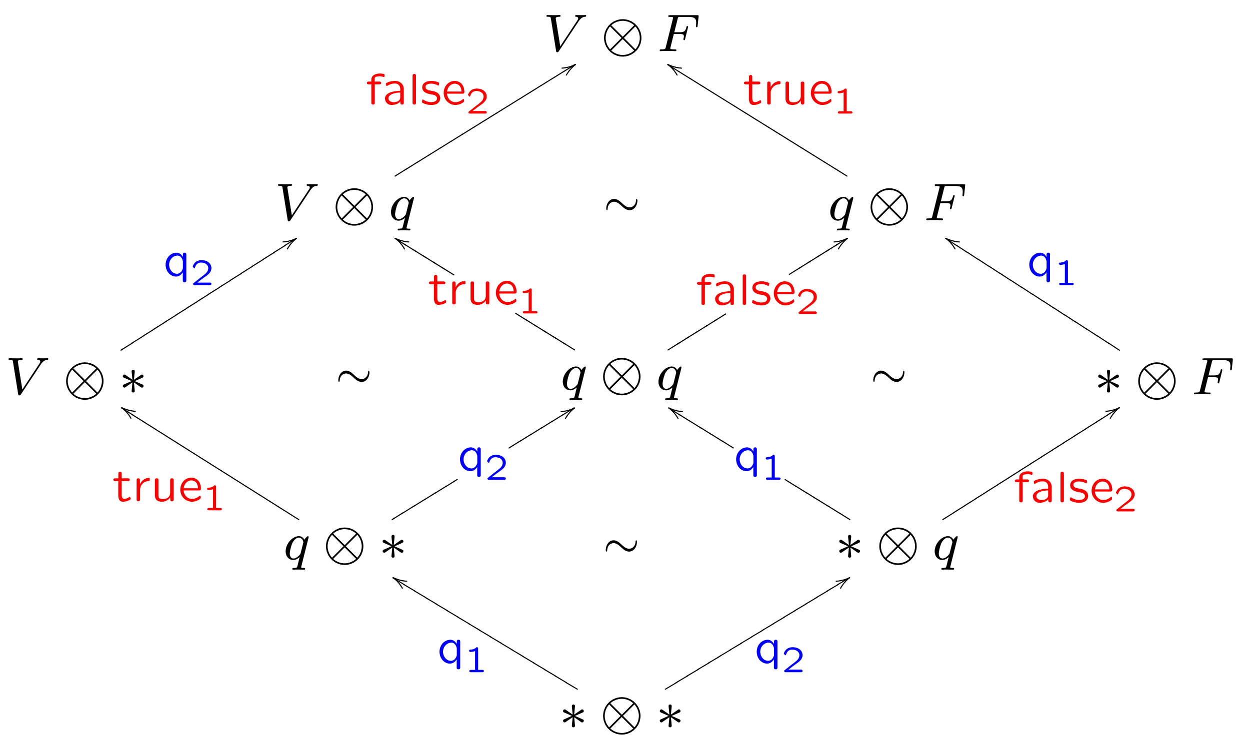
The tensor product of two boolean games $\mathbb{B}_1$ et $\mathbb{B}_2$:



A step towards true concurrency: bend the branches!



True concurrency: tile the diagram!



Asynchronous game semantics

A proof π

trajectories in
asynchronous transition spaces

A proof π

The phenomenon refined: a **truly concurrent** semantics of proofs.

Asynchronous games

An **asynchronous game** is an event structure equipped with a **polarity** function

$$\lambda \quad : \quad M \longrightarrow \{-1, +1\}$$

indicating whether a move is Player ($+1$) or Opponent (-1).

Legal plays

A **legal play** is a path

$$* \xrightarrow{m_1} x_1 \xrightarrow{m_2} x_2 \xrightarrow{m_3} \cdots x_{k-1} \xrightarrow{m_k} x_k$$

starting from the empty position $*$ of the transition space, and satisfying:

$$\forall i \in [1, ..., k], \qquad \lambda(m_i) = (-1)^i.$$

So, a legal play is **alternated** and starts by an **Opponent move.**

Strategies

A **strategy** is a set of **legal plays of even length**, such that:

— σ contains **the empty play,**

— σ is **closed under even-length prefix**

$$s \cdot m \cdot n \in \sigma \;\Rightarrow\; s \in \sigma,$$

— σ is **deterministic**

$$s \cdot m \cdot n_1 \in \sigma \quad \text{and} \quad s \cdot m \cdot n_2 \in \sigma \quad \Rightarrow \quad n_1 = n_2.$$

A strategy plays according to the current play.

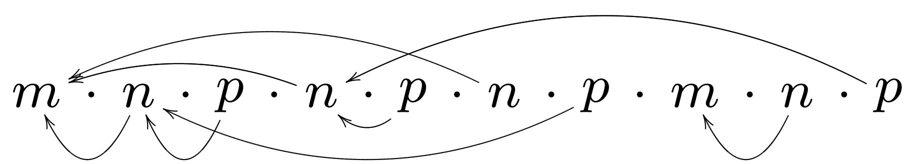
Innocence: strategies with partial information

Full abstraction result [Martin Hyland, Luke Ong, Hanno Nickau, 1994]

Innocence characterizes the interactive behaviour of λ-terms.

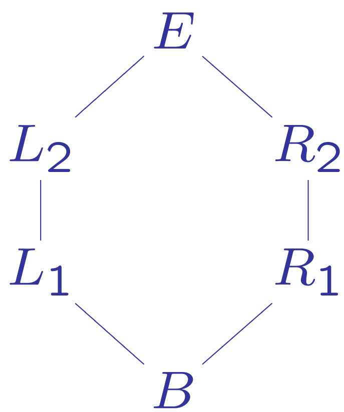
An innocent strategy plays according to the current view.

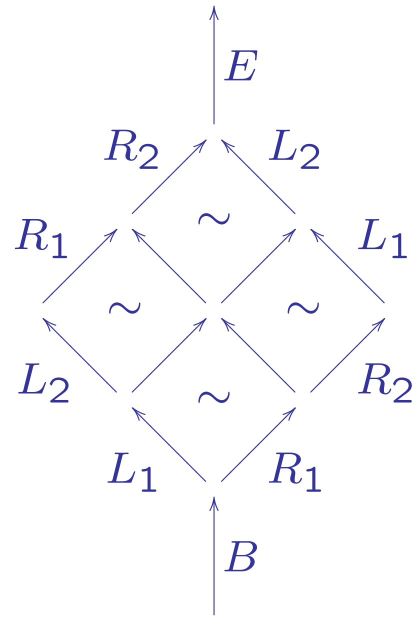
Where are the pointers in asynchronous games?

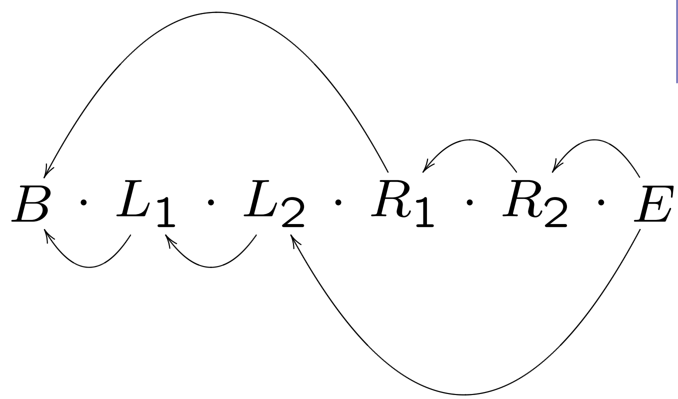


Play = sequence of moves with pointers

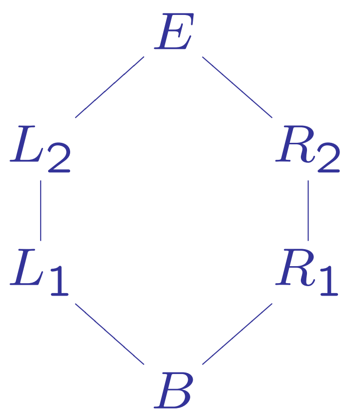
Event structure = generalized arena

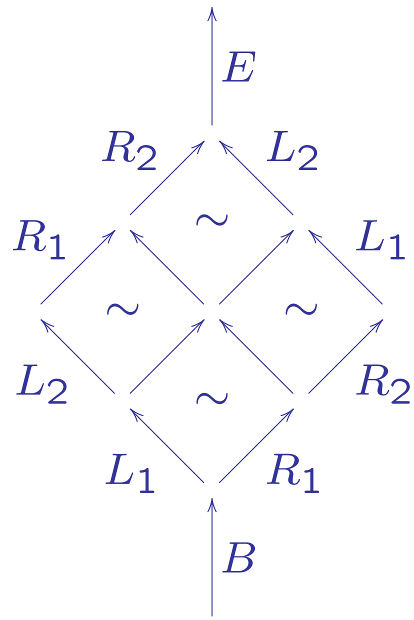


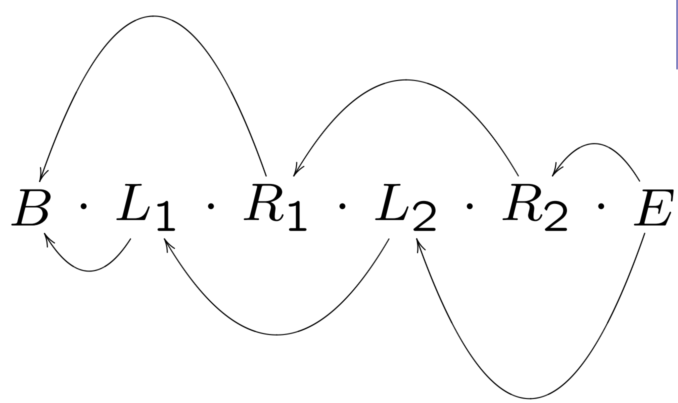




Event structure = generalized arena

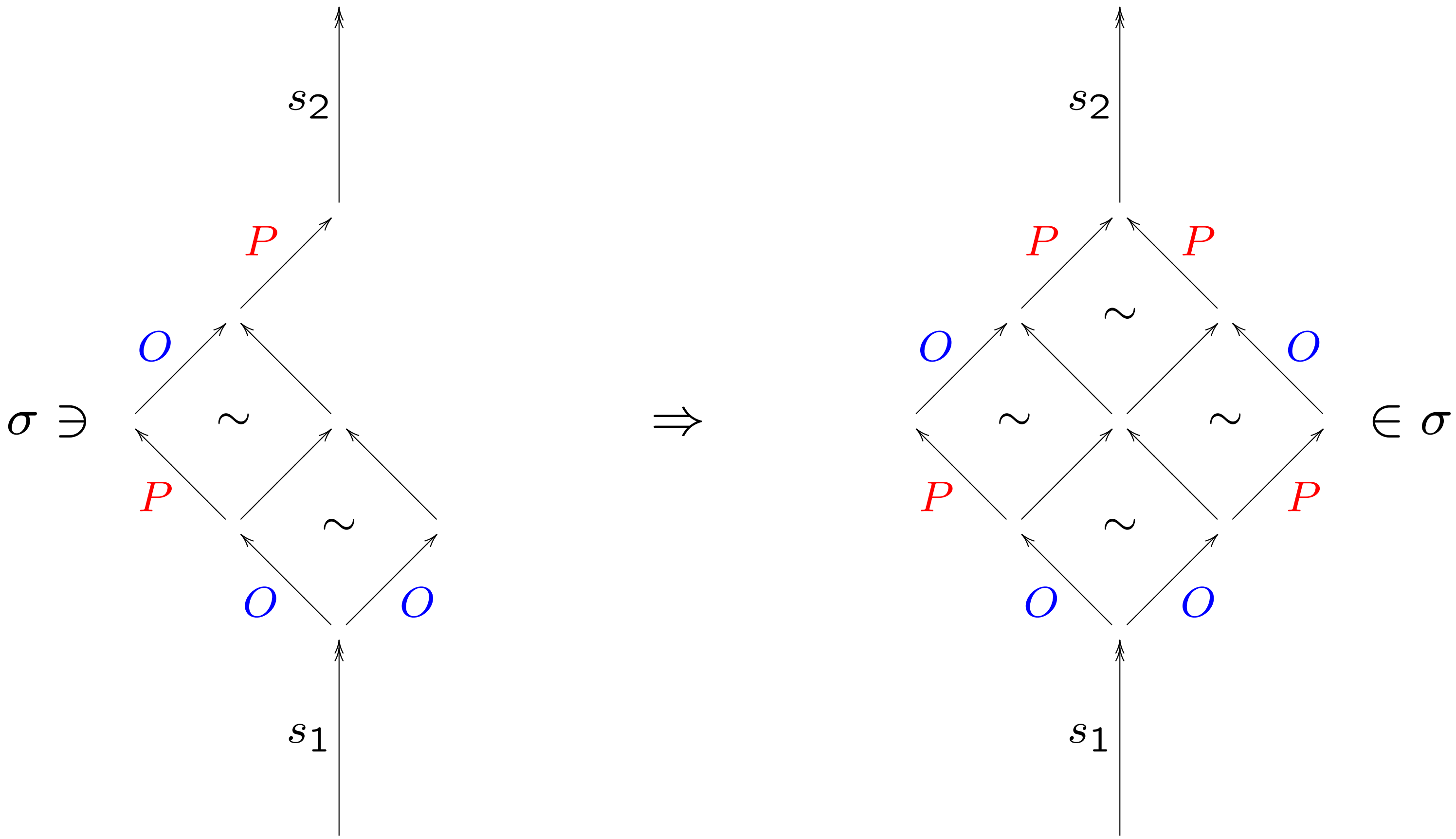




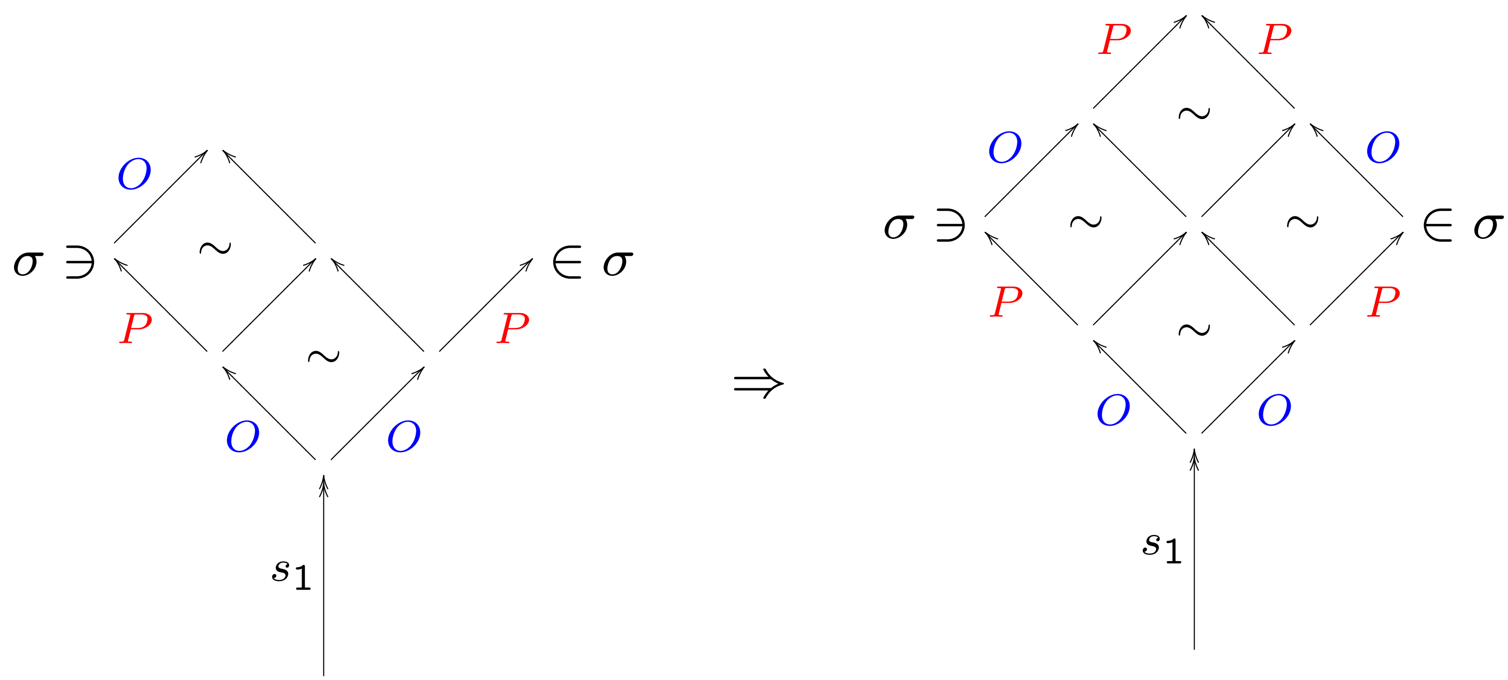


From this follows a reformulation of innocence...

Backward innocence



Forward innocence



Innocent strategies are positional

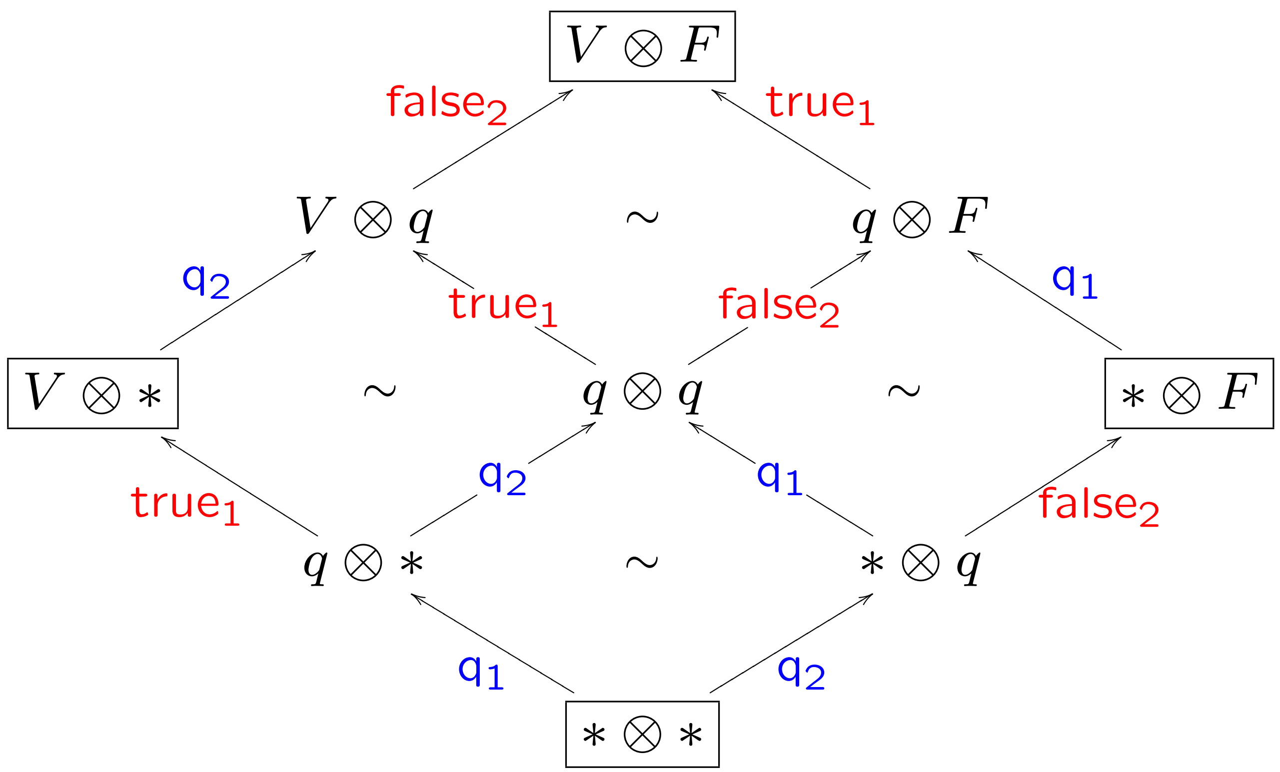
Definition. A strategy σ is **positional** when for every two plays s_1 and s_2 with same target x:

$$s_1 \in \sigma \quad \text{and} \quad s_2 \in \sigma \quad \text{and} \quad s_1 \cdot t \in \sigma \quad \Rightarrow \quad s_2 \cdot t \in \sigma$$

Theorem (by an easy diagrammatic proof)
Every innocent strategy σ is positional

More: An innocent strategy is characterized by the positions it reaches.

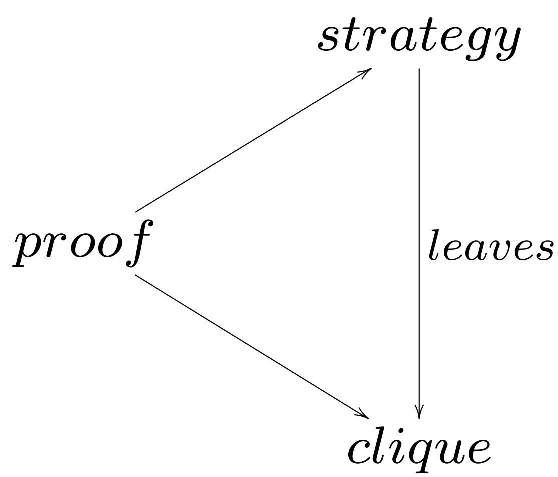
An illustration: the strategy (true $\otimes$ false)



Strategies become **closure operators** on complete lattices as in Abramsky-M. concurrent games.

From asynchronous games to coherence spaces

The diagram commutes



for every proof of a multiplicative additive formula.

Part II (b)

Multiplicatives in asynchronous games

The free dialogue category

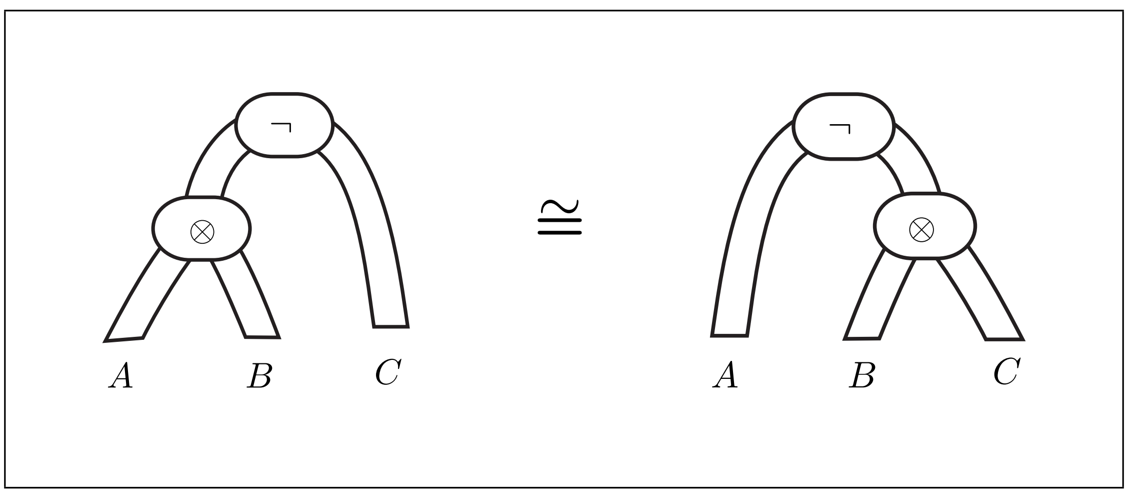
Dialogue categories

A **symmetric monoidal** category $\mathscr{C}$ equipped with a functor

$$\neg \quad : \quad \mathscr{C}^{op} \longrightarrow \mathscr{C}$$

and a natural bijection

$$\varphi_{A,B,C} \quad : \quad \mathscr{C}(A \otimes B, \neg C) \cong \mathscr{C}(A, \neg(B \otimes C))$$



The free dialogue category

The objects of the category **free-dialogue**($\mathscr{C}$) are families of **dialogue games**
constructed by the grammar

$$A, B \quad ::= \quad X \quad | \quad A \oplus B \quad | \quad A \otimes B \quad | \quad \neg A \quad | \quad 1$$

where X is an object of the category $\mathscr{C}$.

The morphisms are **total** and **innocent strategies** on dialogue games.

As we will see: proofs are 3-dimensional variants of knots...

A theorem for free

There exists a functor

$$leaves \quad : \quad \textbf{free-dialogue}(\mathscr{C}) \longrightarrow \mathbf{Coh}$$

which preserves the sum, the tensor, and transports the non-involutive negation of the category $\mathscr{G}$ into the involutive negation of the category $\mathbf{Coh}$.

This functor collapses the dynamic semantics into a static one

Tensor logic

tensor logic	=	a logic of tensor and negation
	=	linear logic without $A \cong \neg\neg A$
	=	the very essence of polarization

Offers a synthesis of linear logic, games and continuations

Research program: recast every aspect of linear logic in this setting

Part III

Exponentials in orbital games

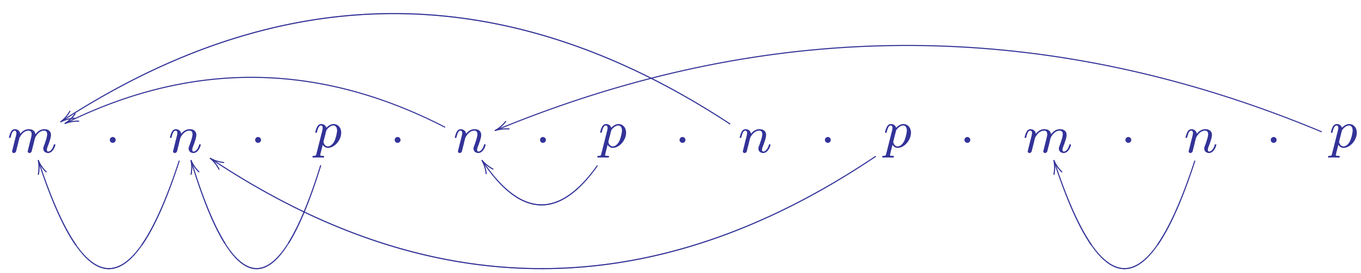
Uniformity formulated as interactive group invariance

Exponentials

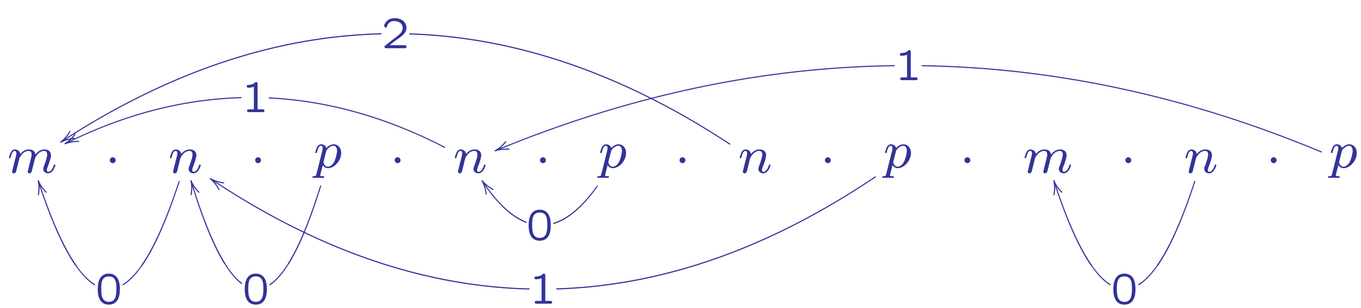
$$!A \quad = \quad \bigotimes_{n \in \mathbb{N}} \quad A$$

Justification vs. copy indexing

In the presence of repetition, the backtracking policy of arena games

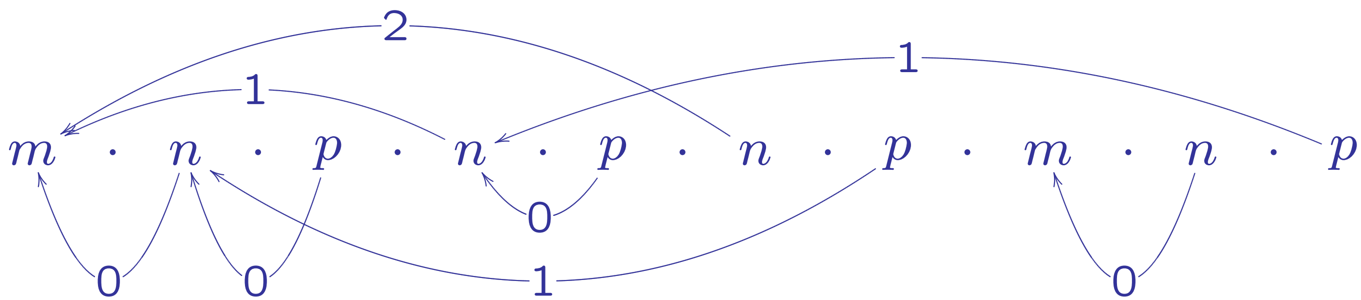


may be alternatively formulated by indexing threads



Justification vs. copy indexing

The justified play with copy indexing



may be then seen as a play in an asynchronous game

$$(m, 0) \cdot (n, 00) \cdot (p, 000) \cdot (n, 01) \cdot (p, 010) \cdots$$

$$\cdots (n, 02) \cdot (p, 001) \cdot (m, 1) \cdot (n, 10) \cdot (p, 011)$$

An associativity problem

The following diagram does not commute

$$
\begin{array}{ccccc}
!A & \xrightarrow{d} & !A \otimes !A & \xrightarrow{d \otimes !A} & (!A \otimes !A) \otimes !A \\
\downarrow{\scriptstyle d} & & & & \downarrow{\scriptstyle \alpha} \\
!A \otimes !A & & \xrightarrow[!A \otimes d]{\qquad\qquad} & & !A \otimes (!A \otimes !A)
\end{array}
$$

Hence, comultiplication is not associative.

Abramsky, Jagadeesan, Malacaria games (1994)

However, this diagram does commute... up to thread indexing !

$$
\begin{array}{ccccc}
!A & \xrightarrow{\;d\;} & !A \otimes !A & \xrightarrow{\;d \otimes !A\;} & (!A \otimes !A) \otimes !A \\
\downarrow{\scriptstyle d} & & \sim & & \downarrow{\scriptstyle \alpha} \\
!A \otimes !A & & \xrightarrow{\;!A \otimes d\;} & & !A \otimes (!A \otimes !A)
\end{array}
$$

So, the game $!A$ defines a pseudo-comonoid instead of a comonoid...

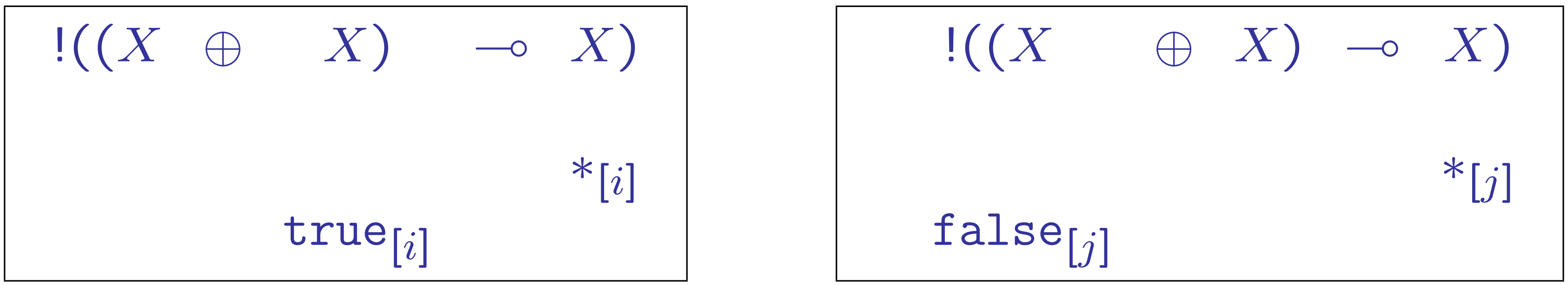
A main difficulty

The dereliction strategies ε_i are equal up to reindexing

$!((X$	$\oplus$	$X)$	$\multimap$	$X)$	$\multimap$	$(X$	$\oplus$	$X)$	$\multimap$	X
										$*$
				$*_{[i]}$						
		$\mathtt{true}_{[i]}$								
								$\mathtt{true}$		

A non uniform taster

The strategy `taster` defined as



tastes the difference between ε_i and ε_j in the sense that

$$\varepsilon_i \circ \texttt{taster} \quad \neq \quad \varepsilon_j \circ \texttt{taster}.$$

Orbital games

An asynchronous game equipped with

– two groups G_A and H_A,

– a left group action on moves $G_A \times M_A \longrightarrow M_A$

– a right group action on moves $M_A \times H_A \longrightarrow M_A$

preserving the asynchronous structure, and such that the left and right actions commute:

$$\forall m \in M_A, \forall g \in G_A, \forall h \in H_A, \qquad (g \centerdot m) \centerdot h = g \centerdot (m \centerdot h).$$

Alternatively

An orbital game is an asynchronous game A equipped with

– a class G_A of automorphisms of A closed under composition,

– a class H_A of automorphisms of A closed under composition,

such that

$$A \xrightarrow{g} A \xrightarrow{h} A \quad = \quad A \xrightarrow{h} A \xrightarrow{g} A$$

The two definitions are essentially the same...

An equivalence relation on plays

Two plays s and t are equal up to reindexing

$$s \quad \approx \quad t$$

when there exists $g \in G_A$ and H_A such that

$$t \quad = \quad g \centerdot s \centerdot h.$$

A simulation preorder between strategies (AJM)

A strategy σ is simulated by a strategy τ when for every pair of plays

$$s \quad \approx \quad s'$$

and for all moves m, n, m' such that

$$s \cdot m \cdot n \in \sigma \quad \text{and} \quad s' \in \tau \quad \text{and} \quad s \cdot m \approx_A s' \cdot m'$$

there exists a move n' such that

$$s \cdot m \cdot n \approx_A s' \cdot m' \cdot n' \quad \text{and} \quad s' \cdot m' \cdot n' \in \tau.$$

$$\boxed{\sigma \quad \precapprox^{\mathbf{sim}} \quad \tau}$$

Interactive invariance

A strategy σ is covered by a strategy τ when

$$\forall s \in \sigma, \quad \forall h \in H_T, \quad \exists g \in G_T, \qquad g \cdot s \cdot h \in \tau.$$

$$\boxed{\sigma \quad \precapprox^{\mathbf{inv}} \quad \tau}$$

Proposition

Suppose that σ and τ are strategies of an orbital game. Then,

$$\sigma \quad \precapprox^{\mathbf{sim}} \quad \tau \quad \Longleftrightarrow \quad \sigma \quad \precapprox^{\mathbf{inv}} \quad \tau$$

This leads to a 2-category of orbital games and uniform strategies, where $!A$ is a pseudo-comonoid.

Projection to coherence spaces

The functor

$$\mathbf{Orbital} \quad \longrightarrow \quad \mathbf{Rel}$$

projects a position to its orbit in the orbital game.

In particular, an indexed family of positions in the game

$$!A \quad = \quad \bigotimes_{n \in \mathbb{N}} \quad A$$

is transported to a multiset of positions.

The locative information is lost on the way...

Interactive invariance on the syntax

Exponential boxes are replaced by «mille-feuilles» whose uniformity is captured by interactive reindexing.

$$(\lambda x.x(i))\vec{P} \quad \mapsto \quad P_i$$

Innocence precedes uniformity...

A link to complexity

Construct the free dialogue category with pseudo-comonoids.

Part IV

A bialgebraic definition of traces

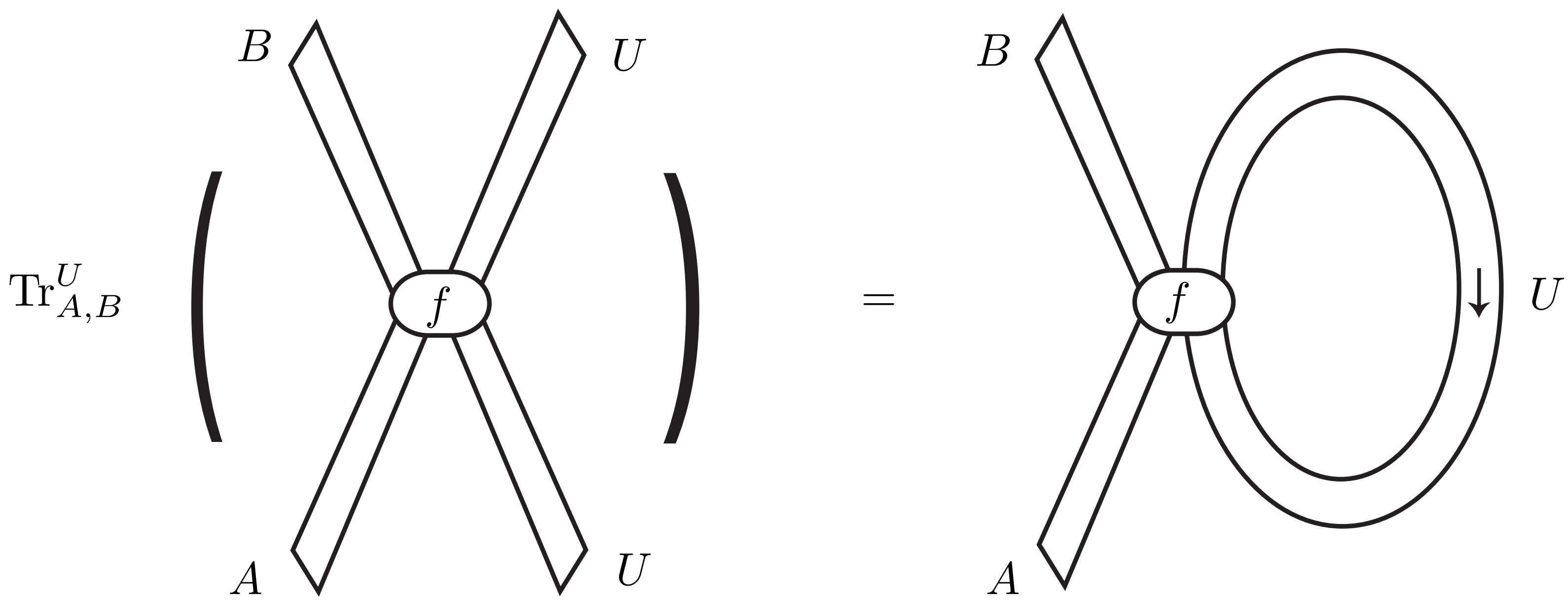
Traced monoidal categories

A **trace** in a balanced category $\mathscr{C}$ is an operator

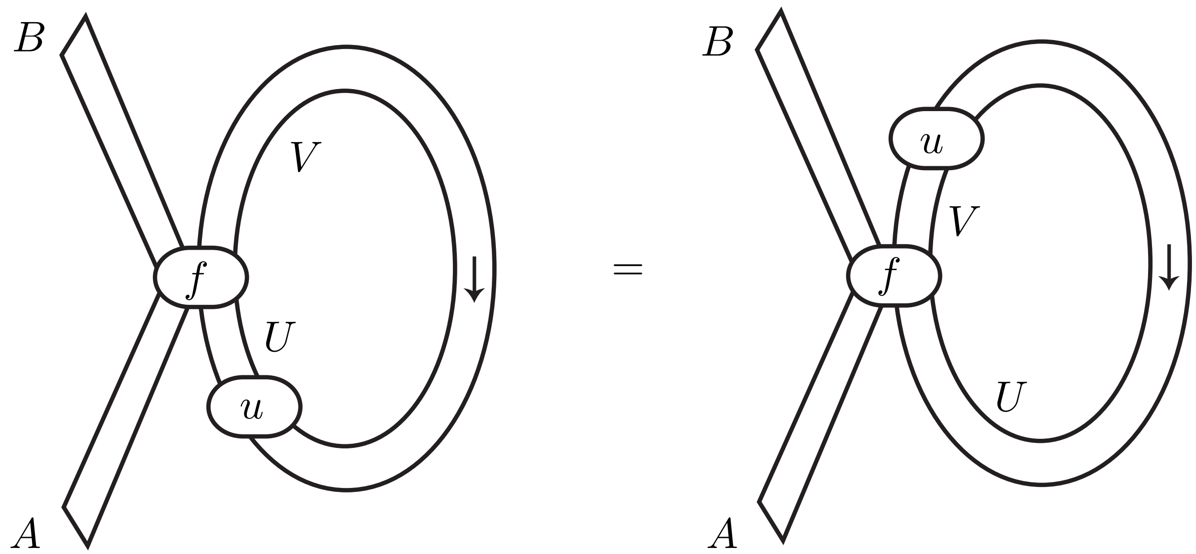
$$\mathbf{Tr}_{A,B}^{U} \qquad \frac{A \otimes U \longrightarrow B \otimes U}{A \longrightarrow B}$$

depicted as **feedback** in string diagrams:

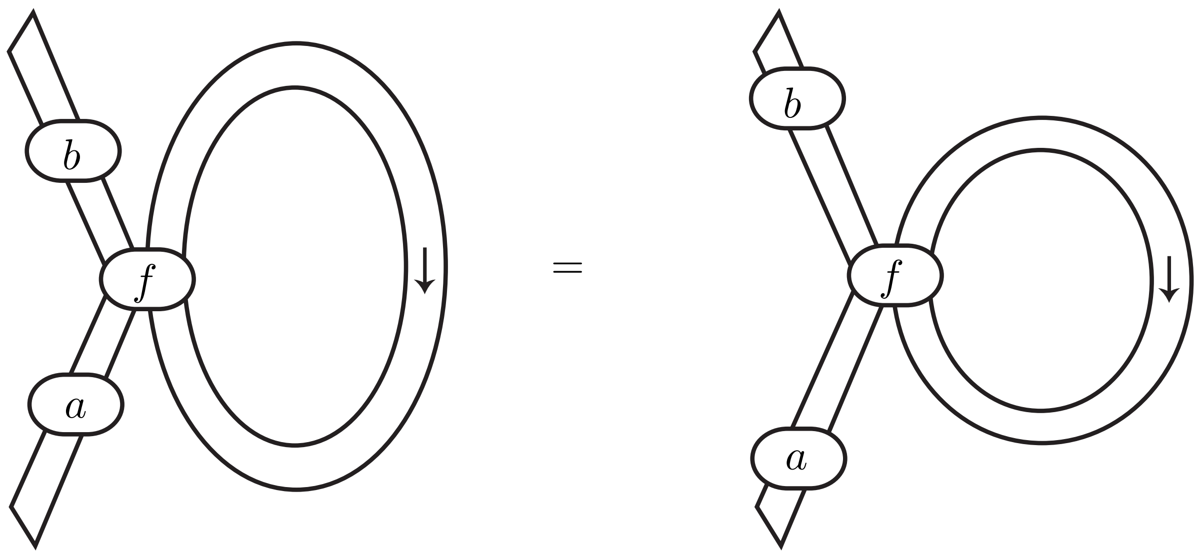
Trace operator



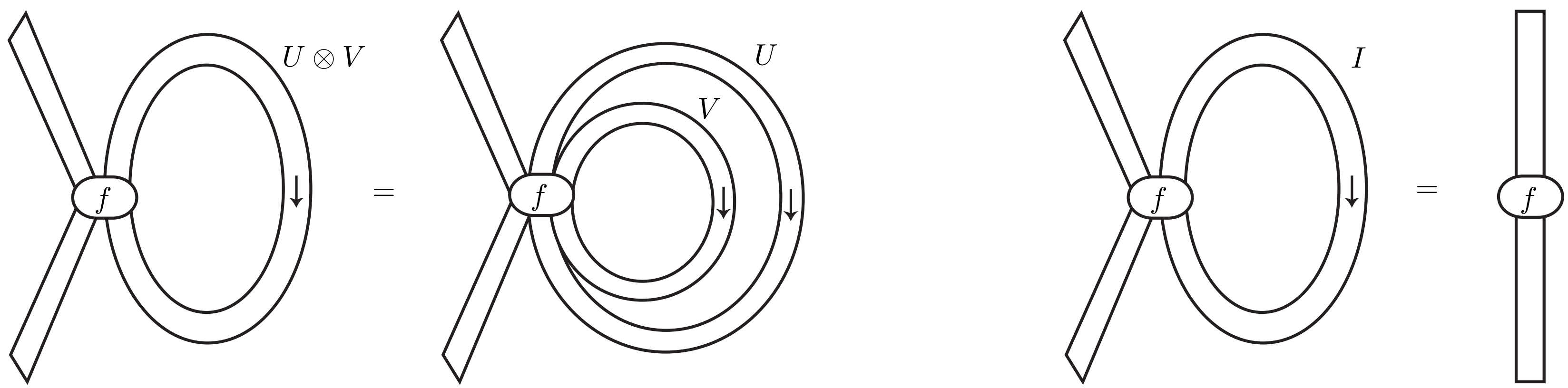
Sliding (naturality in U)



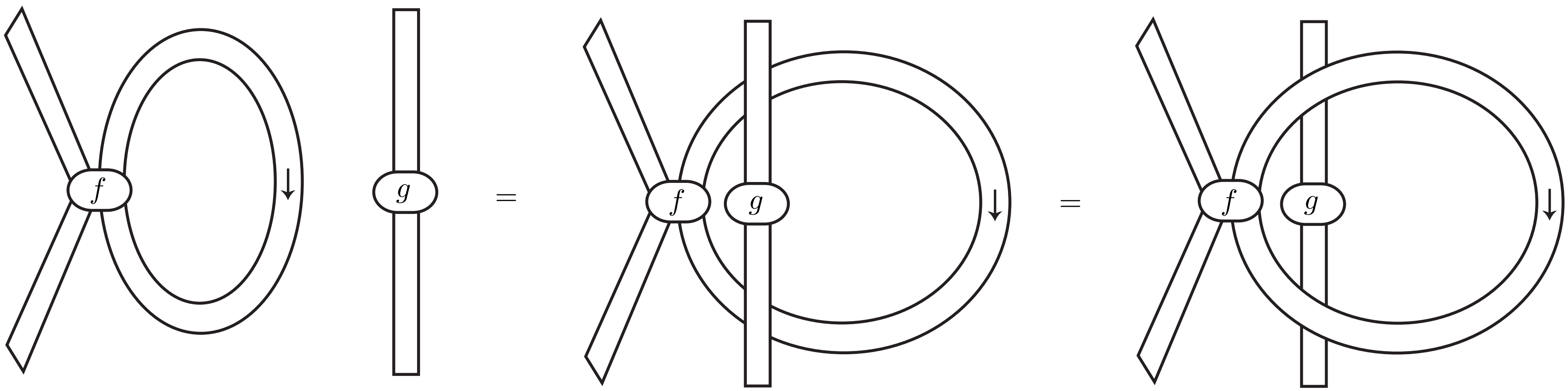
Tightening (naturality in A, B)



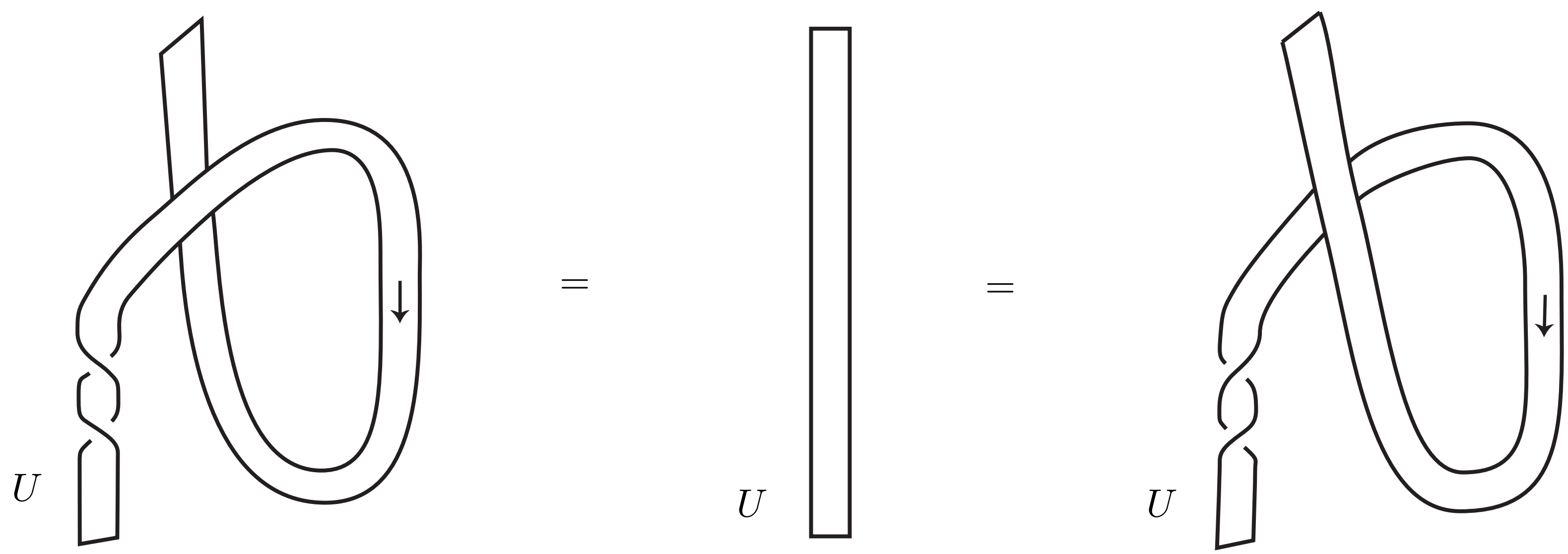
Vanishing (monoidality in U)



Superposing



Yanking



Trace