

A Generalisation of Pre-logical Predicates to Simply Typed Formal Systems

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Abstract. We generalise the notion of pre-logical predicates [HS02] to arbitrary simply typed formal systems and their categorical models. We establish the basic lemma of pre-logical predicates and composability of binary pre-logical relations in this generalised setting. This generalisation takes place in a categorical framework for typed higher-order abstract syntax and semantics [Fio02,MS03].

1 Introduction

Pre-logical predicates (relations) [HS02] are a generalisation of logical predicates. They are defined for the *simply typed lambda calculus* and its set-theoretic environmental models called *lambda applicative structures* [Mit96]. Two important properties are enjoyed by pre-logical predicates but not logical predicates. One is that pre-logical predicates are *equivalent* to predicates satisfying the basic lemma (interpretation of all terms respects predicates — this is the key to many applications of logical relations), and the other is that binary pre-logical relations are closed under relational composition.

We aim to generalise pre-logical predicates from the simply typed lambda calculus to arbitrary *simply typed formal systems* (we just say *typed formal system* below) and their categorical models, then show that the above important properties hold in this generalised setting.

This generalisation enables us to extend pre-logical predicates systematically to other calculi, such as lambda calculus with various type constructors and variable binders, and calculi other than lambda calculus, such as logics and process calculi. This opens up the possibility of characterising observational equivalence [HS02] and constructive data refinement [HLST00] in various non-lambda calculi.

There are three underlying elements on which pre-logical predicates are defined: syntax (normally the simply typed lambda calculus), semantics (set-theoretic environmental models) and predicates (as subsets of carrier sets). We generalise these three elements along the following dimensions:

- We generalise syntax to an arbitrary typed formal system described by a *typed binding signature* [MS03]. A typed formal system is a formal system whose inference rules fit within the following scheme:

$$\frac{\Gamma, \Gamma_1 \vdash M_1 : \tau_1 \quad \cdots \quad \Gamma, \Gamma_m \vdash M_m : \tau_m}{\Gamma \vdash o(\Gamma_1.M_1, \dots, \Gamma_m.M_m) : \tau}$$

This is general enough to subsume various simple type systems and calculi such as the simply typed lambda calculus, many-sorted first-order logic, pi-calculus, etc.

- We generalise from set-theoretic to category-theoretic semantics. Following the principle of categorical semantics, we give a semantics of a typed formal system in a Cartesian category \mathbb{C} by mapping types to objects and terms to morphisms in \mathbb{C} .
- As we move to category theory, we need to change the notion of predicates from subsets to appropriate category-theoretic constructs. We use *subscones*, which is a mild generalisation of the injective scones of [MS93].

We represent all three elements as objects and morphisms in the category of *presentation models* \mathbf{M}_T , where T is the set of types [MS03]. In this category, the collection of well-formed terms modulo α -equivalence is represented as the initial algebra of the endofunctor corresponding to a typed binding signature.

After this generalisation, we formulate pre-logical predicates and predicates satisfying the basic lemma, and show their equivalence. Then we show the composability of binary pre-logical relations.

We look at three examples of pre-logical predicates, i) the relationship between pre-logical predicates for combinatory algebra and those for lambda calculus, ii) the connection between pre-logical predicates and lax logical predicates [PPST00] and iii) a characterisation of elementary submodels of first-order classical logic by a pre-logical relation.

Structure of This Paper The generalisation of pre-logical predicates takes place in the following way. In section 2, we first introduce a category of presentation models \mathbf{M}_T , and typed binding signatures as a description of typed formal systems. We give a categorical semantics of typed formal systems in a very weak sense. We introduce a formulation of predicates with respect to this semantics, using subscones. All three elements (syntax, semantics and predicates) are expressed in category \mathbf{M}_T . Then we formulate pre-logical predicates and predicates satisfying the basic lemma in section 3. The basic idea of the formulation of pre-logical predicates is that the inverse image of a predicate along the meaning function has an algebra structure. We show that predicates satisfying the basic lemma and pre-logical predicates are equivalent. Composition of binary pre-logical relations is discussed in section 4. In section 5, we look at three examples of pre-logical predicates. Proofs are attached in the appendix.

Related Work First, we briefly consider formulations of logical predicates. Logical predicates (and relations) have been widely used as a tool to study properties of the simply typed lambda calculus. Ma and Reynolds [MR92] formulated logical predicates as Cartesian closed functors from the free CCC \mathbf{L} to $\mathbf{Pred}(G)$. Hermida [Her93] pointed out that $\mathbf{Pred}(G)$ can be replaced with the total category \mathbb{E} of a fibration $p : \mathbb{E} \rightarrow \mathbb{B}$, provided that CCC structures on \mathbb{B} can be lifted to \mathbb{E} . Plotkin et al. [PPST00] introduced a weakening of Ma and Reynolds' formulation called *Lax logical predicates*, which are functors from \mathbf{L} to $\mathbf{Pred}(G)$ preserving only finite products. The basic lemma still holds for lax logical predicates, and furthermore the converse holds. In this sense lax logical predicates and pre-logical predicates are the same. They extended lax logical

predicates from the lambda calculus to the language described by a finitary monad over \mathbf{Cat} extending finite product structure. Lax logical predicates are also extended to the computational lambda calculus [KP99]. Binary lax logical relations are closed under composition.

Kinoshita, et al. [KOPT97] proposed a generalisation of logical relations called *L-relations*. Their framework is also parameterised by a finitary monad L over \mathbf{Cat} , which allows us to generalise the language from the lambda calculus. They used category objects in \mathbf{Cat} to formulate the composition of L-relations.

Leiß[Lei01] extended pre-logical predicates to system $F\omega$, and characterised observational equivalence in terms of existence of binary pre-logical relation.

An application of binary pre-logical relations is to characterise observational equivalence between two models of a language [Lei01,HS02,Kat03].

This work refers to the framework by Fiore [Fio02] and Miculan and Scagnetto [MS03] on a categorical model of typed higher-order abstract syntax. This framework is a natural extension of the one considered in [FPT99,Hof99] to take types into account.

Convention We identify a set and its discrete category. We assume that all categories appeared in this paper are locally small. By a Cartesian category we mean a category with chosen finite products. We fix a countably infinite set of variables X (ranged over by x, y, z). For a finite set A , by $|A|$ we mean the number of elements in A . We use \vec{A} for a sequence of meta variables, like A_1, \dots, A_n .

2 Preliminaries

Category of Presentation Models We introduce the category of *presentation models* [MS03] plus some auxiliary categories for the following sections. We represent all three elements involved in the notion of pre-logical predicates (syntax, semantics and predicates) in this category.

Let T be the set of types, whose elements are ranged over by τ, σ . A *context* (ranged over by Γ) is a function from a finite subset of X to T . A *context renaming* from Γ to Γ' is a function $f : \text{dom}(\Gamma) \rightarrow \text{dom}(\Gamma')$ such that $\Gamma' \circ f = \Gamma$. They form the *category of contexts* \mathbf{C}_T^1 with the initial object given by the empty context $! : \emptyset \rightarrow T$ and binary coproducts given by cotupling of contexts $[\Gamma, \Gamma'] : \text{dom}(\Gamma) + \text{dom}(\Gamma') \rightarrow T$. By Γ, Γ' we mean the coproduct of contexts Γ and Γ' whose domains are disjoint. We fix a variable $x \in X$ and define $\langle - \rangle : T \rightarrow \mathbf{C}_T$ by $\langle \tau \rangle = \{x \mapsto \tau\}$. We assume that each variable $x \in \text{dom}(\Gamma)$ has an index number denoted by $\gamma(x) \in \{1, \dots, |\text{dom}(\Gamma)|\}$.

We define the *ambient category* $\mathbf{S}_T = \mathbf{Set}^{\mathbf{C}_T}$. Category \mathbf{S}_T has small limits and colimits, and has a *context extension operator* $(\delta_\tau A)(\Gamma) = A(\Gamma + \langle \tau \rangle)$. In fact $\delta_\tau A$ is isomorphic to $V_\tau \Rightarrow A$, where $V_\tau(\Gamma)$ is the presheaf of *variables of type τ* , defined to be $\mathbf{C}_T(\langle \tau \rangle, \Gamma) \cong \{x \mid \Gamma(x) = \tau\}$, thus δ_τ has a left adjoint. Moreover it has a right

¹ Category \mathbf{C}_T can be described as the comma category $I \downarrow T$ where $I : \mathbf{X} \hookrightarrow \mathbf{Set}$ is the inclusion functor of the full subcategory \mathbf{X} of \mathbf{Set} whose objects are finite subsets of X [Fio02,MS03]. It is a free co-Cartesian category generated from T .

adjoint ([MS03], proposition 2), thus preserves both limits and colimits. We write $\delta_{\vec{\tau}}$ for the composition $\delta_{\tau_1} \circ \dots \circ \delta_{\tau_n}$.

The category of *presentation models* \mathbf{M}_T is defined to be $(\mathbf{S}_T)^T \cong \mathbf{Set}^{\mathbf{C}_T \times T}$. It also has small limits and colimits.

Syntax: Typed Binding Signature A *typed binding signature* (ranged over by Π) is a tuple (T, O) where T is the set of *types* (ranged over by τ, σ) and O is the set of *operators* (ranged over by o), each of which is a pair of an *operator symbol* s and its *arity* $((\vec{\tau}_1, \sigma_1), \dots, (\vec{\tau}_m, \sigma_m), \tau) \in (T^+)^* \times T$. We write $s^{(\vec{\tau}_1, \sigma_1), \dots, (\vec{\tau}_m, \sigma_m) \rightarrow \tau}$ for such a pair in O ², and $o^\tau \in O$ for an operator whose result type is τ . A *typed first-order signature* (ranged over by Σ) is just a typed binding signature (T, O) such that for all $s^{(\vec{\tau}_1, \sigma_1), \dots, (\vec{\tau}_m, \sigma_m) \rightarrow \tau} \in O$, $\vec{\tau}_i = \epsilon$. It coincides with the notion of many-sorted signature.

A typed binding signature Π specifies a *typed formal system*. We first define *raw- Π term* (ranged over by M, N) by the BNF $M ::= x^\tau \mid o(\vec{x}^\tau.M, \dots, \vec{x}^\tau.M)$. In this BNF, $\vec{x}^\tau.M$ means binding of variables \vec{x}^τ in M . As usual, we identify α -equivalent terms. The typed formal system is a system to derive judgment $\Gamma \vdash_\Pi M : \tau$, where Γ is an object in \mathbf{C}_T . The system consists of the following rules for variables and operators.

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash_\Pi x^\tau : \tau} \quad \frac{\Gamma, \vec{x}_1 : \vec{\tau}_1 \vdash_\Pi M_1 : \sigma_1 \quad \dots \quad \Gamma, \vec{x}_n : \vec{\tau}_n \vdash_\Pi M_m : \sigma_m}{\Gamma \vdash_\Pi s^{(\vec{\tau}_1, \sigma_1), \dots, (\vec{\tau}_m, \sigma_m) \rightarrow \tau}(\vec{x}_1^{\vec{\tau}_1}.M_1, \dots, \vec{x}_m^{\vec{\tau}_m}.M_m) : \tau}$$

Example 1. 1. Let B be a set. By $\mathbf{Typ}^{\Rightarrow}(B)$ we mean the set defined by the BNF $\tau ::= b \mid \tau \Rightarrow \tau$ where $b \in B$. The typed binding signature Π_λ for the simply typed lambda calculus is defined to be $(\mathbf{Typ}^{\Rightarrow}(B), \{\text{lam}^{(\tau, \tau') \rightarrow \tau \Rightarrow \tau'}, \text{app}^{\tau \Rightarrow \tau', \tau \rightarrow \tau'}\})$ where τ, τ' ranges over $\mathbf{Typ}^{\Rightarrow}(B)$.

2. The typed first-order signature for combinatory logic is

$$\Sigma_{CL} = (\mathbf{Typ}^{\Rightarrow}(B), \{\text{app}^{\tau \Rightarrow \tau', \tau \rightarrow \tau'}, \mathbf{S}^{(\tau \Rightarrow \tau' \Rightarrow \tau'') \Rightarrow (\tau \Rightarrow \tau') \Rightarrow \tau \Rightarrow \tau''}, \mathbf{K}^{\tau \Rightarrow \tau' \Rightarrow \tau}\})$$

where τ, τ', τ'' ranges over $\mathbf{Typ}^{\Rightarrow}(B)$.

3. Let $\Sigma = (T_0, O_0)$ be a typed first-order signature. The typed binding signature for first-order classical logic over Σ is

$$\Pi_{\Sigma\text{-fol}} = (T_0 \cup \{\Omega\}, O_0 \cup \{\text{exists}^{(\tau, \Omega) \rightarrow \Omega} \mid \tau \in T_0\} \cup \{\text{not}^{\Omega \rightarrow \Omega}, \text{or}^{\Omega, \Omega \rightarrow \Omega}\})$$

The typed formal system described by Π determines an object $S_\Pi(\tau)(\Gamma) = \{M \mid \Gamma \vdash_\Pi M : \tau\}$ in \mathbf{M}_T . This object can be characterised as an initial algebra of the functor associated to Π by $(\Pi A)\tau = V_\tau + \coprod_{s^{(\vec{\tau}_1, \sigma_1), \dots, (\vec{\tau}_m, \sigma_m) \rightarrow \tau} \in O} (\prod_{i=1}^m \delta_{\vec{\tau}_i}(A\sigma_i))$ together with the Π -algebra structure $\iota_\Pi : \Pi S_\Pi \rightarrow S_\Pi$ corresponding to the inference rules ([MS03], theorem 1).

² This definition of typed binding signature is a special case of the one in [MS03] where the set of types allowed for variables is equal to the set of all types.

Semantics: Very Weak Categorical Model We formulate a semantics of a typed formal system $\Pi = (T, O)$ by a morphism to the object in \mathbf{M}_T which reflects a Cartesian category. The notion of semantics considered here is very weak in the sense that it does not exploit any categorical structure other than finite products. The semantics keeps the basic principle of categorical model theory: that is, types are interpreted as objects and terms are interpreted as morphisms.

An *interpretation of types* is just a functor $F : T \rightarrow \mathbb{C}$ where \mathbb{C} is a Cartesian category. We extend it to the functor $F^* : \mathbf{C}_T \rightarrow \mathbf{C}^{op}$ by $F^* \Gamma = \prod_{i=1}^{|\text{dom}(\Gamma)|} F(\Gamma(\gamma^{-1}(i)))$, which preserves finite products in $(\mathbf{C}_T)^{op}$. We write $s_{\Gamma, \Gamma'} : F^* \Gamma \times F^* \Gamma' \rightarrow F^*(\Gamma + \Gamma')$ for the natural isomorphism. For an interpretation of types $F : T \rightarrow \mathbb{C}$, we define the *clone of typed operations* H^F by $H^F(\tau)(\Gamma) = \mathbb{C}(F^* \Gamma, F\tau)$. Let \mathbb{D} be a Cartesian category. For a functor $G : \mathbb{C} \rightarrow \mathbb{D}$ preserving finite products strictly, we define a morphism $H^G : H^F \rightarrow H^{GF}$ in \mathbf{M}_T by $H^G(\tau)(\Gamma)(f : F^* \Gamma \rightarrow F\tau) = Gf : (GF)^* \Gamma = GF^* \Gamma \rightarrow GF\tau$. A *categorical interpretation of Π* consists of a Cartesian category \mathbb{C} , an interpretation of types $F : T \rightarrow \mathbb{C}$ and a morphism $m : S_\Pi \rightarrow H^F$ in \mathbf{M}_T called *interpretation of terms*, which assigns to a well-formed term $\Gamma \vdash_\Pi M : \tau$ a morphism $m(M) : F^* \Gamma \rightarrow F\tau$. We use the notation $\mathbb{C}[\![-]\!]_F$ to represent a categorical interpretation. We define the *product* of categorical interpretations $(\mathbb{C}, F_1, m_1), (\mathbb{D}, F_2, m_2)$ to be $(\mathbb{C} \times \mathbb{D}, \langle F_1, F_2 \rangle, (m_1, m_2))$ where (m_1, m_2) is defined by $(m_1, m_2)(\tau)(\Gamma) = (m_1(\tau)(\Gamma), m_2(\tau)(\Gamma))$.

Often, H^F is equipped with a Π -algebra structure. In this case we can obtain an interpretation of terms by the initiality of S_Π . This is the *initial algebra semantics* for typed binding signature ([FPT99,MS03]).³

To specify a Π -algebra structure over H^F , it is sufficient to specify a morphism $u_o : \prod_{i=1}^m \delta_{\vec{\tau}_i}(H^F \sigma_i) \rightarrow H^F \tau$ in \mathbf{S}_T for each operator $o \in O$ of arity $(\vec{\tau}_1, \sigma_1), \dots, (\vec{\tau}_m, \sigma_m) \rightarrow \tau$. Together with the mapping $v^\tau : V_\tau \rightarrow H^F \tau$ defined to be $v^\tau(\Gamma)(x) = \pi_{\gamma(x)}$, we obtain a Π -algebra structure over H^F by $[v^-, u_{o^-}]_{o^- \in O} : (\Pi H^F) \rightarrow H^F$.

Example 2. (Continued from example 1)

1. Let $F_\lambda : \mathbf{Typ}^{\Rightarrow}(B) \rightarrow \mathbb{C}$ be an interpretation of types satisfying $F_\lambda(\tau \Rightarrow \tau') = F_\lambda(\tau) \Rightarrow F_\lambda(\tau')$. The morphisms in \mathbf{S}_T given by $(u_{\text{lam}(\tau, \tau') \rightarrow \tau \Rightarrow \tau'})_\Gamma(f) = \lambda(f \circ s_{\Gamma, \langle \tau \rangle})$ and $(u_{\text{app} \tau \Rightarrow \tau', \tau \rightarrow \tau'})_\Gamma(f, g) = @ \circ \langle f, g \rangle$ yield a Π_λ -algebra structure over H^{F_λ} . The initial algebra semantics coincides with the standard semantics of the simply typed lambda calculus in \mathbb{C} .
2. Let $\Sigma = (T_0, O_0)$ be a typed first-order signature. A *many-sorted Σ -algebra* \mathcal{A} consists of a T_0 -indexed family of sets $A : T_0 \rightarrow \mathbf{Set}$ called *carrier sets* and an assignment of a function $o_{\mathcal{A}} : A^{\tau_1} \times \dots \times A^{\tau_n} \rightarrow A^\tau$ to each operator $o \in O_0$ of arity $\tau_1, \dots, \tau_n \rightarrow \tau$. To each operator $o \in O_0$, we assign a morphism $(u_o)_\Gamma(f_1, \dots, f_n) = o_{\mathcal{A}} \circ \langle f_1, \dots, f_n \rangle$ in \mathbf{S}_T . This yields a Σ -algebra structure over H^A and the interpretation of terms, namely $\mathcal{A}[\![-]\!] : S_\Sigma \rightarrow H^A$.

³ We note that interpretations of terms are not restricted to algebra morphisms. The reason is to cover the interpretation of terms which is obtained by composition of morphisms of different algebras. This case is considered in example 5.

3. Let $\Sigma = (T_0, O_0)$ be a typed first-order signature and \mathcal{A} be a many-sorted Σ -algebra. We give a categorical semantics of $\Pi_{\Sigma\text{-fol}}$ in \mathbf{Set} , which coincides with the standard interpretation of the first-order classical logic in the model constructed over \mathcal{A} . The interpretation of types $I_{\mathcal{A}} : T_0 \cup \{\Omega\} \rightarrow \mathbf{Set}$ is given by $I_{\mathcal{A}}(\Omega) = \mathbf{2} = \{\top, \perp\}$ and $I_{\mathcal{A}}(\tau) = A^\tau$ for all $\tau \in T_0$. To give an interpretation of terms, we specify the following morphisms for each operator.

$$\begin{aligned} (u_{s^{\tau_1, \dots, \tau_n \rightarrow \tau}})_\Gamma(f_1, \dots, f_n) &= s_{\mathcal{A}} \circ \langle f_1, \dots, f_n \rangle \quad (s^{\tau_1, \dots, \tau_n \rightarrow \tau} \in O_0) \\ (u_{\text{exists}(\tau, \Omega \rightarrow \Omega)})_\Gamma(f)(\rho) &= \top \iff \exists x \in A^\tau . f(\rho, x) = \top \\ (u_{\text{not}(\Omega \rightarrow \Omega)})_\Gamma(f)(\rho) &= \top \iff f(\rho) = \perp \\ (u_{\text{or}(\Omega, \Omega \rightarrow \Omega)})_\Gamma(f_1, f_2)(\rho) &= \top \iff f_1(\rho) = \top \vee f_2(\rho) = \top \end{aligned}$$

This gives the standard set-theoretic semantics of first-order classical logic over \mathcal{A} , namely $\mathcal{I}_{\mathcal{A}}[-] : S_{\Pi_{\Sigma\text{-fol}}} \rightarrow H^{I_{\mathcal{A}}}$.

Predicates: Subcone We introduce the notion of predicates over a categorical interpretation of types. When types are interpreted in set theory, the natural notion of predicate is simply a subset of each carrier set. In categorical settings, carrier sets are replaced by objects, and the notion of predicates is more subtle.

We write $\mathbf{Sub}(\mathbb{D})$ for the category of subobjects in a category \mathbb{D} , and $p_{\mathbb{D}} : \mathbf{Sub}(\mathbb{D}) \rightarrow \mathbb{D}$ for the forgetful functor. First we recall *injective scones* in [MS93]. The injective scone of a Cartesian category \mathbb{C} is the category obtained by pulling back $p_{\mathbf{Set}}$ along the global section functor $\mathbb{C}(1, -)$ ([Jac99], example 1.5.2). In this approach, the notion of predicates over an object C in \mathbb{C} is represented as subsets of global elements of C .

In this paper we use the *subcone* approach [Laf88, MR92, MS93, PPST00], which is a mild generalisation of injective scones. We replace \mathbf{Set} with a category \mathbb{D} with finite limits and the global section functor with finite-product preserving functor. We define the *category* $\mathbf{Pred}(G)$ of G -predicates by pulling back $p_{\mathbb{D}}$ along G .

$$\begin{array}{ccc} \mathbf{Pred}(G) & \longrightarrow & \mathbf{Sub}(\mathbb{D}) \\ \pi_G \downarrow & & \downarrow p_{\mathbb{D}} \\ \mathbb{C} & \xrightarrow{G} & \mathbb{D} \end{array}$$

Category $\mathbf{Pred}(G)$ has finite products which are strictly preserved by π_G ⁴. We also define the category $\mathbf{Rel}_2(G)$ of binary G -relations to be $\mathbf{Pred}(\text{prod} \circ G \times G)$, where $\text{prod} : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ gives the binary products in \mathbb{C} .

We adopt the following notational convention. Let P and Q be objects in $\mathbf{Pred}(G)$ and $f : \pi_G(P) \rightarrow \pi_G(Q)$ be a morphism in \mathbb{C} . We write $f : P \rightarrow Q$ if there exists a morphism $g : P \rightarrow Q$ in $\mathbf{Pred}(G)$ such that $\pi_G(g) = f$.

Let $F : T \rightarrow \mathbb{C}$ and $P : T \rightarrow \mathbf{Pred}(G)$ be interpretations of types. We say that P is a G -predicate over F (written $P \subseteq_G F$) if $\pi_G \circ P = F$. A *binary G -relation* P between $F_1, F_2 : T \rightarrow \mathbb{C}$ (written $P \subseteq_G F_1, F_2$) is just a $(\text{prod} \circ G \times G)$ -predicate over $\langle F_1, F_2 \rangle$. For a predicate $P \subseteq_G F$, there exists a monomorphism $H^{\pi_G} : H^P \hookrightarrow H^F$.

⁴ We give a proof in terms of fibred category theory. In fact $p_{\mathbb{D}}$ is a fibration with fibred finite limits, thus so is π_G (see [Jac99], section 1.8). Then it follows from lemma 8.5.2 of [Jac99].

3 Pre-logical Predicates

In this section, we fix a Cartesian category \mathbb{C} , a category \mathbb{D} with finite limits, a finite product preserving functor $G : \mathbb{C} \rightarrow \mathbb{D}$ and a binding signature Π .

Let $\mathbb{C}[-]_F$ be a categorical interpretation of Π and $P \subseteq_G F$ be a predicate. We consider taking pullback of H^{π_G} along $\mathbb{C}[-]_F$ in \mathbf{M}_T .

$$\begin{array}{ccc}
 S_{\Pi}^P & \xrightarrow{\quad} & H^P \\
 \downarrow i & (*) & \downarrow H^{\pi_G} \\
 S_{\Pi} & \xrightarrow{\mathbb{C}[-]_F} & H^F
 \end{array}$$

The vertex S_{Π}^P can be calculated as $S_{\Pi}^P(\tau)(\Gamma) = \{M \mid \Gamma \vdash_{\Pi} M : \tau \wedge \mathbb{C}[M]_F : P^* \Gamma \rightarrow P\tau\}$. This represents the collection of terms whose meanings by $\mathbb{C}[-]_F$ respects the predicate P . Thus when this is isomorphic to S_{Π} , the meanings of all the well-formed terms by $\mathbb{C}[-]_F$ respects the predicate P .

Definition 1. Let $\mathbb{C}[-]_F$ be a categorical interpretation of Π . We say that a predicate $P \subseteq_G F$ satisfies the basic lemma for Π along $\mathbb{C}[-]_F$ if in diagram (*), S_{Π}^P is isomorphic to S_{Π} . This is equivalent to say that there exists a necessarily unique morphism $p : S_{\Pi} \rightarrow H^P$ (convention: we use the small letter of the predicate) such that $H^{\pi_G} \circ p = \mathbb{C}[-]_F$.

Example 3. (Continued from example 2)

1. Let $P \subseteq_{\mathbb{C}(1, -)} F_{\lambda}$ be a predicate satisfying the basic lemma for Π_{λ} along $\mathbb{C}[-]_{F_{\lambda}}$. It is equivalent to the $\mathbf{Typ}^{\Rightarrow}(B)$ -indexed family of subsets $P\tau \subseteq \mathbb{C}(1, F_{\lambda}\tau)$ such that for all $\rho \in P^*\Gamma$ and $\Gamma \vdash_{\Pi_{\lambda}} M : \tau$, we have $\mathbb{C}[M]_{F_{\lambda}} \circ \rho \in P\tau$.
2. Let $P \subseteq_{\text{IdSet}} A$ be a predicate satisfying the basic lemma for Σ along $\mathcal{A}[-]$. It is equivalent to the T_0 -indexed family of subsets $P\tau \subseteq A\tau$ satisfying $\mathcal{A}[M]\rho \in P\tau$ for all $\Gamma \vdash_{\Sigma} M : \tau$ and $\rho \in P^*\Gamma$.
3. Let Σ be a first-order signature, \mathcal{A}, \mathcal{B} be many-sorted Σ -algebras and $P \subseteq_{\text{IdSet}} I_{\mathcal{A}}, I_{\mathcal{B}}$ be a binary relation satisfying the basic lemma for $\Pi_{\Sigma\text{-fol}}$ along $\mathcal{I}_{\mathcal{A}}[-] \times \mathcal{I}_{\mathcal{B}}[-]$ and $P\Omega = \text{id}_{\Omega}$. It is equivalent to a $T_0 \cup \{\Omega\}$ -indexed family of subsets $P\tau \subseteq A\tau \times B\tau$ such that for all $\Gamma \vdash_{\Pi_{\Sigma\text{-fol}}} M : \tau$ and $(\rho, \rho') \in P^*\Gamma$, $(\mathcal{I}_{\mathcal{A}}[M]\rho, \mathcal{I}_{\mathcal{B}}[M]\rho') \in P\tau$ when $\tau \in T_0$ and $\mathcal{I}_{\mathcal{A}}[M]\rho = \mathcal{I}_{\mathcal{B}}[M]\rho$ when $\tau = \Omega$. The latter implies that \mathcal{A} and \mathcal{B} are *elementary equivalent*.

Now we introduce the notion of pre-logical predicates.

Definition 2. Let $\mathbb{C}[-]_F$ be a categorical interpretation of Π . We call a predicate $P \subseteq_G F$ pre-logical for Π along $\mathbb{C}[-]_F$ if in diagram (*) there exists a necessarily unique Π -algebra $(S_{\Pi}^P, \alpha : \Pi S_{\Pi}^P \rightarrow S_{\Pi}^P)$ such that the projection i induced by pullback is a Π -algebra morphism to the initial algebra (S_{Π}, ι_{Π}) .

An elementary description of P being pre-logical is that a) for all $\Gamma \vdash_{\Pi} x^{\tau} : \tau$, $\mathbb{C}[x^{\tau}]_F : P^*\Gamma \rightarrow P\tau$ and b) for all operator $o \in O$ of arity $(\vec{\tau}_1, \sigma_1), \dots, (\vec{\tau}_m, \sigma_m) \rightarrow$

τ and well-formed terms $\Gamma, \overrightarrow{x_i : \tau_i} \vdash_{\Pi} M_i : \sigma_i$ ($1 \leq i \leq n$), $\mathbb{C}[[M_i]]_F : P^*(\Gamma, \overrightarrow{x_i : \tau_i}) \rightarrow P\sigma_i$ for all $1 \leq i \leq n$ implies $\mathbb{C}[[o(\overrightarrow{x_1^{\tau_1}}.M_1, \dots, \overrightarrow{x_n^{\tau_n}}.M_n)]]_F : P^*\Gamma \rightarrow P\tau$. Normally a) is satisfied as variables are interpreted by projections. For operators having no variable binding, if the interpretation of terms $\mathbb{C}[-]_F$ satisfies a) and the semantic substitution lemma, i.e. $\mathbb{C}[[M[M_i/x_i]]]_F = \mathbb{C}[[M]]_F \circ \langle \mathbb{C}[[M_i]]_F \rangle$, then the condition b) can be rewritten to $\mathbb{C}[[s^{\tau_1, \dots, \tau_n \rightarrow \tau}(x_1^{\tau_1}, \dots, x_n^{\tau_n})]]_F \in P^*(\overrightarrow{x : \tau}) \rightarrow P\tau$.

Example 4. (Continued from example 2)

1. A predicate $P \subseteq_{\mathbb{C}(1, -)} F_\lambda$ is pre-logical for Π_λ along $\mathbb{C}[-]_{F_\lambda}$ if for all $f \in P(\tau \Rightarrow \tau')$ and $g \in P\tau$, $\textcircled{\circ} \langle f, g \rangle \in P\tau'$, and for all $\Gamma, x : \tau \vdash_{\Pi_\lambda} M : \tau'$, $\forall \rho \in P^*(\Gamma, x : \tau) \cdot \mathbb{C}[[M]]_{F_\lambda} \circ \rho \in P\tau'$ implies $\forall \rho \in P^*\Gamma \cdot \mathbb{C}[[\lambda x : \tau. M]]_{F_\lambda} \circ \rho \in P(\tau \Rightarrow \tau')$.
2. A predicate $P \subseteq_{\text{Id}_{\text{Set}}} A$ is pre-logical for Σ along $\mathcal{A}[-]$ if for all $c^{\tau_1, \dots, \tau_n \rightarrow \tau} \in O_0$ and $x_i \in P^{\tau_i}$, $c_{\mathcal{A}}^{\tau_1, \dots, \tau_n \rightarrow \tau}(x_1, \dots, x_n) \in P^\tau$. An *algebraic predicate* [HS02] is just a pre-logical predicate for Σ_{CL} along $\mathcal{U}[-]$ for a typed combinatory algebra \mathcal{U} (i.e. a many-sorted Σ_{CL} algebra).
3. A predicate $P \subseteq_{\text{Id}_{\text{Set}}} I_{\mathcal{A}}$ is pre-logical for $\Pi_{\Sigma\text{-fol}}$ along $\mathcal{I}_{\mathcal{A}}[-]$ if for all first-order operator (including or, not) $s^{\tau_1, \dots, \tau_n \rightarrow \tau}$ and $x_i \in P^{\tau_i}$, $s_{\mathcal{A}}^{\tau_1, \dots, \tau_n \rightarrow \tau}(x_1, \dots, x_n) \in P^\tau$ holds, and for all $\tau \in T_0$ and $\Gamma, x : \tau \vdash M : \Omega$, $\forall \rho \in P^*(\Gamma, x : \tau) \cdot \mathcal{I}_{\mathcal{A}}[[M]]\rho \in P\Omega$ implies $\forall \rho \in P^*\Gamma \cdot \mathcal{I}_{\mathcal{A}}[[\text{exists}(x.M)]]\rho \in P\Omega$.

Theorem 1 (The Basic Lemma of Pre-logical Predicates). *Let $\mathbb{C}[-]_F$ be a categorical interpretation of Π . A predicate $P \subseteq_G F$ is pre-logical if and only if P satisfies the basic lemma.*

Proof. (if) If P satisfies the basic lemma, we have an isomorphism $f : S_\Pi^P \cong S_\Pi$. Then $f : (S_\Pi^P, f^{-1} \circ \iota_\Pi \circ (\Pi f)) \rightarrow (S_\Pi, \iota_\Pi)$ is a Π -algebra morphism. Therefore P is pre-logical.

(only if) Suppose there exists a Π -algebra (S_Π^P, α) . Let $! : (S_\Pi, \iota_\Pi) \rightarrow (S_\Pi^P, \alpha)$ be the unique morphism from the initial Π -algebra. From the universal property of initial Π -algebra, we have $i \circ ! = id$. Now we have $i \circ ! \circ i = i = i \circ id$, and since i is mono, $! \circ i = id$. Therefore (S_Π^P, α) and (S_Π, ι_Π) are isomorphic, thus S_Π and S_Π^P are so. \square

This theorem is a categorical re-formulation of the inductive proof of the basic lemma for pre-logical relations in [HS02]. From now on we identify pre-logical predicates and predicates satisfying the basic lemma.

We give one sufficient condition for P being pre-logical. Below we identify a monomorphism in \mathbf{M}_T and an object in $\mathbf{Sub}(\mathbf{M}_T)$. First, we can *lift* the endofunctor (of a typed binding signature) Π to the one over $\mathbf{Sub}(\mathbf{M}_T)$, namely $\tilde{\Pi}$. Here lifting means that $\tilde{\Pi}$ satisfies $p_{\mathbf{M}_T} \circ \tilde{\Pi} = \Pi \circ p_{\mathbf{M}_T}$ (see [Jac99], section 9.2). This is because all the constructs of Π have liftings over $\mathbf{Sub}(\mathbf{S}_T)$. Functor $p_{\mathbf{M}_T}$ is a subobject fibration, thus admits comprehension ([Jac99], example 4.6.3). It is easy to see that $\tilde{\Pi} \circ \top \cong \top \circ \Pi$, where $\top : \mathbf{M}_T \rightarrow \mathbf{Sub}(\mathbf{M}_T)$ is the right adjoint of $p_{\mathbf{M}_T}$ giving fibred terminal objects. Thus an initial Π -algebra is inductive ([Jac99], definition 9.2.6, proposition 9.2.7), i.e. id_{S_Π} is an initial $\tilde{\Pi}$ -algebra.

Proposition 1. *Let $P \subseteq_G F$ be a predicate and suppose that $H^{\pi_G} : H^P \rightarrow H^F$ has a $\tilde{\Pi}$ -algebra structure. Then P satisfies the basic lemma for Π along the initial algebra semantics of Π in H^F .*

4 Composability of Pre-logical Relations

We move to the composability of binary pre-logical relations. Binary pre-logical relations are closed under relational composition, which is not enjoyed by logical relations [HS02]. We give here a categorical account of composability of pre-logical relations. In this section we fix a typed binding signature Π , a Cartesian category \mathbb{C} , a category \mathbb{D} with finite limits, a finite-product preserving functor $G : \mathbb{C} \rightarrow \mathbb{D}$ and categorical interpretations $\mathbb{C}[-]_{F_i}$ ($1 \leq i \leq 3$) of Π . We write $\text{fst}, \text{snd} : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ for projections.

First, we assume that a composition operator c over $\mathbf{Rel}_2(G)$ is available. This operator is partial, and defined over *composable pairs of relations*, i.e. a pair (R, S) of objects in $\mathbf{Rel}_2(G)$ such that $\text{snd}(\pi_G(R)) = \text{fst}(\pi_G(S))$. The composition operator yields an object $c(R, S)$ in $\mathbf{Rel}_2(G)$ such that $\text{fst}(\pi_G(c(R, S))) = \text{fst}(\pi_G(R))$ and $\text{snd}(\pi_G(c(R, S))) = \text{snd}(\pi_G(S))$, and a morphism $c(f, g) : c(R, S) \rightarrow c(R', S')$ for composable pairs of relations $(R, S), (R', S')$ and morphisms $f : R \rightarrow R', g : S \rightarrow S'$ in $\mathbf{Rel}_2(G)$. It is natural to assume that $\mathbf{Rel}_2(G)$ has identity relation, and the composition operator satisfies the laws of identity and associativity. To summarise, we assume that we have a *category object* in \mathbf{Cat} :

$$\mathbf{Rel}_c(G) \xrightarrow{c} \mathbf{Rel}_2(G) \begin{array}{c} \xrightarrow{\partial_1 = \text{fst} \circ \pi_G} \\ \xleftarrow{id} \\ \xrightarrow{\partial_2 = \text{snd} \circ \pi_G} \end{array} \mathbb{C} \quad (**)$$

where $\mathbf{Rel}_c(G)$ is the category of composable pairs of relations obtained by pulling back ∂_2 along ∂_1 . Using category objects in \mathbf{Cat} to formulate the composition of relations is due to [KOPT97].

For $R \subseteq_G F_1, F_2$ and $S \subseteq_G F_2, F_3$, we define their composition $c(R, S)(\tau)$ to be $c(R\tau, S\tau)$. It is clear that $c(R, S) \subseteq_G F_1, F_3$.

Theorem 2 (Composability of Pre-logical Relations). *Let $R \subseteq_G F_1, F_2$ and $S \subseteq_G F_2, F_3$ be pre-logical binary relations for Π along $\mathbb{C}[-]_{F_1} \times \mathbb{C}[-]_{F_2}$ and $\mathbb{C}[-]_{F_2} \times \mathbb{C}[-]_{F_3}$ respectively. Then $c(R, S) \subseteq_G F_1, F_3$ is pre-logical for Π along $\mathbb{C}[-]_{F_1} \times \mathbb{C}[-]_{F_3}$.*

Proof. We find a morphism $h : S_{\Pi} \rightarrow H^{c(R, S)}$ such that $H^{\pi_G} \circ h = \mathbb{C}[-]_{F_1} \times \mathbb{C}[-]_{F_3}$ where $H^{\pi_G} : H^{c(R, S)} \rightarrow H^{\langle F_1, F_3 \rangle}$ is the morphism in \mathbf{M}_T . We give h by $h(\tau)(\Gamma)(M) = c(r(\tau)(\Gamma)(M), s(\tau)(\Gamma)(M))$ for all well-formed terms $\Gamma \vdash_{\Pi} M : \tau$, where $r : S_{\Pi} \rightarrow H^R$ and $s : S_{\Pi} \rightarrow H^S$ are morphisms which exist by definition of the basic lemma (see definition 1). \square

When do we have a category object as $(**)$ above? Recall that composition of relations can be expressed by the $c(R, S)(x, z) = \exists y. R(x, y) \wedge S(y, z)$. The standard interpretation of this formula in set theory gives the composition of binary relations. Now we replace set theory with *regular fibration* [Jac99], which is a preordered fibration $p : \mathbb{E} \rightarrow \mathbb{B}$ such that \mathbb{B} is Cartesian and p has fibred finite products, fibred

equality and simple coproducts satisfying Frobenius and Beck-Chevalley (for details, see [Jac99]). A regular fibration provides a categorical model of the $\exists\wedge\top$ -fragment of predicate logic. Interpreting the above formula in this model gives rise to a composition operation, which enjoys the identity and associativity laws.

Proposition 2. *Assume that $p_{\mathbb{D}} : \mathbf{Sub}(\mathbb{D}) \rightarrow \mathbb{D}$ is a regular fibration. Then we can construct a category object as $(**)$ above in \mathbf{Cat} .*

5 Examples

Example 5. In this example, we examine the relationship between pre-logical predicates for combinatory algebras and pre-logical predicates for the simply typed lambda calculus in our framework. This is a revisit of proposition 3.3 in [HS02].

The standard abstraction mechanism $\lambda^*x.M$ in combinatory logic (see definition 7.1.5, [Bar84]) induces a Π_λ -algebra structures over $S_{\Sigma_{CL}}$. From the universal property of initial Π_λ -algebra, there is a unique Π_λ -algebra morphism, namely $(-)_CL : S_{\Pi_\lambda} \rightarrow S_{\Sigma_{CL}}$, which coincides with the standard lambda-to-CL translation (definition 7.3.1, [Bar84]). The composition $\mathcal{U}[(-)_CL]$ gives an interpretation of the simply typed lambda calculus in a combinatory algebra \mathcal{U} . In general this is not a Π_λ -algebra morphism. Conversely, giving the standard representation of S, K combinators in S_{Π_λ} equips it with a Σ_{CL} algebra structure. Then there exists a unique Σ_{CL} -algebra morphism from an initial Σ_{CL} -algebra, namely $(-)_\lambda : S_{\Sigma_{CL}} \rightarrow S_{\Pi_\lambda}$.

Let \mathcal{U} be a combinatory algebra and $P \subseteq_{\text{IdSet}} U$ be a pre-logical predicate for Σ_{CL} along $\mathcal{U}[-]$. Then we have $H^{\pi_G} \circ p \circ (-)_CL = \mathcal{U}[(-)_CL]$, thus P is pre-logical for Π_λ along $\mathcal{U}[(-)_CL]$. This explains that an algebraic predicate relating combinators yields a pre-logical predicate (“if” part of proposition 3.3, [HS02]). Conversely, let $P \subseteq_{\text{IdSet}} U$ be a pre-logical predicate for Π_λ along $\mathcal{U}[(-)_CL]$. It is a pre-logical predicate for Σ_{CL} along $\mathcal{U}[(-)_\lambda]$ — but not for $\mathcal{U}[-]$ in general!

Theorem 3. *There exists a combinatory algebra \mathcal{U}_0 and a pre-logical predicate $P \subseteq_{\text{IdSet}} U_0$ for Π_λ along $\mathcal{U}_0[(-)_CL]$ which is not pre-logical for Π_{CL} along $\mathcal{U}_0[-]$.*

The proof uses the fact that the image of the standard lambda-to-CL translation does not cover the entire set of combinatory logic terms, particularly S and K . To exploit this fact, we take \mathcal{U}_0 as the closed term algebra, and see that the definability predicate by $\mathcal{U}_0[(-)_CL]$, which is pre-logical for Π_λ , is not pre-logical for Π_{CL} along $\mathcal{U}_0[-]$. This means that “only if” part of proposition 3.3, [HS02] is not precise enough. The subtle point is that “to which semantics” it satisfies the basic lemma, and it was missed in [HS02].

When is P a pre-logical predicate for Σ_{CL} along $\mathcal{U}[-]$? One answer is to fix the lambda-to-CL translation $(-)_CL$ to make it surjective. To achieve this, we introduce another abstraction mechanism $\lambda'x.M$ defined to be $\lambda'x.x = SKK$, $\lambda'x.M = KM$ provided $x \notin \text{FV}(M)$, $\lambda'x.\lambda^*y.x = K$, $\lambda'x.\lambda^*y.\lambda^*z.xz(yz) = S$, $\lambda'x.MN = S(\lambda'x.M)(\lambda'x.N)$. The lambda-to-CL translation constructed from this abstraction mechanism, say $(-)_CL'$, covers all the combinators, and moreover satisfies $((M)_\lambda)_{CL'} = M$. Thus a pre-logical predicate for Π_λ along $\mathcal{U}[(-)_{CL'}]$ is a pre-logical predicate for

Σ_{CL} along $\mathcal{U}[-]$. Another answer is to require \mathcal{U} to be a *lambda algebra*, which always satisfies $\mathcal{U}[\lambda(-)_{CL}] = \mathcal{U}[-]$ (see lemma 5.2.3-2, [Bar84]).

Example 6. We examine the connection between lax logical predicates [PPST00] and pre-logical predicates as defined in here. For this, we fix a set of base types B and define the set of types including finite products by the BNF $\mathbf{Typ}^{\Rightarrow \times}(B) \ni \tau ::= b \mid 1 \mid \tau \times \tau \mid \tau \Rightarrow \tau$ where $b \in B$. The signature for the simply typed lambda calculus with finite products is defined by

$$\Pi_{\lambda \times} = (\mathbf{Typ}^{\Rightarrow \times}(B), \{\text{lam}^{(\tau, \tau') \rightarrow \tau \Rightarrow \tau'}, \text{app}^{\tau \Rightarrow \tau', \tau \rightarrow \tau'}, *^1, \text{pair}^{\tau, \tau' \rightarrow \tau \times \tau'}, \text{fst}^{\tau \times \tau' \rightarrow \tau}, \text{snd}^{\tau \times \tau' \rightarrow \tau'}\}).$$

Let \mathbf{L} be the free CCC generated from the set of base types B . An object of \mathbf{L} is a type $\tau \in \mathbf{Typ}^{\Rightarrow \times}(B)$, and a morphism from τ to τ' in \mathbf{L} is a $\beta\eta$ -equivalence class of a well-formed terms $x : \tau \vdash_{\Pi_{\lambda \times}} M : \tau'$. We write $I : \mathbf{Typ}^{\Rightarrow \times}(B) \rightarrow \mathbf{L}$ for the inclusion functor. As we have seen in example 2, since \mathbf{L} is a CCC, it provides a $\Pi_{\lambda \times}$ -algebra structure, thus there exists a unique $\Pi_{\lambda \times}$ -morphism $\mathbf{L}[-]_I : S_{\Pi_{\lambda \times}} \rightarrow H^I$. We note that the mapping $\mathbf{L}[-]_I$ is an epimorphism.

Let \mathbb{C} be a CCC, \mathbb{D} be a CCC with finite limits, $G : \mathbb{C} \rightarrow \mathbb{D}$ be a functor preserving finite products and $\llbracket - \rrbracket : \mathbf{L} \rightarrow \mathbb{C}$ be a strict Cartesian closed functor. A *lax logical predicate* q [PPST00] over $\llbracket - \rrbracket$ is a finite-product preserving functor $q : \mathbf{L} \rightarrow \mathbf{Pred}(G)$ such that $\pi_G \circ q = \llbracket - \rrbracket$.

Theorem 4. *A lax logical predicate $p : \mathbf{L} \rightarrow \mathbf{Pred}(G)$ determines a pre-logical predicate $p \circ I \subseteq_G \llbracket - \rrbracket \circ I$ for $\Pi_{\lambda \times}$ along $H^{\llbracket - \rrbracket} \circ \mathbf{L}[-]_I$. Conversely, if $P \subseteq_G \llbracket - \rrbracket \circ I$ is a pre-logical predicate for $\Pi_{\lambda \times}$ along $H^{\llbracket - \rrbracket} \circ \mathbf{L}[-]_I$, then there exists a lax logical predicate q such that for all $\Gamma \vdash_{\Pi_{\lambda \times}} M : \tau$, $H^q \circ \mathbf{L}[M]_I = p(\tau)(\Gamma)(M)$.*

Example 7. In this example we see a characterisation of elementary submodels in terms of a binary pre-logical relation. Let $\Sigma = (T_0, O_0)$ be a typed first-order signature, \mathcal{B} be a many-sorted Σ -algebra and \mathcal{A} be a subalgebra of \mathcal{B} . For all $\Gamma \vdash_{\Pi_{\Sigma\text{-fol}}} M : \tau$ with $\tau \in T_0$, we have $\mathcal{A}[M] = \mathcal{B}[M]$ because \mathcal{A} is a submodel of \mathcal{B} . However, this may not hold when $\tau = \Omega$ because of existential quantifier. Thus we say \mathcal{A} is an *elementary submodel* of \mathcal{B} (written $\mathcal{A} \preceq \mathcal{B}$) if the above holds for $\tau = \Omega$ as well.

Theorem 5. *$\mathcal{A} \preceq \mathcal{B}$ if and only if the inclusion relation $R \subseteq_{\text{Id}_{\text{Set}}} I_{\mathcal{A}}, I_{\mathcal{B}}$, i.e. $R^\Omega = \text{id}_\Omega$ and $R^\tau = \{(x, x) \mid x \in A^\tau\}$, is pre-logical for $\Pi_{\Sigma\text{-fol}}$ along $\mathcal{I}_{\mathcal{A}}[-] \times \mathcal{I}_{\mathcal{B}}[-]$.*

6 Conclusion

We have given a generalisation of pre-logical predicates to arbitrary typed formal systems, and shown that they are equivalent to predicates satisfying the basic lemma, and that binary pre-logical relations are closed under composition. We represent three underlying components of pre-logical predicates — syntax, semantics and predicates — in the category of presentation models. Then we formulate pre-logical predicates and predicates satisfying the basic lemma, and show their equivalence.

It is interesting to extend our framework for defining formal systems. One direction is to allow type variables so that we can cover type systems such as System F or FPC [FP94]. The other direction is to modify the notion of contexts from the Cartesian one to linear one to cover linear logic. In both cases we also have to switch the notion of models from Cartesian categories to more elaborate categorical structures such as polymorphic fibrations, symmetric monoidal categories, etc.

Acknowledgments I thank Donald Sannella, Daniel Turi and John Power for useful discussions. This work was supported by an LFCS studentship.

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A Proofs

Proof of proposition 1 The initiality of $id_{S_{\Pi}}$ gives a morphism $f : S_{\Pi} \rightarrow H^P$ which is above the Π -algebra morphism $! : S_{\Pi} \rightarrow H^F$, i.e. $H^{\pi_G} \circ f = !$. Thus P satisfies the basic lemma by definition. \square

Proof of proposition 2 To prove this proposition, we use the internal logic of fibrations [Jac99]. From the assumption, the logic provides the $\exists \wedge \top =$ -fragment of predicate logic. For details, see [Jac99]. The following reasoning is done in the internal logic of $p_{\mathbb{D}}$.

An object in $\mathbf{Rel}_2(G)$ is a triple (C, C', P) where C, C' are objects in \mathbb{C} and P is a predicate $x : GC, y : GC' \vdash P(x, y)$ of the internal logic of the fibration. A morphism from (C, C', P) to (D, D', Q) is a pair of morphisms $f : C \rightarrow D, g : C' \rightarrow D'$ in \mathbb{C} such that $x : GC, y : GC' \mid P(x, y) \vdash Q(Gf(x), Gg(y))$ holds.

An object in $\mathbf{Rel}_c(G)$ is a tuple (C, C', C'', P, P') such that (C, C', P) and (C', C'', P') are objects in $\mathbf{Rel}_2(G)$. A morphism from (C, C', C'', P, P') to (D, D', D'', Q, Q') in $\mathbf{Rel}_c(G)$ is a triple $(f : C \rightarrow D, g : C' \rightarrow D', h : C'' \rightarrow D'')$ such that $(f, g) : (C, C', P) \rightarrow (D, D', Q)$ and $(g, h) : (C', C'', P') \rightarrow (D', D'', Q')$ are morphisms in $\mathbf{Rel}_2(G)$.

For an object C in \mathbb{C} , we assign an object $id(C)$ in $\mathbf{Rel}_2(G)$ by $x : GC, y : GC \vdash x = y$. For all $f : C \rightarrow D$ in \mathbb{C} , we can derive a judgment $x : GC, y : GC \vdash x = y \vdash Gf(x) = Gf(y)$ in the internal logic of the fibration. We can extend this assignment to a functor $\mathbb{C} \rightarrow \mathbf{Rel}_2(G)$.

For an object (C, C', C'', P, P') in $\mathbf{Rel}_c(G)$, we define an object $c(C, C', C'', P, P')$ in $\mathbf{Rel}_2(G)$ by $C, C'', x : GC, z : GC'' \vdash \exists y : GC' . P(x, y) \wedge P'(y, z)$ (we omit re-indexing functors along projections for readability).

Let $(f, g, h) : (C, C', C'', P, P') \rightarrow (D, D', D'', Q, Q')$ be a morphism in $\mathbf{Rel}_c(G)$. In the internal logic of the fibration, we have the following derivation (annotation of objects are omitted for readability):

$$\frac{\frac{x, y \mid P(x, y) \vdash Q(Gf(x), Gg(y)) \quad y, z \mid P'(y, z) \vdash Q'(Gg(y), Gh(z))}{x, y, z \mid P(x, y) \wedge P'(y, z) \vdash Q(Gf(x), Gg(y)) \quad x, y, z \mid P(x, y) \wedge P'(y, z) \vdash Q'(Gg(y), Gh(z))}}{x, y, z \mid P(x, y) \wedge P'(y, z) \vdash Q(Gf(x), Gg(y)) \wedge Q'(Gg(y), Gh(z))}}{\frac{x, y, z \mid P(x, y) \wedge P'(y, z) \vdash \exists y . Q(Gf(x), y) \wedge Q'(y, Gh(z))}{x, z \mid \exists y . P(x, y) \wedge P'(y, z) \vdash \exists y . Q(Gf(x), y) \wedge Q'(y, Gh(z))}}$$

Thus c extends to a functor $c : \mathbf{Rel}_c(G) \rightarrow \mathbf{Rel}_2(G)$.

To see that id and c satisfy the laws of category object, such as $c(id(C), (C, C', P)) = (C, C', P)$, we show that the predicates on both sides are provably equal. The calculation is much like that in [Jac99], example 4.3.8. Since $p_{\mathbb{D}}$ is a fibred partial order, provable equality implies equality of objects. Thus the above equation strictly holds. \square

Proof of theorem 3 We use the fact that the image of the standard lambda-to-CL translation does not cover the entire set of combinatory logic terms.

First we write $[M]_w$ for the equivalence class of a combinatory logic term M by weak equivalence $=_w$ (see [Bar84], definition 7.2.1). We define the *closed term combinatory algebra* \mathcal{U}_0 by the tuple $(U_0, \bullet_w, [S]_w, [K]_w)$ where $U_0^\tau = \{[M]_w \mid M \in S_{\Sigma_{CL}}(\emptyset, \tau)\}$ and $\bullet_w^{\tau, \tau'}$ is the application operators defined by $[M]_w \bullet_w^{\tau, \tau'} [N]_w = [MN]_w$ for $[M]_w \in U_0^{\tau \Rightarrow \tau'}$ and $[N]_w \in U_0^\tau$. It is easy to see that the above choice of combinators satisfies the axioms of the combinatory algebra. As we have seen in example 2, we obtain an interpretation of combinatory logic terms in \mathcal{U}_0 , namely $\mathcal{U}_0 \llbracket - \rrbracket$. Explicitly, $\mathcal{U}_0 \llbracket M \rrbracket \{[M_1]_w/x_1, \dots, [M_n]_w/x_n\} = [M[M_1/x_1, \dots, M_n/x_n]]_w$. We interpret simply typed lambda terms by $\mathcal{U}_0 \llbracket (-)_{CL} \rrbracket$.

Now we define the definability predicate $D^\tau \subseteq U_0^\tau$ by $D^\tau = \{\mathcal{U}_0 \llbracket M_{CL} \rrbracket \in U_0^\tau \mid M \in S_{\Pi_\lambda}(\emptyset, \tau)\}$. This is a pre-logical predicate for Π_λ along $\mathcal{U}_0 \llbracket (-)_{CL} \rrbracket$. However, $D^{\tau \Rightarrow \tau' \Rightarrow \tau}$ does not include $[K^{\tau \Rightarrow \tau' \Rightarrow \tau}]_w$ for all τ and τ' , thus is not a pre-logical predicate for Σ_{CL} along $\mathcal{U}_0 \llbracket - \rrbracket$.

It is easy to see that there exists no closed term M such that $M_{CL} = K$ by induction on M . Next we prove the following lemma:

Lemma 1. *For all closed lambda term M and all combinatory term N , $M_{CL} \twoheadrightarrow_w N$ implies there exists a closed lambda term N' such that $N = N'_{CL}$.*

When M begins with a lambda abstraction, M_{CL} is always a normal form. Thus the claim clearly holds by taking $N' = M$. We do not consider the case when M is a variable, since we assume M is closed. So we think of the case when $M = M_0 M_1$ with two closed lambda terms M_0 and M_1 . There are several possible causes of $M_{CL} \twoheadrightarrow_w N$.

- $N = M_{CL}$. We just take $N' = M$.
- There exists a combinatory term L such that $(M_0)_{CL} \twoheadrightarrow_w L$ and $L \twoheadrightarrow_w N$. From IH, there exists a combinatory term L' such that $L = L'_{CL}$. Again from IH, there exists a combinatory term N' such that $N = N'_{CL}$.
- $(M_0)_{CL} \twoheadrightarrow_w N_0$ and $N = N_0(M_1)_{CL}$. From IH, there exists a closed lambda term N'_0 such that $(N'_0)_{CL} = N_0$. Thus $N = (N'_0)_{CL}(M_1)_{CL} = (N'_0 M_1)_{CL}$.
- $(M_1)_{CL} \twoheadrightarrow_w N_1$ and $N = (M_0)_{CL} N_1$. The proof is similar to the above case.
- $(M_0)_{CL} = K N_0$ with a combinatory term N_0 and $N = N_0$. From the definition of lambda-to-CL translation, M_0 should be equal to $\lambda x . N'_0$ where N'_0 is a closed lambda term. Thus $N_0 = (N'_0)_{CL}$.
- $(M_0)_{CL} = S N_0 N_1$ with some combinatory terms N_0, N_1 and $N = N_0(M_1)_{CL}(N_1(M_1)_{CL})$. From the definition of lambda-to-CL translation, M_0 should be equal to $\lambda x . (N'_0 N'_1)$. Then $\lambda x . (N'_0 N'_1) = S(N'_0)_{CL}(N'_1)_{CL}$, which implies $N_0 = (N'_0)_{CL}$ and $N_1 = (N'_1)_{CL}$. Thus we take $N' = N'_0 M_1 (N'_1 M_1)$.

Thus there exists no term reducing to K in the image of $(-)_{CL}$, otherwise K should be in the image of $(-)_{CL}$ (we assume the strong normalisation of \twoheadrightarrow_w). Thus $[K]_w \notin D^{\tau \Rightarrow \tau' \Rightarrow \tau}$. \square

Proof of theorem 4 We only show the converse. The assumption says that there exists a morphism $p : S_{\Pi_{\lambda \times}} \rightarrow H^P$ such that $H^{\pi_G} \circ p = H^{\llbracket - \rrbracket} \circ \mathbf{L} \llbracket - \rrbracket_I$. Recall that $\mathbf{L} \llbracket - \rrbracket_I$ is an epimorphism and H^{π_G} is a monomorphism. In category \mathbf{M}_T , any epimorphism is

orthogonal to monomorphism, thus there exists a unique morphism h such that $H^{\pi_G} \circ h = H^{\llbracket - \rrbracket}$ and $h \circ \mathbf{L}[\llbracket - \rrbracket]_I = p$. Now we define the functor $q : \mathbf{L} \rightarrow \mathbf{Pred}(G)$ in question by $q\tau = P\tau$ and $qf = h(\tau')(\tau)(f)$ for a morphism $f : \tau \rightarrow \tau'$ in \mathbf{L} . We see q is indeed a functor. First q preserves identity, since $\pi_G(q(id_\tau)) = \llbracket id_\tau \rrbracket = id_{\llbracket \tau \rrbracket} = \pi_G(id_{P\tau})$ and π_G is faithful, we have $q(id_\tau) = id_{P\tau}$. Next we show $q(f \circ g) = qf \circ qg$ for all $f : \tau' \rightarrow \tau''$ and $g : \tau \rightarrow \tau'$. We have $\pi_G(q(f \circ g)) = \llbracket f \circ g \rrbracket = \llbracket f \rrbracket \circ \llbracket g \rrbracket = \pi_G(qf \circ qg)$, and since π_G is faithful, we have $q(f \circ g) = qf \circ qg$. It is routine to check $H^q \circ \mathbf{L}[\llbracket M \rrbracket]_I = p(\tau)(\Gamma)(M)$.

Next we show $P(\tau \times \tau') = P\tau \times P\tau'$. We consider well-formed terms $x : \tau \times \tau' \vdash_{\Pi_{\lambda \times}} \text{fst}(x) : \tau$ and $x : \tau \times \tau' \vdash_{\Pi_{\lambda \times}} \text{snd}(x) : \tau'$. We define $j = \langle p(\tau)(x : \tau \times \tau')(\text{fst}(x)), p(\tau')(x : \tau \times \tau')(\text{snd}(x)) \rangle$. Since $H^{\pi_G} \circ p = \llbracket - \rrbracket \circ \mathbf{L}[\llbracket - \rrbracket]_I$, we can show that $\pi_G j = \text{id}_{\llbracket \tau \times \tau' \rrbracket}$, which implies $j = \text{id}_{P(\tau \times \tau')}$ since π_G is faithful. This means that the comparison map $P(\tau \times \tau') \rightarrow P\tau \times P\tau'$ is identity, thus $P(\tau \times \tau') = P\tau \times P\tau'$ holds. \square

Proof of theorem 5 Assume $\mathcal{A} \preceq \mathcal{B}$. We only have to show that for all $\Gamma \vdash_{\Pi_{\Sigma\text{-fol}}} M : \tau$ where $\tau \in T_0$ and $\rho \in A^*\Gamma$, $\mathcal{I}_{\mathcal{A}}[\llbracket M \rrbracket]\rho = \mathcal{I}_{\mathcal{B}}[\llbracket M \rrbracket]\rho$. This is clear, since M consists of operators in Σ , and \mathcal{A} is a subalgebra of \mathcal{B} . Conversely, assume that the basic lemma holds. Then for all $\Gamma \vdash_{\Pi_{\Sigma\text{-fol}}} M : \Omega$ and $\rho \in A^*\Gamma$, we have $(\mathcal{I}_{\mathcal{A}}[\llbracket M \rrbracket]\rho, \mathcal{I}_{\mathcal{B}}[\llbracket M \rrbracket]\rho) \in R\Omega = \text{id}_\Omega$. Thus $\mathcal{A} \preceq \mathcal{B}$ holds. \square