No-counterexample Interpretations of Logic and the Geometry of Interaction

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The Plan of the Talk

- I will first explain the motivation.
- Then I will mostly explain the no-counterexample interpretation (NCI) according to Tait's work.
- Finally I will add a small observation of mine and present NCI in a trace-like graphical representation.

Introduction

- The functional interpretations of logic have a flavor of game.
- The values for existential quantiers are positive and those for universal quantifiers are negative.
- In the negation the positive and the negative change the roles.
- I want to relate them to GoI.

GoI and Cut-elimination

- GoI is supposed to model the dynamics of cut-elimination.
- For the consistency proof the cut-elimination of propositional logic is not so interesting….
- All techniques of the consistency proof is to handle the alternating quantifiers.

The Consistency Proof of PA

- The epsilon substitution method by Hilbert and Ackermann.
- The Cut-elimination method by Gentzen
- The Dialectica interpretation by Goedel
- No-counterexample interpretation by Kreisel.

The Pre-history

- Gentzen’s first version of the consistency proof is in terms of “reduction”.
- Goedel described Gentzen’s idea in his Zilsel lecture, essentially as a no-counter example interpretation.
- It can be stated in terms of game, recently revived by Coquand

Our Convention

- Consider the sentences in a classical first-order logic.
- Quantified sentences are regarded as infinitary disjunctions and conjunctions.
- Negations are pushed inside by the De Morgan duality.
- In the games, the player’s moves are blue and the opponents’ are red.

The Henkin Hintikka Game

- Start with a sentence.
- The Player and the Opponent form a new sentence from the sentence in the previous stage.
- Ends with an atomic sentence.
- The Player wins if the atomic sentence is true. The opponent wins otherwise.

The Moves in the Henkin Hintikka Game

Done ϕ_j

$$\Uparrow$$

Σ Δ Π ϕ_k ϕ_k

The Gentzen Game

- Start with a list of sentences in the prenex normal form.
- The Player and the Opponent form a new list of sentences from the list in the previous stage.
- The player wins if the list contains a true prime (atomic) sentence.

The Moves in the Gentzen Game

$$A_1, p_1, \ldots, p_2, \ldots, \phi_j, \ldots, (\prod \phi_k), \ldots\ldots$$

$$\Uparrow$$

$$\downarrow$$

$$A_1, p_2, \ldots, \Delta_i, \ldots, \sum \phi_k, \ldots\ldots$$

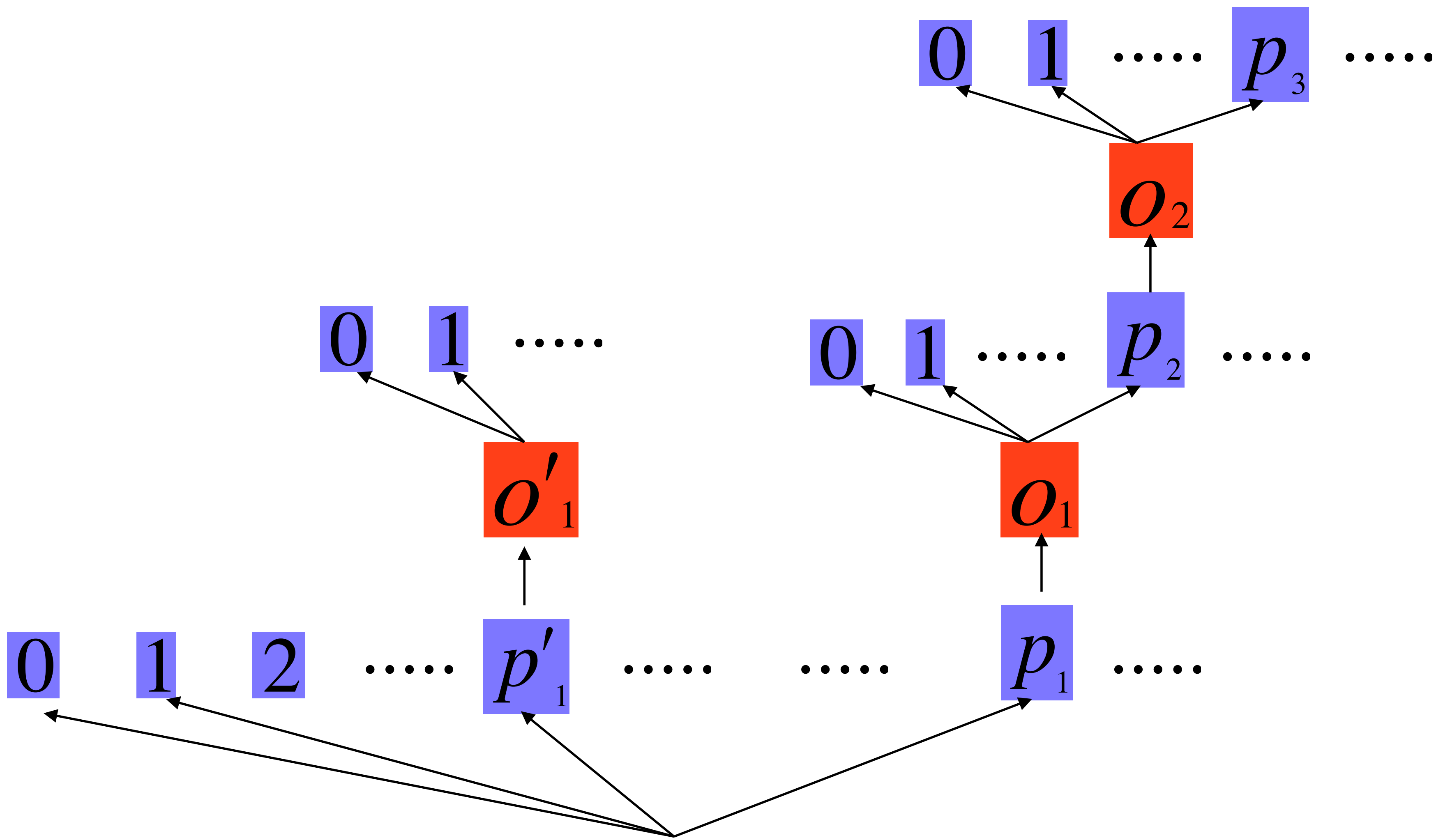
Some Restriction

- The Player does not repeat the same instantiation, in other words, always chooses a different disjunct from the given disjunction.
- We regard quantifier free sentences as prime (atomic).
- This restriction is not crucial with respect to the expressive power.

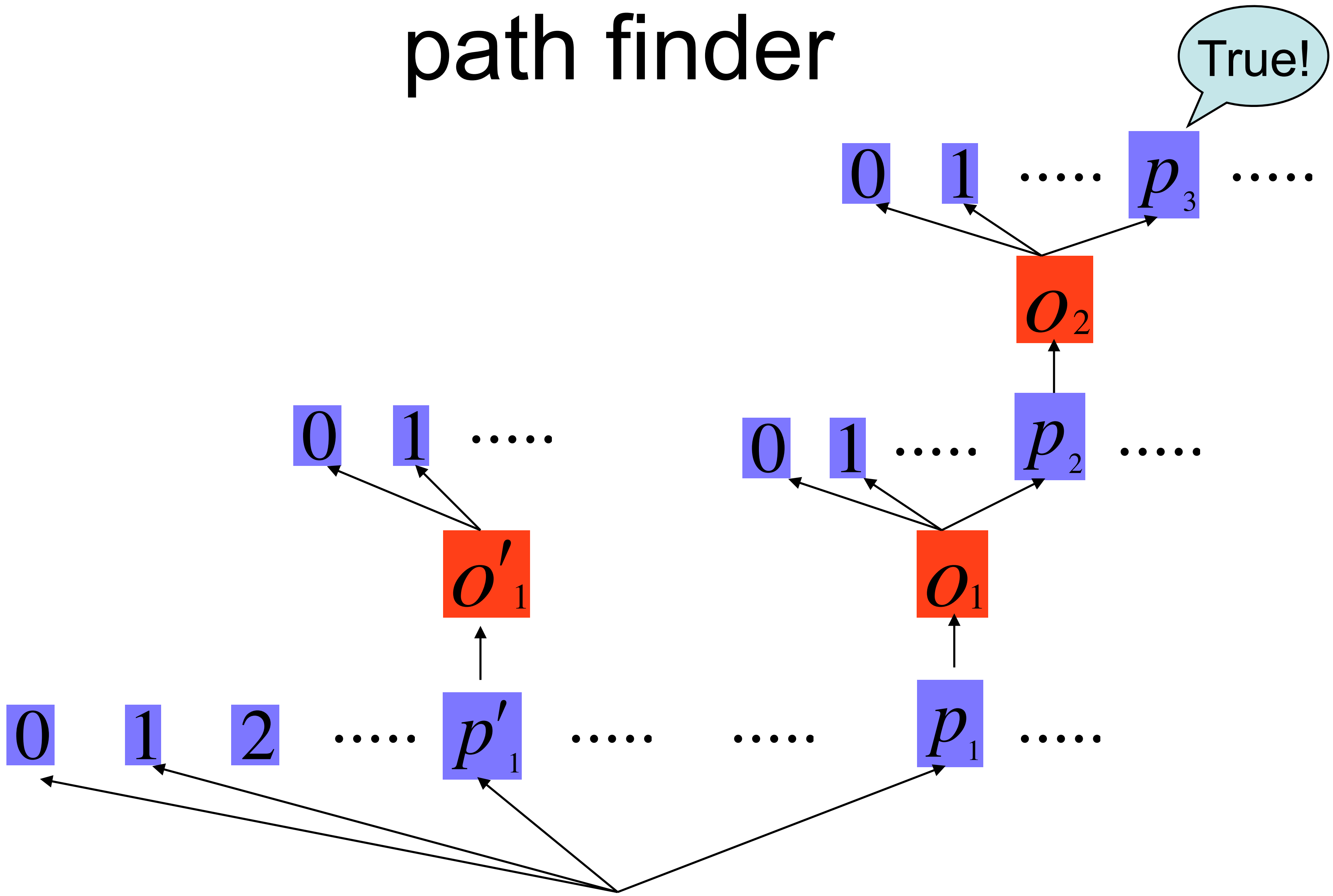
The counter-strategy as a function

- The counter-strategy (trying to falsify) of the Opponent may be seen as a function of the previous moves of the Player.
- The re-instantiation of the existential sentence may be seen as “the change of mind”.

The counter-strategy as a tree



The winning strategy as a path finder



The no-counterexample Interpretation (NCI)

- The universally quantified (negative) variables are replaced by the functions of the preceding existentially quantified (positive) variables.
- For a provable sentence one can find the functionals of those negative functions, yielding the witnesses for the positive variables.

A Brief Histroy of NCI

- NCI was introduced by G. Kreisel,using Herbrand's theorem for FOL and the epsilon substitution for PA.
- The direct proofs are given by Kohlenbach and Tait.
- Tait's work dates back to early 1960's, which had been unpublished since then.

The NCI and the Gentzen Game

$$\exists u_1 \forall x_1 \exists u_2 \forall x_2 A(u_1, x_1, u_2, x_2)$$

$\Downarrow$ Consider a counter-strategy.

$$A(u_1, f_1 u_1, u_2, f_2 u_1 u_2)$$

$\Downarrow$ The winning strategy finds a path.

$$A\left(F_1 \bar{f}, f_1\left(F_1 \bar{f}\right), F_2 \bar{f}, f_2\left(F_1 \bar{f}\right)\left(F_2 \bar{f}\right)\right)$$

where $\bar{f} \equiv f_1 f_2$

Modus Ponens

$$\exists x_1 \forall u_1 \exists x_2 \forall u_2 A(x_1, u_1, x_2, u_2)$$

$$\forall x_1 \exists u_1 \forall x_2 \exists u_2 \exists v \forall y \left[\neg A(x_1, u_1, x_2, u_2) \vee B(v, y)\right]$$

$$\Downarrow$$

$$\exists v \forall y B(v, y)$$

Modus Ponens in NCI

$$A\left(F_1\bar{f},\ f_1\left(F_1\bar{f}\right),\ F_2\bar{f},\ f_2\left(F_1\bar{f}\right)\left(F_2\bar{f}\right)\right)$$

$$\neg A\left(g_1,\ G_1\overline{g},\ g_2\left(G_1\overline{g}\right),\ G_2\overline{g}\right) \vee B\left(H\overline{g},\ g_3\left(G_1\overline{g}\right)\left(H\overline{g}\right)\right)$$

$$\Downarrow$$

$$B\left(H g_1 g_2 h,\ h\left(J h\right)\left(H g_1 g_2 h\right)\right)$$

with g_1, g_2

such that $A\left(g_1,\ G_1 g_1 g_2 h,\ g_2\left(G_1 g_1 g_2 h\right),\ G_2 g_1 g_2 h\right)$

Finding the Counter Strategies

$$
\begin{matrix}
A\Big(& F_1 f_1 f_2, & f_1\left(F_1 f_1 f_2\right), & F_2 f_1 f_2, & f_2\left(F_1 f_1 f_2\right)\left(F_2 f_1 f_2\right) & \Big) \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
A\Big(& g_1, & G_1 g_1 g_2 h, & g_2\left(G_1 g_1 g_2 h\right), & G_2 g_1 g_2 h & \Big)
\end{matrix}
$$

$$\hat{g}_1 = F_1 f_1 f_2$$

$$\hat{f}_1\left(F_1 \hat{f}_1 f_2\right) = G_1 \hat{g}_1 g_2 h$$

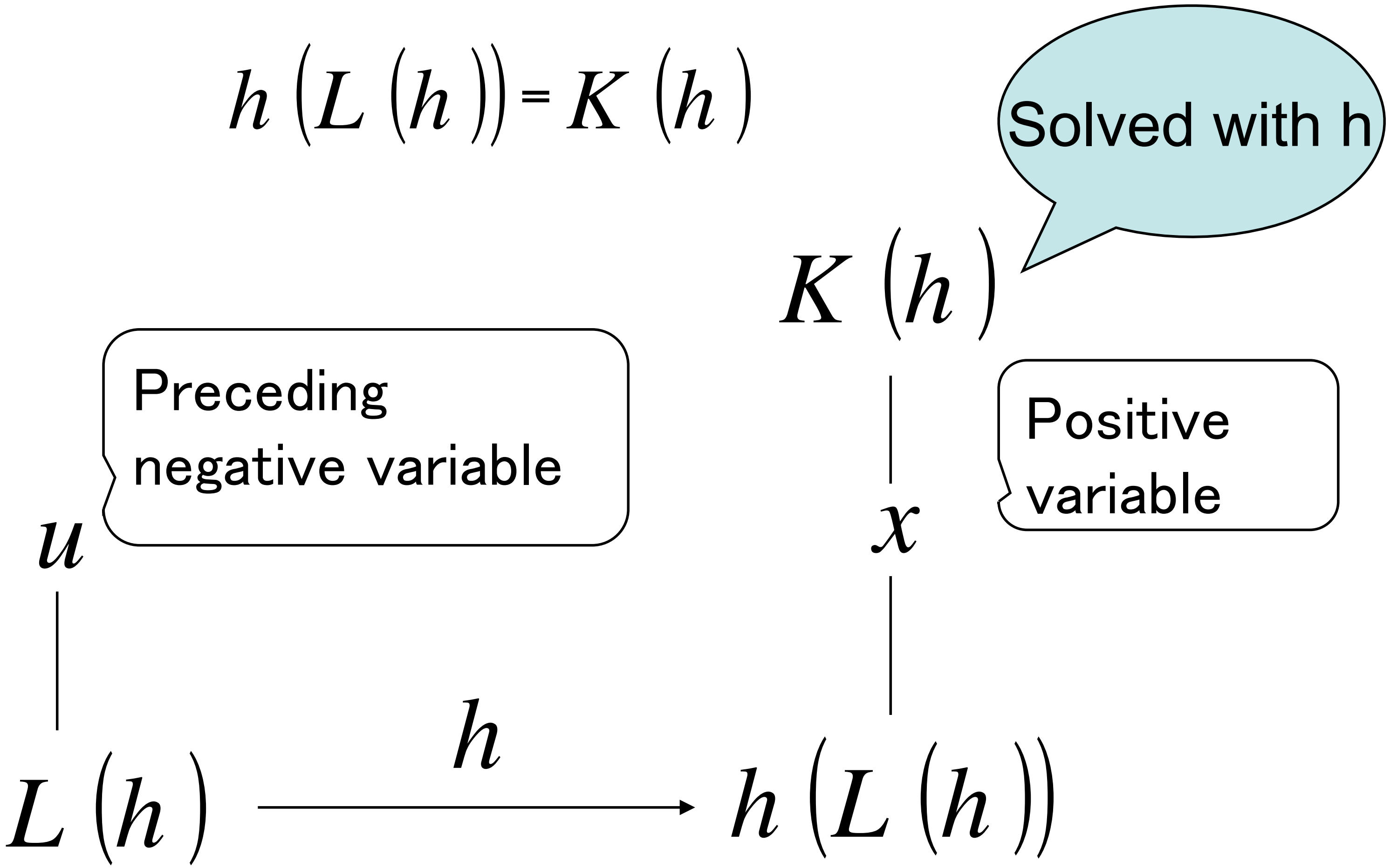
$$\hat{g}_2\left(G_1 \hat{g}_1 \hat{g}_2 h\right) = F_2 \hat{f}_1 f_2$$

$$\hat{f}_2\left(F_1 \hat{f}_1 \hat{f}_2\right)\left(F_2 \hat{f}_1 \hat{f}_2\right) = G_2 \hat{g}_1 \hat{g}_2 h$$

The General Pattern

Find the solution h for

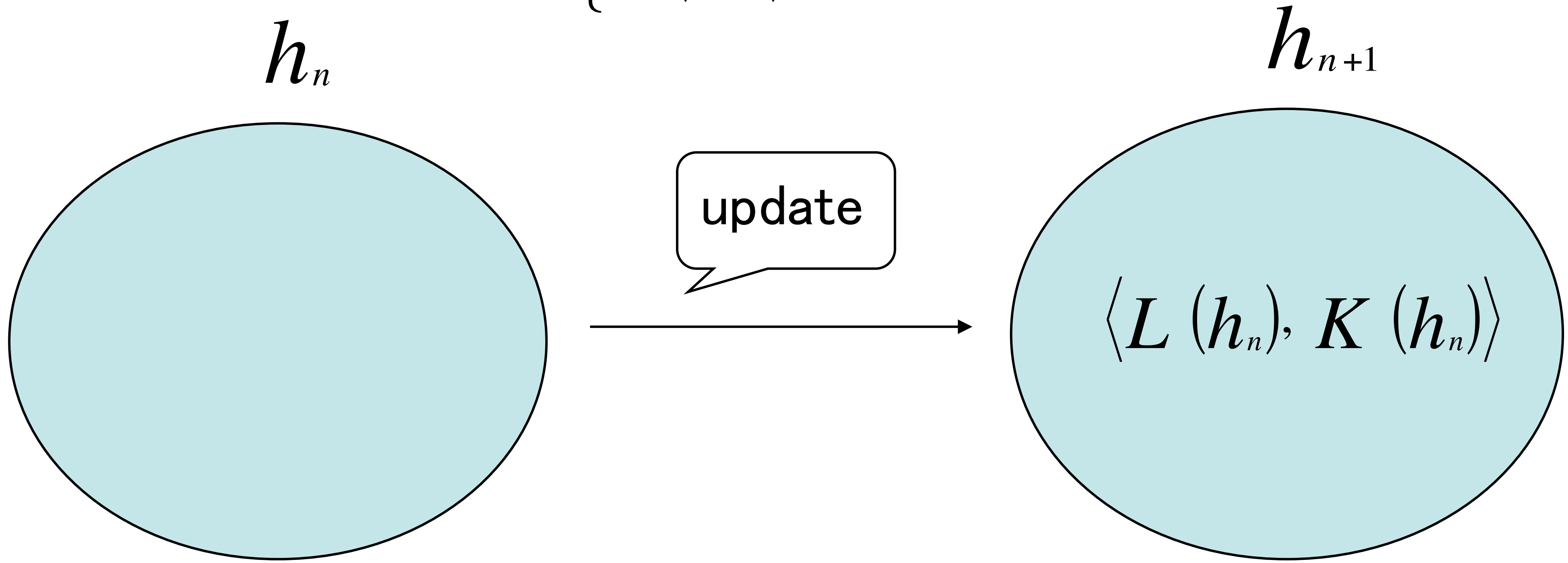
$$h\left(L\left(h\right)\right) = K\left(h\right)$$



The Approximation

$$h_0(m) = 0$$

$$h_{n+1}(m) = \begin{cases} K(h_n) & \text{if } m = L(h_n) \\ h_n(m) & \text{otherwise} \end{cases}$$



The System of α-recursive Functionals

- Tait introduced the system of recursively definable functionals, allowing the recursion along a primitive recursively definable well-founded partial order α.
- One can keep track of how much of the initial segment of input functions is necessary to compute the value of the functional, along α.

The Solution in the System of α-recursive Functionals

Let $\hat{h}$ be h_{n+1} such that

$$L(h_{n+1}) = L(h_n)$$

We have $L(h) = L(h') \Rightarrow K(h) = K(h')$

Hence $h_{n+1}(L(h_{n+1})) = h_{n+1}(L(h_n)) = K(h_n) = K(h_{n+1})$

The Solution as the Fixpoint of the Update Operator

Assume $L(h_{n+1}) = L(h_n)$

For $m = L(h_{n+1})$

$$h_{n+2}(L(h_{n+1})) = K(h_{n+1}) = K(h_n) = h_{n+1}(L(h_n))$$

Otherwise

$$h_{n+2}(m) = h_{n+1}(m)$$

$$h_{n+1}(L(h_{n+1})) = h_{n+2}(L(h_{n+1})) = K(h_{n+1})$$

The Solution is the Fixpoint of the Update Operator

Assume $h = U(h)$

Then $h(L(h)) = U(h)(L(h)) = K(h)$

Assume $h(L(h)) = K(h)$

For $m = L(h)$

$U(h)(m) = K(h) = h(m)$

Otherwise

$U(h)(m) = h(m)$

The Similarity between NCI and GoI

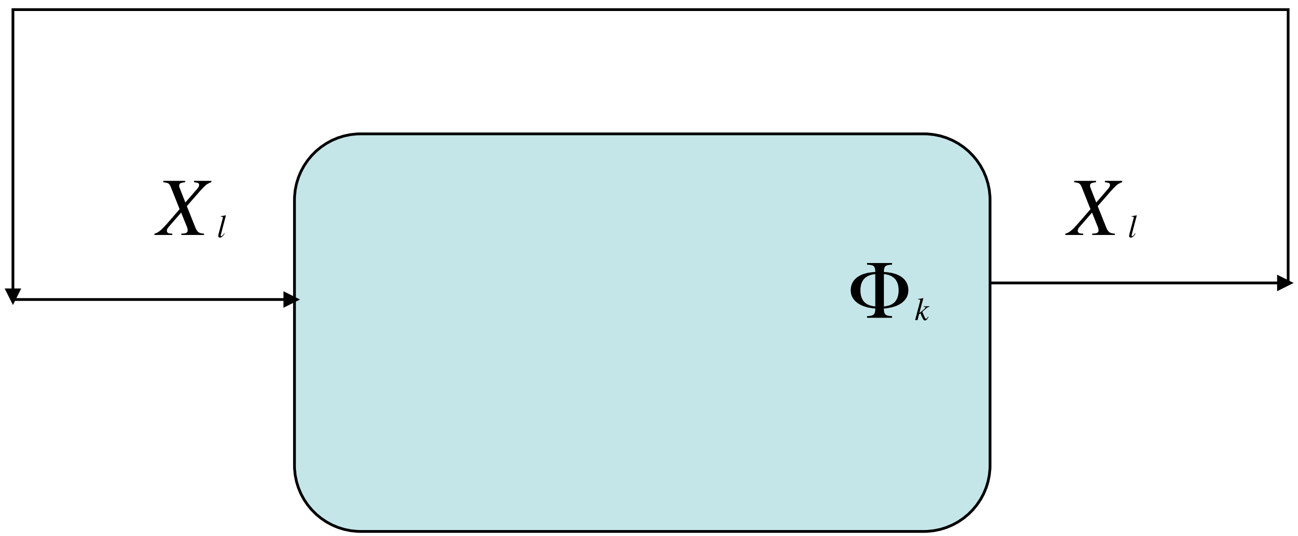
- The morphisms in GoI and the interpretations in NCI are functions from "negatives" to "positives".
- The composition in GoI and Modus Ponens of NCI are formed by taking trace and fixpoint, connecting the corresponding negatives and positives.
- The simple duality is lost in NCI.

Trace-like Operation

Given $\langle \Phi_i, F_j \rangle$ with $\Phi_i, F_j : \langle X_1, \ldots.. X_l, \ldots.. X_1 \rangle \rightarrow X_l$

Take the fixpoint of Φ_k

and substitute it in $\langle \Phi_i, F_j \rangle$

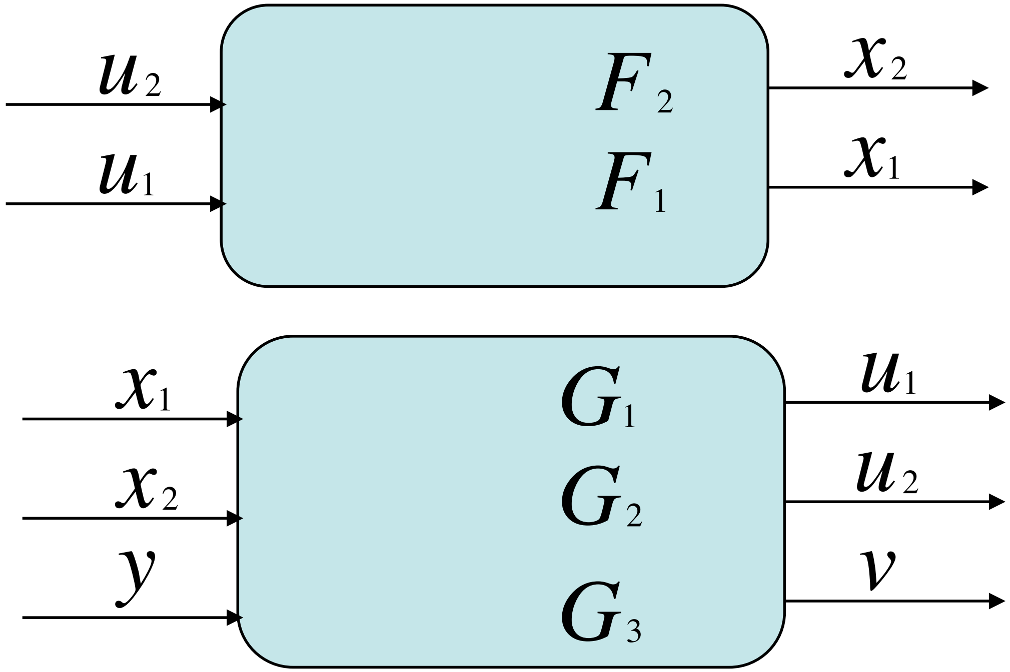


The Counter Strategies in Cyclic Graphs

$$A\left(F_1 f_1 f_2,\quad f_1\left(F_1 f_1 f_2\right),\quad F_2 f_1 f_2,\quad f_2\left(F_1 f_1 f_2\right)\left(F_2 f_1 f_2\right)\right)$$

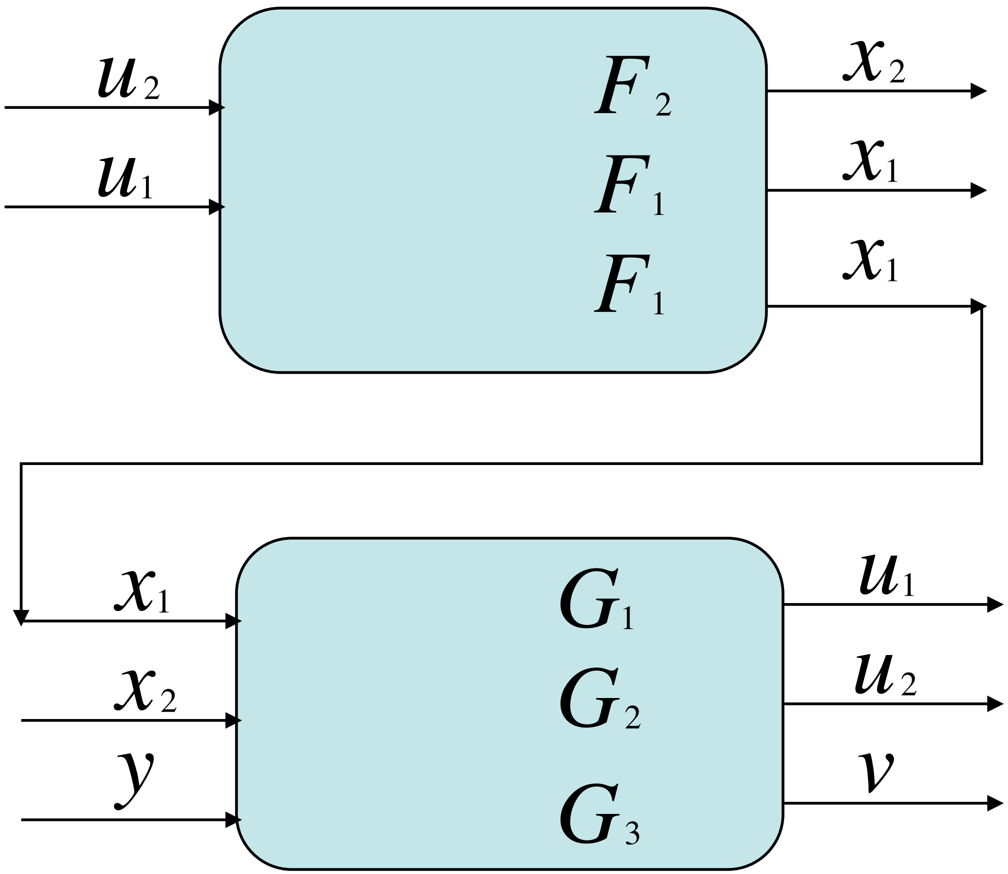
$$A\left(\quad g_1,\qquad G_1 g_1 g_2 h\,,\quad g_2\left(G_1 g_1 g_2 h\right),\quad G_2 g_1 g_2 h\right)$$

$$A\left(\quad x_1,\quad u_1,\quad x_2,\quad u_2\right) \vee B\left(\quad v\,,\quad y\right)$$



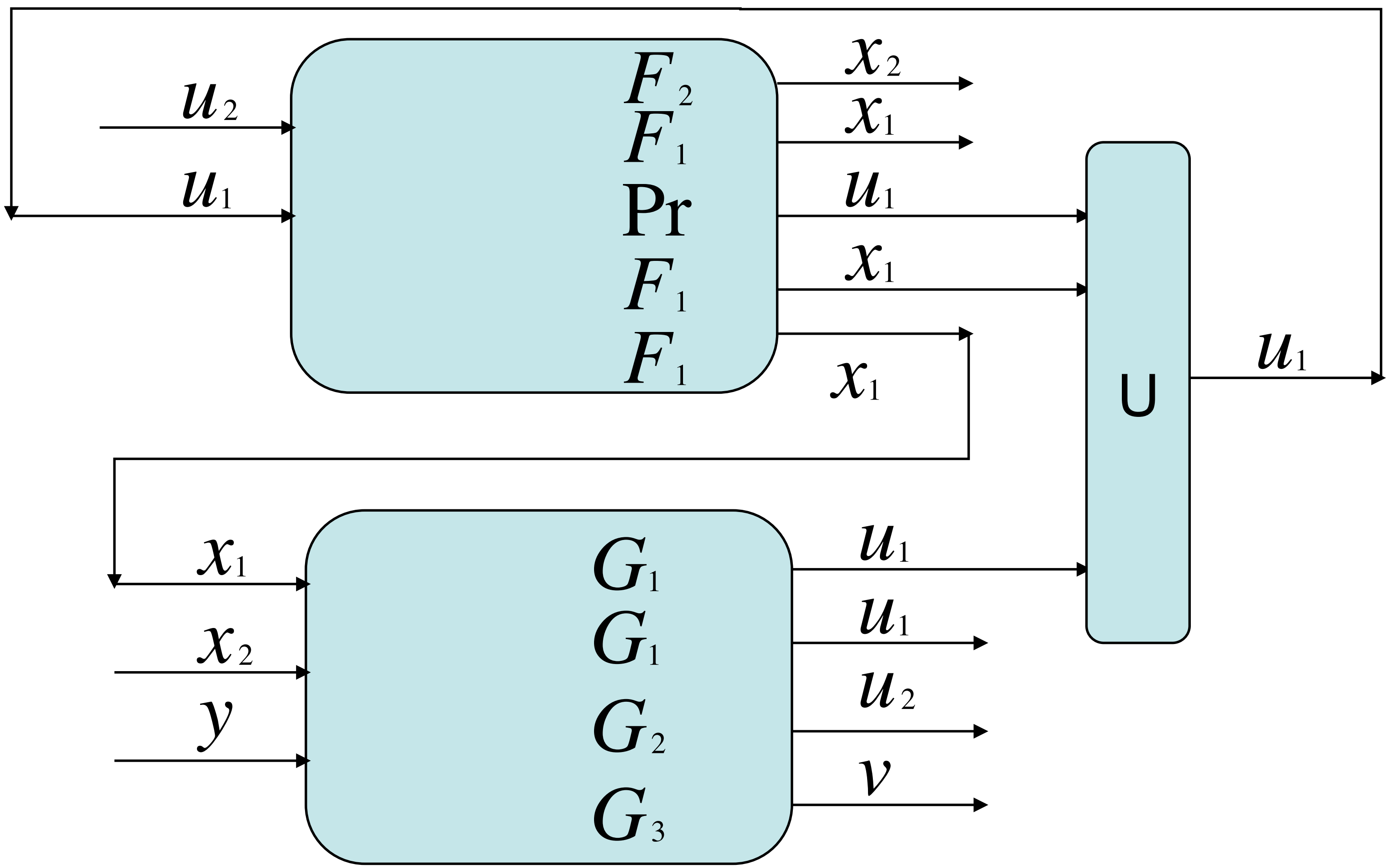
Stage 1

$$\hat{g}_1 = F_1 f_1 f_2$$



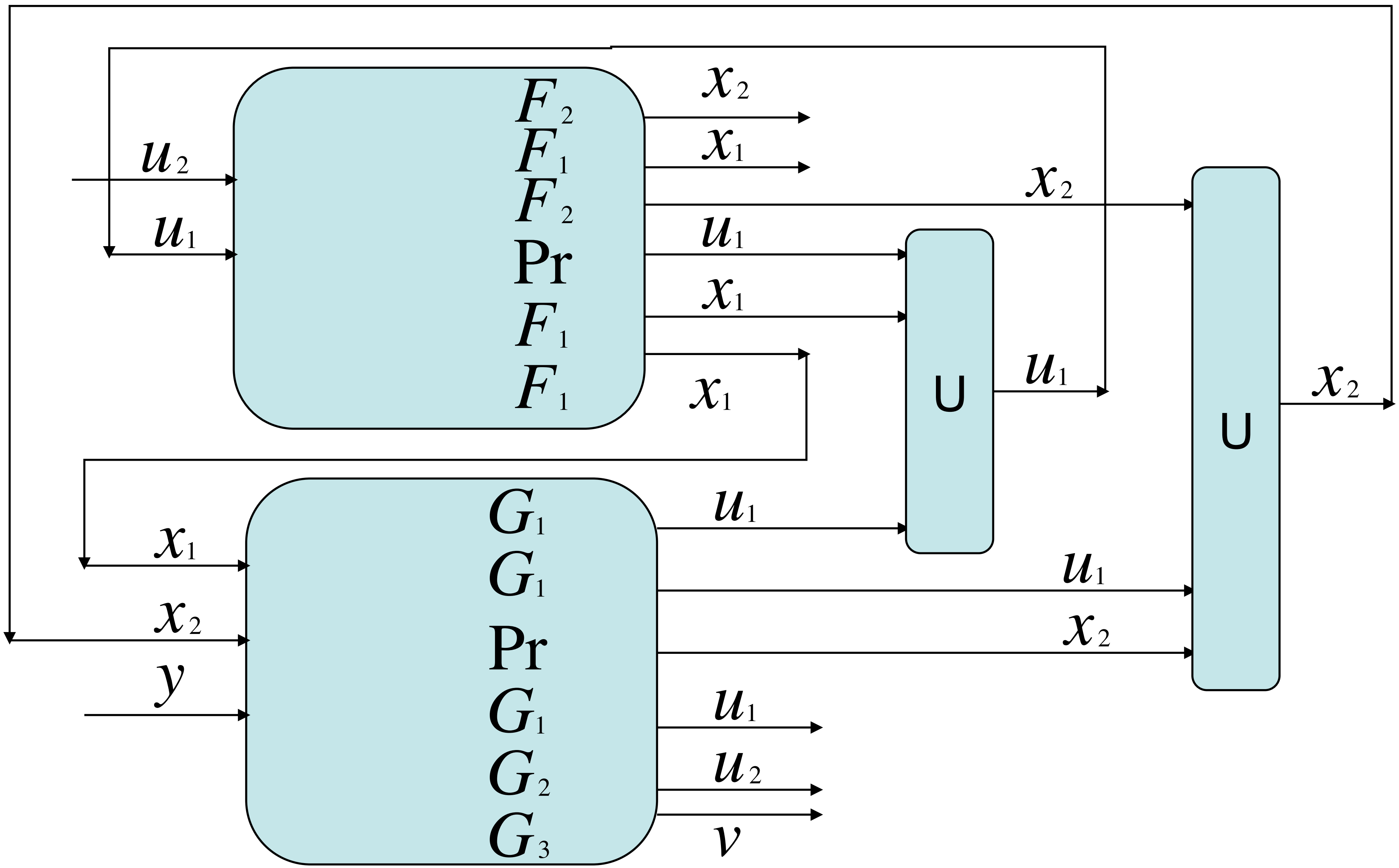
Stage 2

$$\hat{f}_1\left(F_1 \hat{f}_1 f_2\right) = G_1 \hat{g}_1 g_2 h$$



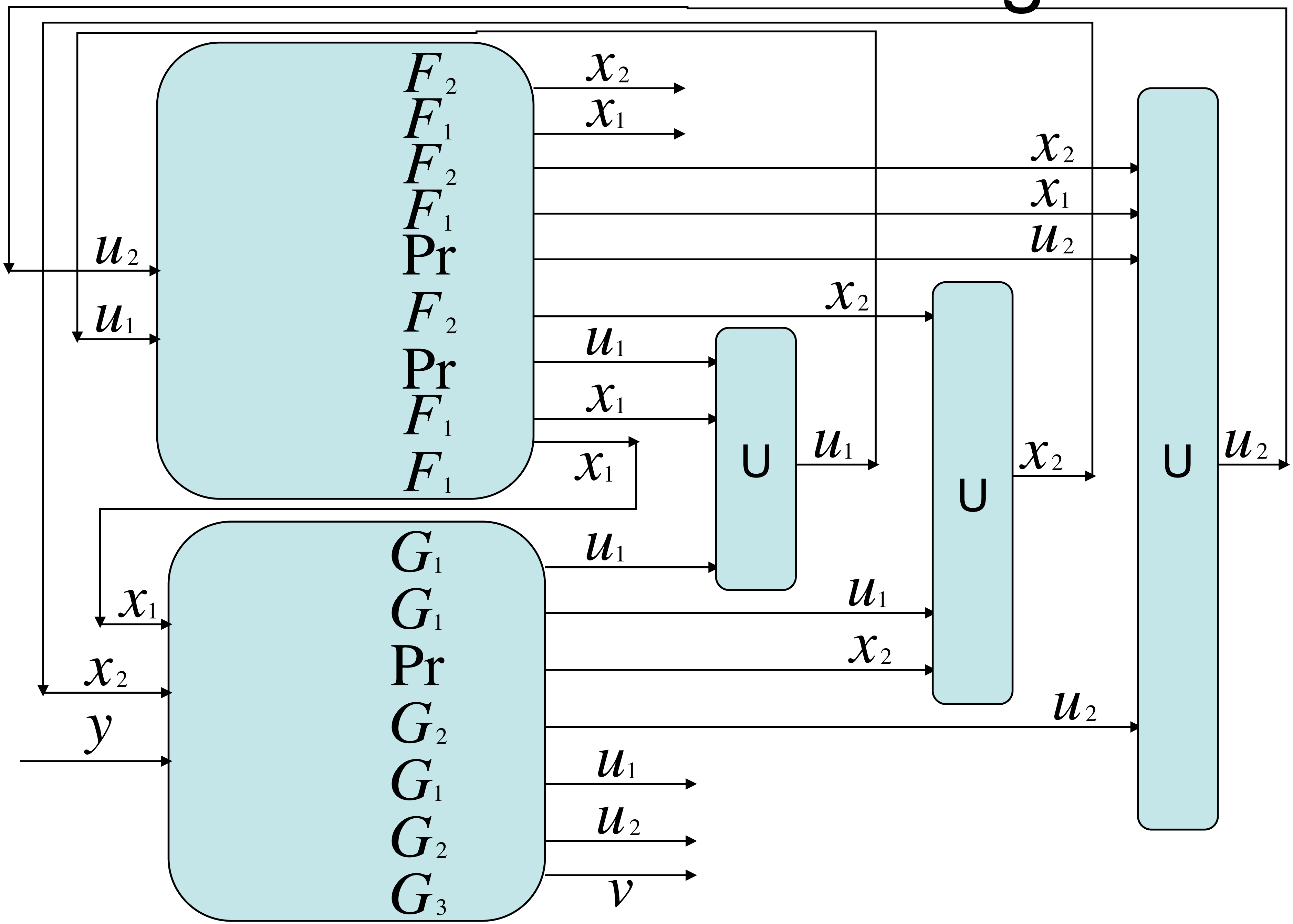
$$\hat{g}_2\left(G_1\hat{g}_1\hat{g}_2h\right) = F_2\hat{f}_1f_2$$

Stage 3

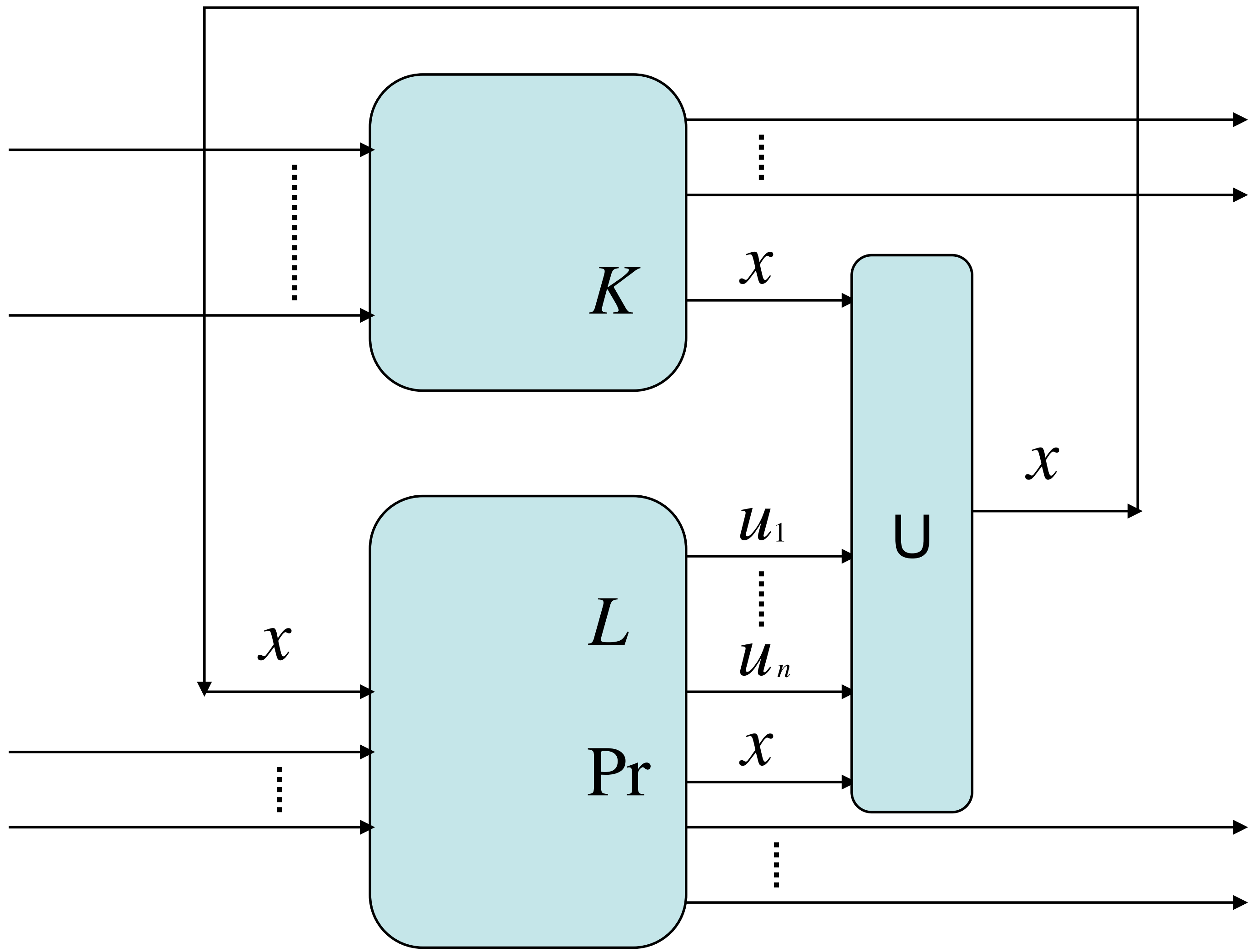


$$\hat{f}_2\left(F_1\hat{f}_1\hat{f}_2\right)\left(F_2\hat{f}_1\hat{f}_2\right) = G_2\hat{g}_1\hat{g}_2 h$$

Stage 4



The General Pattern



The Categorical NCI?

- Adding the propositional structure is just straightforward.
- The “trace” here is partial. We need to find a suitable framework.
- Study the formal properties of our “trace”.

The Dialectica Interpretation and NCI

- In Dialectica we have the duality at the cost of higher-types.
- In Dialectica the cut is composition while in NCI the cut is taking the “trace”.
- NCI is a germ of Dialectica?

Conclusion

- We have seen the similarity between NCI and GoI.
- The predicate logic is quite relevant.
- The unified framework?