Publ. RIMS Kyoto Univ. **4x** (201x), 1–52
DOI 10.4171/PRIMS/*

Combinatorial Belyi Cuspidalization and Arithmetic Subquotients of the Grothendieck-Teichmüller Group

by

Shota Tsujimura

Abstract

In this paper, we develop a certain combinatorial version of the theory of Belyi cuspidalization developed by Mochizuki. Write $\overline{\mathbb{Q}} \subseteq \mathbb{C}$ for the subfield of algebraic numbers $\in \mathbb{C}$. We then apply this theory of combinatorial Belyi cuspidalization to certain natural closed subgroups of the Grothendieck-Teichmüller group associated to the field of p-adic numbers [where p is a prime number] and to stably $\times\mu$-indivisible subfields of $\overline{\mathbb{Q}}$, i.e., subfields for which every finite field extension satisfies the property that every nonzero divisible element in the field extension is a root of unity.

2010 Mathematics Subject Classification: Primary 14H30.
Keywords: anabelian geometry, Belyi cuspidalization, Grothendieck-Teichmüller group.

Introduction

In [AbsTopII], §3 [cf. [AbsTopII], Corollary 3.7], the theory of Belyi cuspidalization was developed and applied to reconstruct the decomposition groups of the closed points of a hyperbolic orbicurve of strictly Belyi type over a mixed characteristic local field [cf. [AbsTopII], Definition 3.5; [AbsTopII], Remark 3.7.2].

In the present paper, we develop a certain combinatorial version of the theory of Belyi cuspidalization developed in [AbsTopII], §3. To begin, let us recall the Grothendieck-Teichmüller group GT, which may be regarded as a closed subgroup of the outer automorphism group of the étale fundamental group Π_X [cf. Notations and Conventions] of $X \stackrel{\text{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$ [cf. [CmbCsp], Definition 1.11, (i); [CmbCsp], Remark 1.11.1], where $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$ denotes the projective line

Communicated by S. Mochizuki. Received March 26, 2019. Revised August 20, 2019; November 23, 2019; February 16, 2020.

S. Tsujimura: RIMS, Kyoto University, Kyoto 606-8502, Japan;
e-mail: stsuji@kurims.kyoto-u.ac.jp

over the field of algebraic numbers $\overline{\mathbb{Q}}$ [cf. Notations and Conventions], minus the three points "0", "1", "∞". Recall, further, that the natural outer action of $G_{\mathbb{Q}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on Π_X determines natural inclusions

$$G_{\mathbb{Q}} \subseteq \mathrm{GT} \subseteq \mathrm{Out}(\Pi_X),$$

and that Π_X is topologically finitely generated and slim [cf., e.g., [MT], Remark 1.2.2; [MT], Proposition 1.4]. By pulling-back the exact sequence of profinite groups

$$1 \longrightarrow \Pi_X \quad (\stackrel{\sim}{\to} \mathrm{Inn}(\Pi_X)) \longrightarrow \mathrm{Aut}(\Pi_X) \longrightarrow \mathrm{Out}(\Pi_X) \longrightarrow 1$$

via the natural inclusion $\mathrm{GT} \subseteq \mathrm{Out}(\Pi_X)$, we obtain an exact sequence of profinite groups

$$1 \longrightarrow \Pi_X \longrightarrow \Pi_X \stackrel{\mathrm{out}}{\rtimes} \mathrm{GT} \longrightarrow \mathrm{GT} \longrightarrow 1$$

[cf. Notations and Conventions].

We shall develop a combinatorial version for $\Pi_X \stackrel{\mathrm{out}}{\rtimes} \mathrm{GT}$ — i.e., which we regard as a sort of group-theoretic version of $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$, where "$\mathbb{Q}$" is replaced by "GT"— of the theory of Belyi cuspidalization. We shall refer to this combinatorial version of the theory of Belyi cuspidalization as the theory of *combinatorial Belyi cuspidalization*. We construct combinatorial Belyi cuspidalizations and, in particular, the "GT analogue" of the set (equipped with a natural action of GT) of decomposition groups of $\Pi_X \stackrel{\mathrm{out}}{\rtimes} \mathrm{GT}$, by applying the technique of *tripod synchronization* developed in [CbTpII], together with the Grothendieck Conjecture for hyperbolic curves over number fields [cf. [Tama1], Theorem 0.4; [LocAn], Theorem A].

Let $U \to X$ be a connected finite étale covering of X, $U \hookrightarrow X$ an open immersion. Then the morphisms $U \to X$, $U \hookrightarrow X$ determine, respectively, the vertical and horizontal arrows in a diagram of outer homomorphisms of profinite groups as follows:

$$\begin{array}{ccc} \Pi_U & \longrightarrow & \Pi_X \\ \downarrow & & \\ \Pi_X. & & \end{array}$$

We shall refer to any pair consisting of

- a diagram obtained in this way;
- an open subgroup of Π_X, which, by a slight of abuse of notation, we denote by $\Pi_U \subseteq \Pi_X$, that belongs to the Π_X-conjugacy class of open subgroups that arises as the image of the vertical arrow of the diagram

as a *Belyi diagram*.

Let $(\Pi, G \subseteq \mathrm{Out}(\Pi))$ be a pair consisting of

- an abstract topological group Π;
- a closed subgroup G of $\mathrm{Out}(\Pi)$.

If there exists an isomorphism of such pairs

$$(\Pi, G \subseteq \mathrm{Out}(\Pi)) \xrightarrow{\sim} (\Pi_X, \mathrm{GT} \subseteq \mathrm{Out}(\Pi_X))$$

[i.e., if there exist isomorphisms $\Pi \xrightarrow{\sim} \Pi_X$ and $G \xrightarrow{\sim} \mathrm{GT}$ of topological groups compatible with the inclusions $G \subseteq \mathrm{Out}(\Pi)$ and $\mathrm{GT} \subseteq \mathrm{Out}(\Pi_X)$], then we shall refer to the pair $(\Pi, G \subseteq \mathrm{Out}(\Pi))$ as a *tripodal pair*.

Let $(\Pi, G \subseteq \mathrm{Out}(\Pi))$ be a tripodal pair; $J \subseteq G$ a closed subgroup of G; Π^* an open subgroup of Π. Then one verifies easily [cf. Lemma 1.2] that, for any sufficiently small normal open subgroup $M \subseteq J$, there exist an outer action of M on Π^* and an open injection $\Pi^* \overset{\mathrm{out}}{\rtimes} M \hookrightarrow \Pi \overset{\mathrm{out}}{\rtimes} J$ such that

(a) the outer action of M preserves and induces the identity automorphism on the set of the conjugacy classes of cuspidal inertia subgroups of Π^* [cf. Theorem A, (i)];

(b) the outer action of M on Π^* extends uniquely [cf. the slimness of Π] to a Π^*-outer action on Π that is compatible with the outer action of J ($\supseteq M$) on Π; the injection $\Pi^* \overset{\mathrm{out}}{\rtimes} M \hookrightarrow \Pi \overset{\mathrm{out}}{\rtimes} J$ is the injection determined by the inclusions $\Pi^* \subseteq \Pi$ and $M \subseteq J$ and the Π^*-outer actions on Π^* and Π.

Then our first main result is the following [cf. Theorem 1.3]:

Theorem A (**Combinatorial Belyi cuspidalization for a tripod**). *Fix a Belyi diagram*

$$\begin{array}{ccc} \Pi_U & \longrightarrow & \Pi_X \\ \big\downarrow & & \\ \Pi_X & & \end{array}$$

that arises from a connected finite étale covering $U \to X$ and an open immersion $U \hookrightarrow X$ [as in the above discussion]. Then:

(i) Let $(\Pi, G \subseteq \mathrm{Out}(\Pi))$ be a **tripodal pair***. Fix an isomorphism of pairs α : $(\Pi, G \subseteq \mathrm{Out}(\Pi)) \xrightarrow{\sim} (\Pi_X, \mathrm{GT} \subseteq \mathrm{Out}(\Pi_X))$. Then the set of subgroups of Π determined, via α, by the cuspidal inertia subgroups of Π_X, may be* **reconstructed***, in a* **purely group-theoretic** *way, from the pair $(\Pi, G \subseteq \mathrm{Out}(\Pi))$. We shall refer to the subgroups of Π constructed in this way as the* **cuspidal inertia subgroups** *of Π. In particular, for each open subgroup $\Pi^* \subseteq \Pi$ of Π,*

the pair $(\Pi, G \subseteq \mathrm{Out}(\Pi))$ *determines a set* $I(\Pi^*)$ *(respectively,* $\mathrm{Cusp}(\Pi^*)$*) of cuspidal inertia subgroups of* Π^* *(respectively, cusps of* Π^**), namely, the set of intersections of* Π^* *with cuspidal inertia subgroups of* Π *(respectively, the conjugacy classes of cuspidal inertia subgroups of* Π^**).*

(ii) Let $N \subseteq \mathrm{GT}$ *be a normal open subgroup. Suppose that we are given an outer action of* N *on* Π_U *and an open injection* $\Pi_U \overset{\text{out}}{\rtimes} N \hookrightarrow \Pi_X \overset{\text{out}}{\rtimes} \mathrm{GT}$ *such that the above conditions (a), (b) in the case of "*$\Pi^* \subseteq \Pi$*", "*$M \subseteq J$*" hold for* $\Pi_U \subseteq \Pi_X$*,* $N \subseteq \mathrm{GT}$*. Then the original* **outer action** *of* $N \subseteq \mathrm{GT}$ *on* Π_X **coincides** *with the outer action of* N *on* Π_X *induced [cf. condition (a)] by the outer action of* N *on* Π_U *and the outer surjection* $\Pi_U \twoheadrightarrow \Pi_X$ *[i.e., the horizontal arrow in the above Belyi diagram].*

(iii) Let

$$C(\Pi) = (\Pi, G \subseteq \mathrm{Out}(\Pi), \Pi^*, \{0, 1, \infty\} \subseteq \mathrm{Cusp}(\Pi), \{0, 1, \infty\} \subseteq \mathrm{Cusp}(\Pi^*))$$

be a 5*-tuple consisting of the following data:*

- *a topological group* Π*;*
- *a closed subgroup* $G \subseteq \mathrm{Out}(\Pi)$ *such that the pair* $(\Pi, G \subseteq \mathrm{Out}(\Pi))$ *is a tripodal pair;*
- *an open subgroup* $\Pi^* \subseteq \Pi$ *of* Π *of genus* 0*, where we observe that the genus of an open subgroup of* Π *may be defined by using the cuspidal inertia subgroups of the open subgroup [cf. (i)];*
- *a subset* $\{0, 1, \infty\} \subseteq \mathrm{Cusp}(\Pi)$ *[cf. (i)] of cardinality* 3 *[equipped with labels "*0*", "*1*", "*∞*"] of the set* $\mathrm{Cusp}(\Pi)$*;*
- *a subset* $\{0, 1, \infty\} \subseteq \mathrm{Cusp}(\Pi^*)$ *[cf. (i)] of cardinality* 3 *[equipped with labels "*0*", "*1*", "*∞*"] of the set* $\mathrm{Cusp}(\Pi^*)$*.*

Suppose that the collection of data $C(\Pi)$ *is isomorphic to the collection of data*

$$\begin{aligned} C(\Pi_X) &= (\Pi_X, \mathrm{GT} \subseteq \mathrm{Out}(\Pi_X), \Pi_U, \\ &\qquad \{0, 1, \infty\} \subseteq \mathrm{Cusp}(\Pi_X), \{0, 1, \infty\} \subseteq \mathrm{Cusp}(\Pi_U)) \end{aligned}$$

determined, in a natural way, by the given Belyi diagram. [Here, we observe that the horizontal arrow in the given Belyi diagram determines, in a natural way, data $\{0, 1, \infty\} \subseteq \mathrm{Cusp}(\Pi_U)$*.] Fix an isomorphism of collections of data* $C(\Pi) \xrightarrow{\sim} C(\Pi_X)$*. Thus, the outer surjection* $\Pi_U \twoheadrightarrow \Pi_X$ *[i.e., the horizontal arrow in the given Belyi diagram], together with the isomorphism* $C(\Pi) \xrightarrow{\sim} C(\Pi_X)$*, determine an outer surjection* $\Pi^* \twoheadrightarrow \Pi$*. Let* $N \subseteq G$ *be a* *normal open subgroup such that the conditions (a), (b) considered above in the case of "$M \subseteq J$" hold for $N \subseteq G$. Then the* **outer surjection** $\Pi^* \twoheadrightarrow \Pi$ *may be* **reconstructed**, *in a* **purely group-theoretic** *way, from the collection of data $C(\Pi)$ as the outer surjection induced by the unique Π-outer surjection $\Pi^* \overset{\text{out}}{\rtimes} N \twoheadrightarrow \Pi \overset{\text{out}}{\rtimes} N$ [i.e., surjection well-defined up to composition with inner automorphisms arising from elements of Π] that lies over the identity morphism of N such that*

- *the kernel of this Π-outer surjection $\Pi^* \overset{\text{out}}{\rtimes} N \twoheadrightarrow \Pi \overset{\text{out}}{\rtimes} N$ is topologically generated by the cuspidal inertia subgroups of Π^* which are not associated to $0, 1, \infty \in \mathrm{Cusp}(\Pi^*)$;*
- *the conjugacy class of cuspidal inertia subgroups of Π^* associated to 0 (respectively, 1, ∞) $\in \mathrm{Cusp}(\Pi^*)$ maps to the conjugacy class of cuspidal inertia subgroups of Π associated to 0 (respectively, 1, ∞) $\in \mathrm{Cusp}(\Pi)$.*

Next, let us consider the situation discussed in Theorem A, (ii). Let J be a closed subgroup of GT. Thus, for each normal open subgroup M of J such that $M \subseteq N \cap J$, we have a diagram

$$\begin{array}{ccc} \Pi_U \overset{\text{out}}{\rtimes} M & \longrightarrow & \Pi_X \overset{\text{out}}{\rtimes} M \\ \downarrow & & \\ \Pi_X \overset{\text{out}}{\rtimes} M & & \end{array}$$

of Π_X*-outer homomorphisms* [i.e., homomorphisms well-defined up to composition with inner automorphisms arising from elements of Π_X] of profinite groups. We shall refer to a diagram obtained in this way as an *arithmetic Belyi diagram.*

Fix an arithmetic Belyi diagram $\mathbb{B}^{\rtimes}$ as above. Write

$$\mathbb{D}(\mathbb{B}^{\rtimes}, M, J)$$

for the set of the images via the natural composite Π_X-outer homomorphism $\Pi_U \overset{\text{out}}{\rtimes} M \twoheadrightarrow \Pi_X \overset{\text{out}}{\rtimes} M \hookrightarrow \Pi_X \overset{\text{out}}{\rtimes} J$ of the normalizers in $\Pi_U \overset{\text{out}}{\rtimes} M$ of cuspidal inertia subgroups of Π_U;

$$\mathbb{D}(\mathbb{B}^{\rtimes}, J)$$

for the quotient set $\left(\sqcup_{M \subseteq J} \mathbb{D}(\mathbb{B}^{\rtimes}, M, J)\right)/\sim$, where M ranges over all sufficiently small normal open subgroups of J, and we write $\mathbb{D}(\mathbb{B}^{\rtimes}, M, J) \ni G_M \sim G_{M^\dagger} \in \mathbb{D}(\mathbb{B}^{\rtimes}, M^\dagger, J)$ if $G_M \cap G_{M^\dagger}$ is open in both G_M and $G_{M^\dagger}$.

Write

$$\mathbb{D}(J)$$

for the quotient set $\left(\sqcup_{\mathbb{B}^{\rtimes}} \mathbb{D}(\mathbb{B}^{\rtimes}, J)\right)/\sim$, where $\mathbb{B}^{\rtimes}$ ranges over all arithmetic Belyi diagrams, and we write $\mathbb{D}({}^{\dagger}\mathbb{B}^{\rtimes}, J) \ni G_{{}^{\dagger}\mathbb{B}^{\rtimes}} \sim G_{{}^{\ddagger}\mathbb{B}^{\rtimes}} \in \mathbb{D}({}^{\ddagger}\mathbb{B}^{\rtimes}, J)$ if $G_{M^{\dagger}} \cap G_{M^{\ddagger}}$ is open in both $G_{M^{\dagger}}$ and $G_{M^{\ddagger}}$ for some representative $G_{M^{\dagger}}$ (respectively, $G_{M^{\ddagger}}$) of $G_{{}^{\dagger}\mathbb{B}^{\rtimes}}$ (respectively, $G_{{}^{\ddagger}\mathbb{B}^{\rtimes}}$). We shall refer to $\mathbb{D}(J)$ as the set of *decomposition subgroup-germs of* $\Pi_X \overset{\text{out}}{\rtimes} J$. One verifies immediately that the natural conjugation action of $\Pi_X \overset{\text{out}}{\rtimes} J$ on itself induces a natural action of $\Pi_X \overset{\text{out}}{\rtimes} J$ on $\mathbb{D}(J)$ [cf. Corollary 1.6].

Write

$$D(J)$$

for the quotient set $\mathbb{D}(J)/\Pi_X$. Thus, $D(J)$ admits a natural action by J. Here, we recall that, by the ["usual"] theory of Belyi cuspidalization developed in [AbsTopII], §3, we have a *natural bijection*

$$D(G_{\mathbb{Q}}) \overset{\sim}{\leftarrow} \overline{\mathbb{Q}} \cup \{\infty\}$$

[cf. Corollary 1.7].

Next, let J_1 and J_2 be closed subgroups of GT. If $J_1 \subseteq J_2 \subseteq \mathrm{GT}$, then one verifies immediately from the definition of $D(J)$ that the inclusion $J_1 \subseteq J_2$ induces, by considering the intersection of subgroups of $\Pi_X \overset{\text{out}}{\rtimes} J_2$ with $\Pi_X \overset{\text{out}}{\rtimes} J_1$, a natural surjection $D(J_2) \twoheadrightarrow D(J_1)$ that is equivariant with respect to the natural actions of J_1 ($\subseteq J_2$) on the domain and codomain [cf. Corollary 1.6]. Thus, we obtain the following commutative diagram

$$\begin{array}{ccc} \mathrm{GT} & \supseteq & G_{\mathbb{Q}} \\ \curvearrowright & & \curvearrowright \\ D(\mathrm{GT}) & \twoheadrightarrow & D(G_{\mathbb{Q}}) \overset{\sim}{\leftarrow} \overline{\mathbb{Q}} \cup \{\infty\} \end{array}$$

[cf. Corollary 1.7]. In particular, since the outer action of GT on Π_X preserves the cuspidal inertia subgroups of Π_X associated to ∞,

if one could prove that the surjection $D(\mathrm{GT}) \twoheadrightarrow D(G_{\mathbb{Q}})$ is a *bijection*, then it would follow that GT *naturally acts on the set* $\overline{\mathbb{Q}}$.

In fact, at the time of writing of the present paper, the author does not know

whether or not the surjection $D(\mathrm{GT}) \twoheadrightarrow D(G_{\mathbb{Q}})$ *is a bijection*,

or indeed, more generally,

whether or not GT *admits a natural action on the set* $\overline{\mathbb{Q}}$.

On the other hand, we obtain the following result concerning the p-adic analogue of this sort of issue [cf. Corollary 2.4]:

Corollary B (**Natural surjection from $\mathrm{GT}_p^{\mathrm{tp}}$ to $G_{\mathbb{Q}_p}$**). *Let p be a prime number; $\overline{\mathbb{Q}}_p$ an algebraic closure of $\mathbb{Q}_p$ [cf. Notations and Conventions]. Write $\mathrm{GT}_p^{\mathrm{tp}}$ for the p-adic version of the Grothendieck-Teichmüller group defined in Definition 2.1 [cf. also Remark 2.1.2]. Then one may construct a surjection $\mathrm{GT}_p^{\mathrm{tp}} \twoheadrightarrow G_{\mathbb{Q}_p} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ whose restriction to $G_{\mathbb{Q}_p}$ is the identity automorphism.*

The key point of the proof of the above corollary is the following theorem [cf. Theorem 2.2]:

Theorem C. (**Determination of moduli of certain types of *p*-adic hyperbolic curves by data arising from geometric tempered fundamental groups**). *We maintain the notation of Corollary B. Write $X \stackrel{\mathrm{def}}{=} \mathbb{P}^1_{\mathbb{C}_p} \setminus \{0,1,\infty\}$, where $\mathbb{C}_p$ denotes the p-adic completion of $\overline{\mathbb{Q}}_p$. Let $Y \to X$ be a connected finite étale covering of X; y, y' elements of $Y(\mathbb{C}_p)$. Write Y_y (respectively, $Y_{y'}$) for $Y \setminus \{y\}$ (respectively, $Y \setminus \{y'\}$); Π_Y^{tp} (respectively, $\Pi_{Y_y}^{\mathrm{tp}}$, $\Pi_{Y_{y'}}^{\mathrm{tp}}$) for the tempered fundamental group of Y (respectively, Y_y, $Y_{y'}$). Suppose that there exists an isomorphism $\Pi_{Y_y}^{\mathrm{tp}} \xrightarrow{\sim} \Pi_{Y_{y'}}^{\mathrm{tp}}$ that fits into a commutative diagram*

$$\begin{array}{ccc} \Pi_{Y_y}^{\mathrm{tp}} & \xrightarrow{\sim} & \Pi_{Y_{y'}}^{\mathrm{tp}} \\ \downarrow & & \downarrow \\ \Pi_Y^{\mathrm{tp}} & = & \Pi_Y^{\mathrm{tp}}, \end{array}$$

where the vertical arrows are the surjections [determined up to composition with an inner automorphism] induced by the natural open immersions of hyperbolic curves. Then $y = y'$.

Finally, we consider yet another interesting class of closed subgroups of GT which act naturally on the set of algebraic numbers $\overline{\mathbb{Q}}$. Let p be a prime number. For any field F and positive integer n, we shall write

$$F^\times \stackrel{\mathrm{def}}{=} F \setminus \{0\}, \quad \mu_n(F) \stackrel{\mathrm{def}}{=} \{x \in F^\times \mid x^n = 1\}, \quad \mu(F) \stackrel{\mathrm{def}}{=} \bigcup_{m \geq 1} \mu_m(F)$$

$$\mu_{p^\infty}(F) \stackrel{\mathrm{def}}{=} \bigcup_{m \geq 1} \mu_{p^m}(F), \quad F^{\times p^\infty} \stackrel{\mathrm{def}}{=} \bigcap_{m \geq 1} (F^\times)^{p^m}, \quad F^{\times \infty} \stackrel{\mathrm{def}}{=} \bigcap_{m \geq 1} (F^\times)^m$$

[cf. Notations and Conventions]. We shall say that the field K is *stably* p-$\times$ (respectively, p-$\times\mu$, $\times$, $\times\mu$)-*indivisible* if, for every finite extension L of K, $L^{\times p^\infty} = \{1\}$ (respectively, $L^{\times p^\infty} \subseteq \mu(L)$, $L^{\times\infty} = \{1\}$, $L^{\times\infty} \subseteq \mu(L)$) [cf. Definition 3.3, (v)]. We shall say that K is *stably* μ_{p^∞} (respectively, *stably* μ)-*finite* if, for every finite extension $K^\dagger$ of K, $\mu_{p^\infty}(K^\dagger)$ (respectively, $\mu(K^\dagger)$) is a finite group [cf. Definition

3.3, (vii)]. First, we observe that such fields exist in great abundance [cf. Lemma 3.4]:

Lemma D (**Basic properties of stably p-$\times$/p-$\times\mu$/$\times$/$\times\mu$-indivisible fields**). *Let p be a prime number, K a field of characteristic $\neq p$.*

(i) If K is p-$\times$ (respectively, $\times$)-indivisible, then K is p-$\times\mu$ (respectively, $\times\mu$)-indivisible. Let $\Box \in \{\times\mu, \times\}$. If K is p-$\Box$-indivisible, then K is $\Box$-indivisible.

(ii) Let $\Box \in \{p\text{-}\times, p\text{-}\times\mu, \times, \times\mu\}$, L an extension field of K. Then if L is $\Box$-indivisible, then K is $\Box$-indivisible.

(iii) Suppose that K is a generalized sub-p-adic field (respectively, sub-p-adic field) [for example, a finite extension of $\mathbb{Q}$ or $\mathbb{Q}_p$ — cf. [AnabTop], Definition 4.11 (respectively, [LocAn], Definition 15.4, (i))]. Then K is stably p-$\times\mu$-indivisible (respectively, stably p-$\times\mu$-indivisible and stably $\times$-indivisible) and stably μ_{p^∞} (respectively, stably μ)-finite.

(iv) Suppose that K is stably μ_{p^∞} (respectively, stably μ)-finite. Let L be an (algebraic) abelian extension of K. Then if K is stably p-$\times\mu$ (respectively, stably $\times\mu$)-indivisible, then L is stably p-$\times\mu$ (respectively, stably $\times\mu$)-indivisible.

(v) Let L be a(n) (algebraic) Galois extension of K. Suppose that L is stably μ_{p^∞} (respectively, stably μ)-finite. Then if K is stably p-$\times\mu$ (respectively, stably $\times\mu$)-indivisible, then L is stably p-$\times\mu$ (respectively, stably $\times\mu$)-indivisible.

(vi) Let L be a(n) (algebraic) pro-prime-to-p Galois extension of K. Then if K is stably p-$\times\mu$-indivisible, then L is stably p-$\times\mu$-indivisible.

Thus, in particular, it follows from Lemma D, (i), (ii), (iii), (iv), (vi), that, if p is a prime number, then any *subfield* of an *abelian* or *pro-prime-to-p Galois* extension of a *finite extension* of $\mathbb{Q}$ or $\mathbb{Q}_p$ is *stably p-$\times\mu$-indivisible*, hence *stably $\times\mu$-indivisible* [cf. Remark 3.4.1].

Let K be a *stably $\times\mu$-indivisible* field of characteristic 0; $\overline{K}$ an algebraic closure of K. Write $G_K \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{K}/K)$. Then we apply the theory of combinatorial Belyi cuspidalization developed in §1 to obtain the following [cf. Corollary 3.9]:

Corollary E. (**Natural homomorphism from the commensurator in** GT **of the absolute Galois group of a stably $\times\mu$-indivisible field to** $G_{\mathbb{Q}}$). *Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{K}$. In the following, we shall use this embedding to regard $\overline{\mathbb{Q}}$ as a subfield of $\overline{K}$. Thus, we obtain a homomorphism $G_K \to G_{\mathbb{Q}}$ ($\subseteq$ GT) [cf. the discussion at the beginning of the Introduction]. Suppose that this homomorphism $G_K \to G_{\mathbb{Q}}$ is injective. In the following, we shall use this injection $G_K \hookrightarrow G_{\mathbb{Q}}$ to regard G_K as a subgroup of $G_{\mathbb{Q}}$, hence also as a subgroup of* GT. *Then one may* *construct a natural surjection*

$$C_{\mathrm{GT}}(G_K) \twoheadrightarrow C_{G_{\mathbb{Q}}}(G_K)\ (\subseteq G_{\mathbb{Q}}).$$

[cf. Notations and Conventions] whose restriction to $C_{G_{\mathbb{Q}}}(G_K)$ *is the identity automorphism.*

The key point of the proof of the above corollary is the injectivity portion of the section conjecture for hyperbolic curves of genus 0 over a *stably* $\times\mu$*-indivisible* field of characteristic 0 [cf. Corollary 3.7]. This injectivity is a consequence of the following [cf. Theorem 3.5]:

Theorem F. (**Weak version of the Grothendieck Conjecture for hyperbolic curves of genus 0 over a stably p-$\times\mu/\times\mu$-indivisible field of characteristic 0**). *Let K be a stably p-$\times\mu$ (respectively, $\times\mu$)-indivisible field of characteristic 0; $\overline{K}$ an algebraic closure of K. Write $G_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{K}/K)$. Let U and V be hyperbolic curves of genus 0 over K;*

$$\phi : \Pi_U \xrightarrow{\sim} \Pi_V$$

an isomorphism of profinite groups such that ϕ lies over the identity automorphism on G_K. We consider the following conditions:

(a) ϕ induces a bijection between the cuspidal inertia subgroups of Π_U and the cuspidal inertia subgroups of Π_V.

(b) Let $I \subseteq \Pi_U$ be a cuspidal inertia subgroup of Π_U. Consider the natural composite

$$\widehat{\mathbb{Z}}(1) \xrightarrow{\sim} I \xrightarrow{\sim} \phi(I) \xleftarrow{\sim} \widehat{\mathbb{Z}}(1)$$

— where "(1)" denotes the Tate twist; the first and final isomorphisms are the natural isomorphisms [obtained by considering the action of each cuspidal inertia subgroup on the roots of a uniformizer of the local ring of the compactified curve at the cusp under consideration]; the middle isomorphism is the isomorphism induced by ϕ. Then this natural composite is the identity automorphism.

Suppose that condition (a) holds (respectively, conditions (a), (b) hold). Then there exists an isomorphism of K-schemes

$$U \xrightarrow{\sim} V$$

that induces a bijection between the cusps of U and V which is compatible with the bijection between cuspidal inertia groups of Π_U and Π_V induced by ϕ.

On the other hand, if one restricts to the case of a finite extension of the maximal abelian extension $\mathbb{Q}^{\mathrm{ab}} \subseteq \overline{\mathbb{Q}}$ of $\mathbb{Q}$, then one may prove the injectivity portion of the section conjecture for arbitrary hyperbolic curves [cf. Corollary 3.2]:

Corollary G. (**The injectivity portion of the Section Conjecture for arbitrary hyperbolic curves over a finite extension of** $\mathbb{Q}^{\mathrm{ab}}$). *Let* $K \subseteq \overline{\mathbb{Q}}$ *be a number field, i.e., a finite extension of* $\mathbb{Q}$*;* Y *a hyperbolic curve over* K*. Write* $K^{\mathrm{cycl}} = K \cdot \mathbb{Q}^{\mathrm{ab}}$*;* $Y_{K^{\mathrm{cycl}}} \stackrel{\mathrm{def}}{=} Y \times_K K^{\mathrm{cycl}}$*;* $G_{K^{\mathrm{cycl}}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/K^{\mathrm{cycl}})$*;* $Y(K^{\mathrm{cycl}})$ *for the set of* K^{cycl}*-valued points of* Y*;* $Y_{\overline{\mathbb{Q}}} \stackrel{\mathrm{def}}{=} Y \times_K \overline{\mathbb{Q}}$*;* $\mathrm{Sect}(\Pi_{Y_{K^{\mathrm{cycl}}}} \twoheadrightarrow G_{K^{\mathrm{cycl}}})$ *for the set of equivalence classes of sections of the natural surjection* $\Pi_{Y_{K^{\mathrm{cycl}}}} \twoheadrightarrow G_{K^{\mathrm{cycl}}}$*, where we consider two such sections to be equivalent if they differ by composition with an inner automorphism induced by an element of* $\Pi_{Y_{\overline{\mathbb{Q}}}}$*. Then the natural map*

$$Y(K^{\mathrm{cycl}}) \to \mathrm{Sect}(\Pi_{Y_{K^{\mathrm{cycl}}}} \twoheadrightarrow G_{K^{\mathrm{cycl}}})$$

is injective.

This paper is organized as follows. In §1, we develop the theory of combinatorial Belyi cuspidalization. In §2, we first show that the moduli of a hyperbolic curve over $\overline{\mathbb{Q}}_p$ of genus 0 with 4 points removed are completely determined by the geometric tempered fundamental group of the curve, regarded as an extension of the geometric tempered fundamental group of the tripod [cf. Notations and Conventions] over $\overline{\mathbb{Q}}_p$ [cf. Theorem C]. This result, together with the theory of combinatorial Belyi cuspidalization developed in §1, implies that there exists a surjection $\mathrm{GT}_p^{\mathrm{tp}} \twoheadrightarrow G_{\mathbb{Q}_p}$ whose restriction to $G_{\mathbb{Q}_p}$ is the identity automorphism [cf. Corollary B]. In §3, we observe that the injectivity portion of the section conjecture for hyperbolic curves [cf. Corollary G] (respectively, hyperbolic curves of genus 0 [cf. Theorem F]) over maximal cyclotomic extensions of number fields (respectively, over stably $\times\mu$-indivisible fields of characteristic 0 [cf. Lemma D]) holds [by a well-known argument!] and prove that, if the natural outer surjection $G_K \to G_{\mathbb{Q}}$ is injective, then there exists a surjection $C_{\mathrm{GT}}(G_K) \twoheadrightarrow C_{G_{\mathbb{Q}}}(G_K)$ whose restriction to $C_{G_{\mathbb{Q}}}(G_K)$ is the identity automorphism [cf. Corollary E].

Notations and Conventions

In this paper, we follow the notations and conventions of [CbTpI].

Fields: The notation $\mathbb{Q}$ will be used to denote the field of rational numbers. The notation $\mathbb{Z}$ will be used to denote the ring of integers of $\mathbb{Q}$. The notation $\mathbb{C}$ will be used to denote the field of complex numbers. The notation $\overline{\mathbb{Q}} \subseteq \mathbb{C}$ will be used to denote the set or field of algebraic numbers $\in \mathbb{C}$. We shall refer to a finite extension field of $\mathbb{Q}$ as a *number field*. If p is a prime number, then the notation $\mathbb{Q}_p$ will be used to denote the p-adic completion of $\mathbb{Q}$; the notation $\mathbb{Z}_p$ will be used to denote the ring of integers of $\mathbb{Q}_p$. We shall refer to a finite extension field of $\mathbb{Q}_p$ as a *p-adic local field*. For any field F, prime number p, and positive integer n, we shall write

$$F^\times \stackrel{\text{def}}{=} F \setminus \{0\}, \quad \mu_n(F) \stackrel{\text{def}}{=} \{x \in F^\times \mid x^n = 1\},$$

$$\mu_{p^\infty}(F) \stackrel{\text{def}}{=} \bigcup_{m \geq 1} \mu_{p^m}(F), \quad \mu(F) \stackrel{\text{def}}{=} \bigcup_{m \geq 1} \mu_m(F),$$

$$F^{\times p^\infty} \stackrel{\text{def}}{=} \bigcap_{m \geq 1} (F^\times)^{p^m}, \quad F^{\times \infty} \stackrel{\text{def}}{=} \bigcap_{m \geq 1} (F^\times)^m.$$

Topological groups: Let G be a topological group and $H \subseteq G$ a closed subgroup of G. Then we shall denote by $Z_G(H)$ (respectively, $N_G(H)$, $C_G(H)$) the *centralizer* (respectively, *normalizer*, *commensurator*) of $H \subseteq G$, i.e.,

$$Z_G(H) \stackrel{\text{def}}{=} \{g \in G \mid ghg^{-1} = h \text{ for any } h \in H\}$$
$$(\text{respectively, } N_G(H) \stackrel{\text{def}}{=} \{g \in G \mid g \cdot H \cdot g^{-1} = H\}$$
$$C_G(H) \stackrel{\text{def}}{=} \{g \in G \mid H \cap g \cdot H \cdot g^{-1} \text{ is of finite index in } H \text{ and } g \cdot H \cdot g^{-1}\}).$$

We shall say that G is *slim* if $Z_G(U) = \{1\}$ for any open subgroup U of G.

Let G be a topological group. Then we shall write $\mathrm{Aut}(G)$ for the group of automorphisms of the topological group G, $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$ for the group of inner automorphisms of G, and $\mathrm{Out}(G) \stackrel{\text{def}}{=} \mathrm{Aut}(G)/\mathrm{Inn}(G)$. We shall refer to an element of $\mathrm{Out}(G)$ as an *outomorphism* of G. Now suppose that G is *center-free* [i.e., $Z_G(G) = \{1\}$]. Then we have a natural exact sequence of groups

$$1 \longrightarrow G \ \ (\stackrel{\sim}{\rightarrow} \mathrm{Inn}(G)) \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \longrightarrow 1.$$

If J is a group, and $\rho : J \to \mathrm{Out}(G)$ is a homomorphism, then we shall denote by

$$G \stackrel{\text{out}}{\rtimes} J$$

the group obtained by pulling back the above exact sequence of groups via ρ. Thus, we have a *natural exact sequence* of groups

$$1 \longrightarrow G \longrightarrow G \stackrel{\text{out}}{\rtimes} J \longrightarrow J \longrightarrow 1.$$

Suppose further that G is *profinite* and *topologically finitely generated*. Then one verifies immediately that the topology of G admits a basis of *characteristic open subgroups*, which thus induces a *profinite topology* on the groups $\mathrm{Aut}(G)$ and

Out(G) with respect to which the above exact sequence relating Aut(G) and Out(G) determines an exact sequence of *profinite groups.* In particular, one verifies easily that if, moreover, J is *profinite*, and $\rho : J \to \mathrm{Out}(G)$ is *continuous*, then the above exact sequence involving $G \overset{\mathrm{out}}{\rtimes} J$ determines an exact sequence of *profinite groups.*

Curves: A smooth hyperbolic curve of genus 0 over a field k with precisely 3 cusps [i.e., points at infinity], all of which are defined over k, will be referred to as a "*tripod*".

Fundamental groups: For a connected Noetherian scheme S, we shall write Π_S for the étale fundamental group of S, relative to a suitable choice of basepoint.

§1. Combinatorial Belyi cuspidalization

In this section, we develop the theory of combinatorial Belyi cuspidalization. First, we introduce the notion of a Belyi diagram as follows.

Definition 1.1.

(i) Write X for $\mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, 1, \infty\}$, where $\mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, 1, \infty\}$ denotes the projective line over the field of algebraic numbers $\overline{\mathbb{Q}}$ [cf. Notations and Conventions], minus the three points "0", "1", "∞". Let $U \to X$ be a connected finite étale covering of X, $U \hookrightarrow X$ an open immersion. Then the morphisms $U \to X$, $U \hookrightarrow X$ determine, respectively, the vertical and horizontal arrows in a diagram of outer homomorphisms of profinite groups as follows:

$$\begin{array}{ccc} \Pi_U & \longrightarrow & \Pi_X \\ \downarrow & & \\ \Pi_X. & & \end{array}$$

We shall refer to any pair consisting of

- a diagram obtained in this way;
- an open subgroup of Π_X, which, by a slight abuse of notation, we denote by $\Pi_U \subseteq \Pi_X$, that belongs to the Π_X-conjugacy class of open subgroups that arises as the image of the vertical arrow of the diagram

as a *Belyi diagram.*

(ii) Recall the Grothendieck-Teichmüller group GT, which may be regarded as a closed subgroup of the outer automorphism group of the étale fundamental group Π_X [cf. Notations and Conventions] of $X = \mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$ [cf. [CmbCsp], Definition 1.11, (i); [CmbCsp], Remark 1.11.1]. Let $(\Pi, G \subseteq \mathrm{Out}(\Pi))$ be a pair consisting of

- an abstract topological group Π;
- a closed subgroup G of $\mathrm{Out}(\Pi)$.

If there exists an isomorphism of such pairs

$$(\Pi, G \subseteq \mathrm{Out}(\Pi)) \xrightarrow{\sim} (\Pi_X, \mathrm{GT} \subseteq \mathrm{Out}(\Pi_X))$$

[i.e., if there exist isomorphisms $\Pi \xrightarrow{\sim} \Pi_X$ and $G \xrightarrow{\sim} \mathrm{GT}$ of topological groups compatible with the inclusions $G \subseteq \mathrm{Out}(\Pi)$ and $\mathrm{GT} \subseteq \mathrm{Out}(\Pi_X)$], then we shall refer to the pair $(\Pi, G \subseteq \mathrm{Out}(\Pi))$ as a *tripodal pair*.

Lemma 1.2. *Let $J \subseteq \mathrm{GT}$ be a closed subgroup of* GT*. Fix a Belyi diagram*

$$\begin{array}{ccc} \Pi_U & \longrightarrow & \Pi_X \\ \big\downarrow & & \\ \Pi_X. & & \end{array}$$

Write $\phi_U : \mathrm{Aut}(\Pi_U) \twoheadrightarrow \mathrm{Out}(\Pi_U)$, $\phi_X : \mathrm{Aut}(\Pi_X) \twoheadrightarrow \mathrm{Out}(\Pi_X)$ for the natural surjections. Then, for any sufficiently small normal open subgroup $M \subseteq J$, there exist an outer action of M on Π_U and an open injection $\Pi_U \overset{\mathrm{out}}{\rtimes} M \hookrightarrow \Pi_X \overset{\mathrm{out}}{\rtimes} J$ such that

(a) the outer action of M preserves and induces the identity automorphism on the set of the conjugacy classes of cuspidal inertia subgroups of Π_U;

(b) the outer action of M on Π_U extends uniquely [cf. the slimness of Π_X] to a Π_U-outer action on Π_X that is compatible with the outer action of J $(\supseteq M)$ on Π_X; the injection $\Pi_U \overset{\mathrm{out}}{\rtimes} M \hookrightarrow \Pi_X \overset{\mathrm{out}}{\rtimes} J$ is the injection determined by the inclusions $\Pi_U \subseteq \Pi_X$ and $M \subseteq J$ and the Π_U-outer actions on Π_U and Π_X.

Proof. First, we recall that Π_X is slim [cf., e.g., [MT], Proposition 1.4]. Write

$$\mathrm{Aut}^{\Pi_U}(\Pi_X) \subseteq \mathrm{Aut}(\Pi_X)$$

for the subgroup of $\mathrm{Aut}(\Pi_X)$ consisting of elements that induce automorphisms of Π_U that fix each of the conjugacy classes of cuspidal inertia subgroups of Π_U;

$$\mathrm{Inn}^{\Pi_U}(\Pi_X) \subseteq \mathrm{Aut}^{\Pi_U}(\Pi_X)$$

for the image of Π_U by the natural isomorphism $\Pi_X \xrightarrow{\sim} \mathrm{Inn}(\Pi_X)$. It follows immediately from the slimness of Π_X [cf., e.g., [MT], Proposition 1.4] that the natural homomorphism $\mathrm{Aut}^{\Pi_U}(\Pi_X) \to \mathrm{Aut}(\Pi_U)$ is injective. This injectivity implies that $\mathrm{Ker}(\mathrm{Aut}^{\Pi_U}(\Pi_X) \to \mathrm{Out}(\Pi_U)) \subseteq \mathrm{Inn}^{\Pi_U}(\Pi_X)$.

Since Π_U is a finite index subgroup of Π_X, and the cardinality of the conjugacy classes of cuspidal inertia subgroups of Π_U is finite, there exists a normal open subgroup M_{Aut} of $\phi_X^{-1}(J) \subseteq \mathrm{Aut}(\Pi_X)$ satisfying the following conditions:

(i) $M_{\mathrm{Aut}} \cap \mathrm{Inn}(\Pi_X) \subseteq \mathrm{Inn}^{\Pi_U}(\Pi_X)$;

(ii) $M_{\mathrm{Aut}} \subseteq \mathrm{Aut}^{\Pi_U}(\Pi_X)$.

Write

$$M_U \subseteq \mathrm{Out}(\Pi_U),$$
$$M \subseteq \mathrm{Out}(\Pi_X),$$
$$M_{U,\mathrm{Aut}} \subseteq \mathrm{Aut}^{\Pi_U}(\Pi_X)/\mathrm{Inn}^{\Pi_U}(\Pi_X)$$

for the respective images of the composites

$$M_{\mathrm{Aut}} \subseteq \mathrm{Aut}^{\Pi_U}(\Pi_X) \hookrightarrow \mathrm{Aut}(\Pi_U) \overset{\phi_U}{\twoheadrightarrow} \mathrm{Out}(\Pi_U),$$
$$M_{\mathrm{Aut}} \subseteq \mathrm{Aut}^{\Pi_U}(\Pi_X) \subseteq \mathrm{Aut}(\Pi_X) \overset{\phi_X}{\twoheadrightarrow} \mathrm{Out}(\Pi_X),$$
$$M_{\mathrm{Aut}} \subseteq \mathrm{Aut}^{\Pi_U}(\Pi_X) \twoheadrightarrow \mathrm{Aut}^{\Pi_U}(\Pi_X)/\mathrm{Inn}^{\Pi_U}(\Pi_X).$$

Then we have a commutative diagram of profinite groups

$$\begin{array}{ccccc}
\mathrm{Aut}(\Pi_U) & \longleftarrow & \mathrm{Aut}^{\Pi_U}(\Pi_X) & \longrightarrow & \mathrm{Aut}(\Pi_X) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Out}(\Pi_U) & \longleftarrow & \mathrm{Aut}^{\Pi_U}(\Pi_X)/\mathrm{Inn}^{\Pi_U}(\Pi_X) & \longrightarrow & \mathrm{Out}(\Pi_X) \\
\uparrow & & \uparrow & & \uparrow \\
M_U & \longleftarrow & M_{U,\mathrm{Aut}} & \longrightarrow & M,
\end{array}$$

where the lower left-hand horizontal arrow is a bijection; the lower right-hand horizontal arrow is a surjection. Finally, it follows immediately from condition (i) that the surjection $M_{U,\mathrm{Aut}} \to M$ in the above commutative diagram is bijective. Now the assertions of Lemma 1.2 follow formally. □

Theorem 1.3 (**Combinatorial Belyi cuspidalization for a tripod**). *Fix a Belyi diagram*

$$\begin{array}{ccc}
\Pi_U & \longrightarrow & \Pi_X \\
\downarrow & & \\
\Pi_X & &
\end{array}$$

that arises from a connected finite étale covering $U \to X$ and an open immersion $U \hookrightarrow X$ [cf. Definition 1.1, (i)]. Then:

(i) *Let* $(\Pi, G \subseteq \mathrm{Out}(\Pi))$ *be a* **tripodal pair**. *Fix an isomorphism of pairs* $\alpha : (\Pi, G \subseteq \mathrm{Out}(\Pi)) \overset{\sim}{\to} (\Pi_X, \mathrm{GT} \subseteq \mathrm{Out}(\Pi_X))$. *Then the set of subgroups of* Π *determined, via* α, *by the cuspidal inertia subgroups of* Π_X, *may be* **reconstructed**, *in a* **purely group-theoretic** *way, from the pair* $(\Pi, G \subseteq \mathrm{Out}(\Pi))$. *We shall refer to the subgroups of* Π *constructed in this way as the* **cuspidal inertia subgroups** *of* Π. *In particular, for each open subgroup* $\Pi^* \subseteq \Pi$ *of* Π, *the pair* $(\Pi, G \subseteq \mathrm{Out}(\Pi))$ *determines a set* $I(\Pi^*)$ *(respectively,* $\mathrm{Cusp}(\Pi^*)$*) of cuspidal inertia subgroups of* Π^* *(respectively, cusps of* Π^**), namely, the set of intersections of* Π^* *with cuspidal inertia subgroups of* Π *(respectively, the conjugacy classes of cuspidal inertia subgroups of* Π^**).*

(ii) *Let* $N \subseteq \mathrm{GT}$ *a normal open subgroup. Suppose that we are given an outer action of* N *on* Π_U *and an open injection* $\Pi_U \overset{\mathrm{out}}{\rtimes} N \hookrightarrow \Pi_X \overset{\mathrm{out}}{\rtimes} \mathrm{GT}$ *such that the conditions (a), (b) in Lemma 1.2 in the case of "*$M \subseteq J$*" hold for* $N \subseteq \mathrm{GT}$. *Then the original* **outer action** *of* $N \subseteq \mathrm{GT}$ *on* Π_X **coincides** *with the outer action of* N *on* Π_X *induced [cf. condition (a)] by the outer action of* N *on* Π_U *and the outer surjection* $\Pi_U \twoheadrightarrow \Pi_X$ *[i.e., the horizontal arrow in the above Belyi diagram].*

(iii) *Let*

$$C(\Pi) = (\Pi, G \subseteq \mathrm{Out}(\Pi), \Pi^*, \{0, 1, \infty\} \subseteq \mathrm{Cusp}(\Pi), \{0, 1, \infty\} \subseteq \mathrm{Cusp}(\Pi^*))$$

be a 5*-tuple consisting of the following data:*

- *a topological group* Π;
- *a closed subgroup* $G \subseteq \mathrm{Out}(\Pi)$ *such that the pair* $(\Pi, G \subseteq \mathrm{Out}(\Pi))$ *is a tripodal pair;*
- *an open subgroup* $\Pi^* \subseteq \Pi$ *of* Π *of genus* 0, *where we observe that the genus of an open subgroup of* Π *may be defined by using the cuspidal inertia subgroups of the open subgroup [cf. (i)];*
- *a subset* $\{0, 1, \infty\} \subseteq \mathrm{Cusp}(\Pi)$ *[cf. (i)] of cardinality* 3 *[equipped with labels "0", "1", "∞"] of the set* $\mathrm{Cusp}(\Pi)$;
- *a subset* $\{0, 1, \infty\} \subseteq \mathrm{Cusp}(\Pi^*)$ *[cf. (i)] of cardinality* 3 *[equipped with labels "0", "1", "∞"] of the set* $\mathrm{Cusp}(\Pi^*)$.

Suppose that the collection of data $C(\Pi)$ *is isomorphic to the collection of data*

$$\begin{aligned} C(\Pi_X) \ = \ & (\Pi_X, \mathrm{GT} \subseteq \mathrm{Out}(\Pi_X), \Pi_U, \\ & \{0,1,\infty\} \subseteq \mathrm{Cusp}(\Pi_X), \{0,1,\infty\} \subseteq \mathrm{Cusp}(\Pi_U)) \end{aligned}$$

determined, in a natural way, by the given Belyi diagram. [Here, we observe that the horizontal arrow in the given Belyi diagram determines, in a natural way, data $\{0,1,\infty\} \subseteq \mathrm{Cusp}(\Pi_U)$*.] Fix an isomorphism of collections of data* $C(\Pi) \xrightarrow{\sim} C(\Pi_X)$*. Thus, the outer surjection* $\Pi_U \twoheadrightarrow \Pi_X$ *[i.e., the horizontal arrow in the given Belyi diagram], together with the isomorphism* $C(\Pi) \xrightarrow{\sim} C(\Pi_X)$*, determine an outer surjection* $\Pi^* \twoheadrightarrow \Pi$*. Let* $N \subseteq G$ *be a normal open subgroup such that similar conditions to the conditions (a), (b) considered in Lemma 1.2 in the case of "*$M \subseteq J$*" hold for* $N \subseteq G$*. Then the* **outer surjection** $\Pi^* \twoheadrightarrow \Pi$ *may be* **reconstructed***, in a* **purely group-theoretic** *way, from the collection of data* $C(\Pi)$ *as the outer surjection induced by the unique* Π*-outer surjection* $\Pi^* \overset{\mathrm{out}}{\rtimes} N \twoheadrightarrow \Pi \overset{\mathrm{out}}{\rtimes} N$ *[i.e., surjection well-defined up to composition with inner automorphisms arising from elements of* Π*] that lies over the identity morphism of* N *such that*

- *the kernel of this* Π*-outer surjection* $\Pi^* \overset{\mathrm{out}}{\rtimes} N \twoheadrightarrow \Pi \overset{\mathrm{out}}{\rtimes} N$ *is topologically generated by the cuspidal inertia subgroups of* Π^* *which are not associated to* $0, 1, \infty \in \mathrm{Cusp}(\Pi^*)$*;*
- *the conjugacy class of cuspidal inertia subgroups of* Π^* *associated to* 0 *(respectively,* 1*,* ∞*)* $\in \mathrm{Cusp}(\Pi^*)$ *maps to the conjugacy class of cuspidal inertia subgroups of* Π *associated to* 0 *(respectively,* 1*,* ∞*)* $\in \mathrm{Cusp}(\Pi)$*.*

Proof. First, we verify assertion (i). Since the outer action of GT on Π_X determined by the inclusion $\mathrm{GT} \subseteq \mathrm{Out}(\Pi_X)$ is l-cyclotomically full [cf. [CmbGC], Definition 2.3, (ii)], assertion (i) follows immediately from [CmbGC], Corollary 2.7, (i), and its proof.

Next, we verify assertion (ii). First, we observe that:

Claim 1.3.A: It suffices to prove assertion (ii) for a sufficiently small normal open subgroup $N^\dagger \subseteq N$.

Indeed, let $\sigma \in N$. Write

- $\rho' : N \to \mathrm{Out}(\Pi_X)$ for the original outer action;
- $\rho'' : N \to \mathrm{Out}(\Pi_X)$ for the outer action of N on Π_X induced [cf. condition (a)] by the outer action of N on Π_U and the outer surjection $\Pi_U \twoheadrightarrow \Pi_X$.

Suppose that $\rho'|_{N^\dagger} = \rho''|_{N^\dagger}$. Write $\rho \stackrel{\mathrm{def}}{=} \rho'|_{N^\dagger}$; $\sigma' \stackrel{\mathrm{def}}{=} \rho'(\sigma)$; $\sigma'' \stackrel{\mathrm{def}}{=} \rho''(\sigma)$. Our goal is to prove that $\sigma' = \sigma''$. Since $N^\dagger$ is a normal subgroup in N, for each $\tau \in N^\dagger$, $\sigma'\rho(\tau)(\sigma')^{-1} = \rho'(\sigma\tau\sigma^{-1}) = \rho''(\sigma\tau\sigma^{-1}) = \sigma''\rho(\tau)(\sigma'')^{-1}$. Thus, $(\sigma'')^{-1}\sigma' \in Z_{\mathrm{Out}(\Pi_X)}(\rho(N))$. By the Grothendieck Conjecture for hyperbolic curves over number fields [cf. [Tama1], Theorem 0.4], $(\sigma'')^{-1}\sigma'$ is induced by a geometric automorphism of X. Since the condition (a) in Lemma 1.2 in the case of "$M \subseteq J$" holds for $N \subseteq \mathrm{GT}$, $(\sigma'')^{-1}\sigma'$ preserves and fixes each conjugacy class of cuspidal inertia subgroups of Π_X. Thus, we conclude that $\sigma' = \sigma''$. This completes the proof of Claim 1.3.A.

Write

- Π_{X_3} for the étale fundamental group of the third configuration space X_3 of X [cf. [MT], Definition 2.1, (i)];
- $\mathrm{pr}_i : \Pi_{X_3} \twoheadrightarrow \Pi_X$ $(i = 1, 2, 3)$ for choices of surjections that induce the natural outer surjections determined by the natural scheme-theoretic projections;
- $U^{\times 3} \stackrel{\mathrm{def}}{=} U \times U \times U$, $X^{\times 3} \stackrel{\mathrm{def}}{=} X \times X \times X$, $\Pi_U{}^{\times 3} \stackrel{\mathrm{def}}{=} \Pi_U \times \Pi_U \times \Pi_U$, $\Pi_X{}^{\times 3} \stackrel{\mathrm{def}}{=} \Pi_X \times \Pi_X \times \Pi_X$;
- $V_3 \stackrel{\mathrm{def}}{=} X_3 \times_{X^{\times 3}} U^{\times 3}$, where the fiber product is with respect to the open immersion $X_3 \hookrightarrow X^{\times 3}$ that arises from the definition of the configuration space X_3 and the finite étale covering $U^{\times 3} \to X^{\times 3}$ determined by the given connected finite étale covering $U \to X$.

Next, we make the following *observations*:

- the projection $V_3 \to U^{\times 3}$ is an open immersion that factors as the composite of an open immersion $V_3 \hookrightarrow U_3$ and the open immersion $U_3 \hookrightarrow U^{\times 3}$ that arises from the definition of the configuration space U_3;
- by choosing a suitable basepoint of V_3, we may regard Π_{V_3} as the open subgroup $\Pi_{V_3} \subseteq \Pi_{X_3}$ given by forming the inverse image of the open subgroup $\Pi_U^{\times 3} \subseteq \Pi_X^{\times 3}$ (determined by the open subgroup $\Pi_U \subseteq \Pi_X$) via the surjection $\Pi_{X_3} \twoheadrightarrow \Pi_X^{\times 3}$ determined by $\mathrm{pr}_i : \Pi_{X_3} \twoheadrightarrow \Pi_X$ $(i = 1, 2, 3)$;
- the open immersion $V_3 \hookrightarrow U_3$ induces a natural outer surjection $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$;
- the open immersion $U_3 \hookrightarrow X_3$ determined by the open immersion $U \hookrightarrow X$ induces a natural outer surjection $\Pi_{U_3} \twoheadrightarrow \Pi_{X_3}$;
- we have natural inclusions $N \subseteq \mathrm{GT} \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_{X_3}) \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_X)$ [cf. [CmbCsp], Definition 1.11, (i); [CmbCsp], Remark 1.11.1; [CmbCsp], Theorem 4.1, (i); [CmbCsp], Corollary 4.2, (i), (ii)].

For each $\sigma \in N \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_{X_3})$, let $\tilde{\sigma}_3 \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_{X_3})$ be a lifting of the image $\sigma_3 \in \mathrm{Out}^{\mathrm{FC}}(\Pi_{X_3})$ of σ such that the automorphisms of Π_X induced by $\tilde{\sigma}_3$ via the surjections $\mathrm{pr}_i : \Pi_{X_3} \twoheadrightarrow \Pi_X$ $(i = 1, 2, 3)$ coincide and stabilize the subgroup $\Pi_U \subseteq \Pi_X$ [cf. our hypotheses on N]. Thus, it follows from the various *observations* made above concerning the open subgroup $\Pi_{V_3} \subseteq \Pi_{X_3}$ that $\tilde{\sigma}_3$ induces an automorphism $\tilde{\sigma}_{V_3}$ of Π_{V_3}.

Next, we verify the following assertion:

> Claim 1.3.B: There exists a normal open subgroup $N^\dagger$ of GT such that $N^\dagger \subseteq N$, and, moreover, the following condition holds:
>
> > For each element $\sigma \in N^\dagger$, $\tilde{\sigma}_{V_3} \in \mathrm{Aut}(\Pi_{V_3})$ preserves the kernel of the outer surjection $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$ (respectively, $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3} \twoheadrightarrow \Pi_{X_3}$) induced by the open immersion $V_3 \hookrightarrow U_3$ (respectively, the composite of open immersions $V_3 \hookrightarrow U_3 \hookrightarrow X_3$).
>
> In particular, $\tilde{\sigma}_{V_3} \in \mathrm{Aut}(\Pi_{V_3})$ induces outer automorphisms of Π_{U_3} and Π_{X_3} compatible with the outer surjections $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$ and $\Pi_{U_3} \twoheadrightarrow \Pi_{X_3}$, respectively.

Write

- I_{X_3} for the set of inertia subgroups $\subseteq \Pi_{X_3}$ associated to the irreducible divisors contained in the complement of the interior of the third log configuration space of X [cf. [MT], Definition 2.1, (i)];
- $I_{V_3} \stackrel{\mathrm{def}}{=} \{I \cap \Pi_{V_3} \ (\subseteq \Pi_{X_3}) \mid I \in I_{X_3}\}$;
- I_{U_3} for the set of images of elements of I_{V_3} by the outer surjection $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$;
- $|I_{X_3}|$ (respectively, $|I_{V_3}|$) for the set of Π_{X_3}- (respectively, Π_{V_3}-)conjugacy classes of elements of I_{X_3} (respectively, I_{V_3}).

Next, we make the following *observations*:

- $\tilde{\sigma}_3$ acts on I_{X_3} and induces the identity automorphism of $|I_{X_3}|$ [cf. condition (a) in Lemma 1.2; [CmbCsp], Proposition 1.3, (vi)];
- for each $\sigma \in N$, the action of $\tilde{\sigma}_3$ on I_{X_3} induces a natural action of $\tilde{\sigma}_{V_3}$ on I_{V_3}, and hence on $|I_{V_3}|$;
- since, for each $\sigma \in N$, $\tilde{\sigma}_3$ is completely determined [cf. condition (a) in Lemma 1.2; the fact that U is of genus 0; the definition of $\tilde{\sigma}_3$] up to composition with an inner automorphism of Π_{X_3} arising from Π_{V_3}, we conclude that the natural action of $\tilde{\sigma}_3$ on I_{V_3} determines a natural action of N on $|I_{V_3}|$;
- $|I_{X_3}|$ and $|I_{V_3}|$ are finite sets.

Thus, it follows immediately from the above *observations* that, if we take $N^\dagger$ to be a sufficiently small normal open subgroup of GT, then $\tilde{\sigma}_{V_3}$ induces the identity automorphism of $|I_{V_3}|$ for each $\sigma \in N^\dagger$. Since the kernel of the outer surjection $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$ (respectively, $\Pi_{U_3} \twoheadrightarrow \Pi_{X_3}$) is topologically normally generated by a certain collection of elements of I_{V_3} (respectively, I_{U_3}), we obtain the desired conclusion. This completes the proof of Claim 1.3.B.

By applying Claim 1.3.A and Claim 1.3.B, we may assume [by replacing N by a suitable normal open subgroup of GT] that, for each element $\sigma \in N$, $\tilde{\sigma}_{V_3} \in \mathrm{Aut}(\Pi_{V_3})$ induces outer automorphisms $\sigma_{V_3} \in \mathrm{Out}(\Pi_{V_3})$, $\sigma_{U_3} \in \mathrm{Out}(\Pi_{U_3})$, and $\sigma_{X_3} \in \mathrm{Out}(\Pi_{X_3})$ compatible with the outer surjections $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$ and $\Pi_{U_3} \twoheadrightarrow \Pi_{X_3}$, respectively. Our goal is to prove that

$$\sigma_3 = \sigma_{X_3} \in \mathrm{Out}(\Pi_{X_3}).$$

Note that $\sigma_{X_3} \in \mathrm{Out}^{\mathrm{F}}(\Pi_{X_3})$ by construction. Since $\mathrm{Out}^{\mathrm{F}}(\Pi_{X_3}) = \mathrm{Out}^{\mathrm{FC}}(\Pi_{X_3})$ [cf. [CbTpII], Theorem A, (ii)], $\sigma_{X_3} \in \mathrm{Out}^{\mathrm{FC}}(\Pi_{X_3})$.

In the following discussion, we *fix a surjection* $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$ (respectively, $\Pi_{U_3} \twoheadrightarrow \Pi_{X_3}$) that induces the outer surjection $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$ (respectively, $\Pi_{U_3} \twoheadrightarrow \Pi_{X_3}$) of Claim 1.3.B.

Next, write C for the set of 3-central tripods in Π_{X_3} [cf, [CbTpII], Definition 3.7, (ii)]; C_V for the set of 3-central tripods Π^{ctpd} of Π_{X_3} that satisfy the following condition:

> $\Pi^{\mathrm{ctpd}} \subseteq \Pi_{V_3}$; the image of Π^{ctpd} ($\subseteq \Pi_{V_3}$) by the surjection $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$ is a 3-central tripod of Π_{U_3}.

Then:

> Claim 1.3.C: The natural action of Π_{V_3} by conjugation on C_V is *transitive*; moreover,
>
> $$C \supseteq C_V = \{\Pi^{\mathrm{ctpd}} \in C \mid \Pi^{\mathrm{ctpd}} \cap \mathrm{Ker}(\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}) = \{1\}\} \neq \emptyset.$$

Write $\Delta \subseteq X^{\times 3}$ (respectively, $\Delta_U \subseteq U^{\times 3}$) for the image of X (respectively, U) under the diagonal embedding $X \hookrightarrow X^{\times 3}$ (respectively, $U \hookrightarrow U^{\times 3}$). Note that it follows immediately from the definition of the subgroup $\Pi_{V_3} \subseteq \Pi_{X_3}$ [cf. also [CbTpII], Definitions 3.3, (ii); 3.7, (ii)] that every $\Pi^{\mathrm{ctpd}} \in C$ is contained in Π_{V_3}, and that any two subgroups $\in C$ are Π_{X_3}-conjugate. Moreover, one verifies immediately that the Π_{V_3}-conjugacy classes of subgroups $\in C$ are in natural bijective correspondence with the irreducible [or, equivalently, connected] components of the inverse image of Δ by the finite étale covering $U^{\times 3} \to X^{\times 3}$. Thus, by considering the Π_{V_3}-conjugacy class of subgroups $\in C$ corresponding to Δ_U, we obtain that $C_V \neq \emptyset$. On the other hand, by considering the scheme-theoretic geometry of tripods that give rise to Π_{V_3}-conjugacy classes of subgroups $\in C$ that do *not* correspond to Δ_U, we conclude that such subgroups $\in C$ have *nontrivial intersection* with the kernel of the surjection $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$. This completes the proof of Claim 1.3.C.

Let $\Pi^{\mathrm{ctpd}} \in C_V$. Write Π_U^{ctpd} for the image of Π^{ctpd} by the surjection $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$; Π_X^{ctpd} for the image of Π_U^{ctpd} by the surjection $\Pi_{U_3} \twoheadrightarrow \Pi_{X_3}$. Thus, Π_U^{ctpd} is a 3-central tripod of Π_{U_3}, and Π_X^{ctpd} is a 3-central tripod of Π_{X_3} [hence Π_{X_3}-conjugate to Π^{ctpd}].

By the theory of tripod synchronization [cf. [CbTpII], Theorem C, (ii), (iii)] and the injectivity of $\mathrm{Out}^{\mathrm{FC}}(\Pi_{X_3}) \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_X)$ [cf. [CmbCsp], Theorem 4.1, (i)], we obtain injective tripod homomorphisms

$$T : \mathrm{Out}^{\mathrm{FC}}(\Pi_{X_3})^{\mathrm{cusp}} \to \mathrm{Out}(\Pi^{\mathrm{ctpd}}), \quad T_X : \mathrm{Out}^{\mathrm{FC}}(\Pi_{X_3})^{\mathrm{cusp}} \to \mathrm{Out}(\Pi_X^{\mathrm{ctpd}})$$

[cf. [CmbCsp], Definition 1.1, (v)], which are related to one another via composition with the isomorphism $\zeta : \mathrm{Out}(\Pi^{\mathrm{ctpd}}) \xrightarrow{\sim} \mathrm{Out}(\Pi_X^{\mathrm{ctpd}})$ induced by the geometric outer isomorphism $\Pi^{\mathrm{ctpd}} \xrightarrow{\sim} \Pi_X^{\mathrm{ctpd}}$ [cf. [CbTpII], Definition 3.4, (ii)] determined by the composite surjection $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3} \twoheadrightarrow \Pi_{X_3}$. Since $\tilde{\sigma}_{V_3}$ preserves the Π_{V_3}-conjugacy class of $\Pi^{\mathrm{ctpd}} \subseteq \Pi_{V_3}$ [cf. Claims 1.3.B, 1.3.C; [CbTpII], Theorem C, (ii)], we conclude that $\zeta(T(\sigma_3)) = T_X(\sigma_{X_3})$. This completes the proof of assertion (ii).

Finally, we verify assertion (iii). The existence of a Π-outer surjection $\Pi^* \overset{\mathrm{out}}{\rtimes} N \twoheadrightarrow \Pi \overset{\mathrm{out}}{\rtimes} N$ as in the statement of assertion (iii) follows immediately from assertion (ii) and the various definitions involved. Since $G_{\mathbb{Q}} \subseteq \mathrm{GT} \xrightarrow{\sim} G$, the uniqueness of a Π-outer surjection $\Pi^* \overset{\mathrm{out}}{\rtimes} N \twoheadrightarrow \Pi \overset{\mathrm{out}}{\rtimes} N$ as in the statement of assertion (iii) follows immediately from the Grothendieck Conjecture for hyperbolic curves over number fields [cf. [Tama1], Theorem 0.4], applied to the case of $\mathbb{P}^1_{\mathbb{Q}} \backslash \{0, 1, \infty\}$. This completes the proof of assertion (iii), hence also the proof of Theorem 1.3. □

Definition 1.4. Let $J \subseteq \mathrm{GT}$ be a closed subgroup of GT. In the situation of Theorem 1.3, (ii), for each normal open subgroup M of J satisfying $M \subseteq N \cap J$, we obtain a diagram

$$\begin{array}{ccc} \Pi_U \overset{\mathrm{out}}{\rtimes} M & \longrightarrow & \Pi_X \overset{\mathrm{out}}{\rtimes} M \\ \downarrow & & \\ \Pi_X \overset{\mathrm{out}}{\rtimes} M & & \end{array}$$

of Π_X-*outer homomorphisms* [i.e., homomorphisms well-defined up to composition with inner automorphisms arising from elements of Π_X] of profinite groups. We shall refer to a diagram obtained in this way as an *arithmetic Belyi diagram*.

Definition 1.5.

(i) Fix an arithmetic Belyi diagram $\mathbb{B}^{\rtimes}$ as in Definition 1.4. Write

$$\mathbb{D}(\mathbb{B}^{\rtimes}, M, J)$$

for the set of the images via the natural composite Π_X-outer homomorphism $\Pi_U \overset{\text{out}}{\rtimes} M \twoheadrightarrow \Pi_X \overset{\text{out}}{\rtimes} M \hookrightarrow \Pi_X \overset{\text{out}}{\rtimes} J$ of the normalizers in $\Pi_U \overset{\text{out}}{\rtimes} M$ of cuspidal inertia subgroups of Π_U;

$$\mathbb{D}(\mathbb{B}^{\rtimes}, J)$$

for the quotient set $\big(\sqcup_{M \subseteq J} \mathbb{D}(\mathbb{B}^{\rtimes}, M, J)\big)/\sim$, where M ranges over all sufficiently small normal open subgroups of J, and we write $\mathbb{D}(\mathbb{B}^{\rtimes}, M, J) \ni G_M \sim G_{M^\dagger} \in \mathbb{D}(\mathbb{B}^{\rtimes}, M^\dagger, J)$ if $G_M \cap G_{M^\dagger}$ is open in both G_M and $G_{M^\dagger}$.

(ii) Write

$$\mathbb{D}(J)$$

for the quotient set $\big(\sqcup_{\mathbb{B}^{\rtimes}} \mathbb{D}(\mathbb{B}^{\rtimes}, J)\big)/\sim$, where $\mathbb{B}^{\rtimes}$ ranges over all arithmetic Belyi diagrams, and we write $\mathbb{D}({}^\dagger\mathbb{B}^{\rtimes}, J) \ni G_{{}^\dagger\mathbb{B}^{\rtimes}} \sim G_{{}^\ddagger\mathbb{B}^{\rtimes}} \in \mathbb{D}({}^\ddagger\mathbb{B}^{\rtimes}, J)$ if $G_{M^\dagger} \cap G_{M^\ddagger}$ is open in both $G_{M^\dagger}$ and $G_{M^\ddagger}$ for some representative $G_{M^\dagger}$ (respectively, $G_{M^\ddagger}$) of $G_{{}^\dagger\mathbb{B}^{\rtimes}}$ (respectively, $G_{{}^\ddagger\mathbb{B}^{\rtimes}}$). We shall refer to $\mathbb{D}(J)$ as the set of *decomposition subgroup-germs of* $\Pi_X \overset{\text{out}}{\rtimes} J$.

(iii) We shall refer to the technique of constructing decomposition subgroup-germs of $\Pi_X \overset{\text{out}}{\rtimes} J$ as in (ii) as *combinatorial Belyi cuspidalization.*

Corollary 1.6. *In the situation of Definition 1.5:*

(i) The natural conjugation action of $\Pi_X \overset{\text{out}}{\rtimes} J$ *on itself induces a natural action of* $\Pi_X \overset{\text{out}}{\rtimes} J$ *on* $\mathbb{D}(J)$.

(ii) Write

$$D(J)$$

for the quotient set $\mathbb{D}(J)/\Pi_X$. *Then* $D(J)$ *admits a natural action by* J.

(iii) Let J_1 *and* J_2 *be closed subgroups of* GT. *If* $J_1 \subseteq J_2 \subseteq \mathrm{GT}$, *then the inclusion* $J_1 \subseteq J_2$ *induces, by considering the intersection of subgroups of* $\Pi_X \overset{\text{out}}{\rtimes} J_2$ *with* $\Pi_X \overset{\text{out}}{\rtimes} J_1$, *a natural surjection*

$$D(J_2) \twoheadrightarrow D(J_1)$$

that is equivariant with respect to the natural actions of J_1 $(\subseteq J_2)$ *on the domain and codomain.*

Proof. First, we verify assertion (i). Let $\sigma \in \Pi_X \overset{\text{out}}{\rtimes} J$ $(\subseteq \mathrm{Aut}(\Pi_X))$. Fix an arithmetic Belyi diagram $\mathbb{B}^{\rtimes}$

$$\begin{array}{ccc} \Pi_U \overset{\text{out}}{\rtimes} M & \longrightarrow & \Pi_X \overset{\text{out}}{\rtimes} M \\ \big\downarrow & & \\ \Pi_X \overset{\text{out}}{\rtimes} M. & & \end{array}$$

Next, we *observe* that σ, the inclusion $\Pi_U \subseteq \Pi_X$, and the outer action of M on Π_U determine

- an open subgroup $\Pi_{U^\sigma} \overset{\text{def}}{=} \sigma(\Pi_U)\sigma^{-1} \subseteq \Pi_X$ that belongs to the Π_X-conjugacy class of open subgroups that arises as the image of the outer injection $\Pi_{U^\sigma} \hookrightarrow \Pi_X$ determined by some connected finite étale covering $U^\sigma \to X$;
- an isomorphism $\Pi_U \overset{\sim}{\to} \Pi_{U^\sigma}$ [induced by conjugating by σ] that induces a bijection of the set of cuspidal inertia subgroups;
- an outer action [induced by conjugating by σ] of M on Π_{U^σ};
- a collection of data [induced by conjugating by σ]

$$\begin{aligned} C(\Pi_X)^\sigma \overset{\text{def}}{=} \; & (\Pi_X, \mathrm{GT} \subseteq \mathrm{Out}(\Pi_X), \Pi_{U^\sigma}, \\ & \{0, 1, \infty\} \subseteq \mathrm{Cusp}(\Pi_X), \{0, 1, \infty\} \subseteq \mathrm{Cusp}(\Pi_{U^\sigma})) \end{aligned}$$

  [cf. Theorem 1.3, (i), (iii)];
- an isomorphism $C(\Pi_X) \overset{\sim}{\to} C(\Pi_X)^\sigma$ [induced by conjugating by σ].

Since M is a normal subgroup of J, by conjugating by σ, we obtain an automorphism $\sigma_M : \Pi_X \overset{\text{out}}{\rtimes} M \overset{\sim}{\to} \Pi_X \overset{\text{out}}{\rtimes} M$ and an isomorphism $\sigma_M|_{\Pi_U} : \Pi_U \overset{\text{out}}{\rtimes} M \overset{\sim}{\to} \Pi_{U^\sigma} \overset{\text{out}}{\rtimes} M$ compatible with the natural inclusions $\Pi_U \overset{\text{out}}{\rtimes} M \hookrightarrow \Pi_X \overset{\text{out}}{\rtimes} M$ and $\Pi_{U^\sigma} \overset{\text{out}}{\rtimes} M \hookrightarrow \Pi_X \overset{\text{out}}{\rtimes} M$. Thus, it follows immediately from the above *observations*, together with Theorem 1.3, (ii), (iii), that we obtain a commutative diagram of profinite groups

$$\begin{array}{ccccc} \Pi_X \overset{\text{out}}{\rtimes} M & \longleftarrow & \Pi_U \overset{\text{out}}{\rtimes} M & \longrightarrow & \Pi_X \overset{\text{out}}{\rtimes} M \\ {\scriptstyle \sigma_M}\big\downarrow\wr & & {\scriptstyle \sigma_M|_{\Pi_U}}\big\downarrow\wr & & {\scriptstyle \sigma_M}\big\downarrow\wr \\ \Pi_X \overset{\text{out}}{\rtimes} M & \longleftarrow & \Pi_{U^\sigma} \overset{\text{out}}{\rtimes} M & \longrightarrow & \Pi_X \overset{\text{out}}{\rtimes} M, \end{array}$$

where the upper horizontal arrows "$\leftarrow$", "$\rightarrow$" are, respectively, the vertical and horizontal arrows of $\mathbb{B}^{\rtimes}$; the arrow $\Pi_X \overset{\text{out}}{\rtimes} M \leftarrow \Pi_{U^\sigma} \overset{\text{out}}{\rtimes} M$ is the natural inclusion discussed above; the arrow $\Pi_{U^\sigma} \overset{\text{out}}{\rtimes} M \to \Pi_X \overset{\text{out}}{\rtimes} M$ is the Π_X-outer surjection induced [cf. Theorem 1.3, (ii), (iii)] by the outer surjection $\Pi_{U^\sigma} \to \Pi_X$ determined by the open immersion $U^\sigma \hookrightarrow X$ that maps the cusp 0 (respectively, 1, ∞) of U^σ to the cusp 0 (respectively, 1, ∞) of X. Thus, by the above *observations* and the definition of $\mathbb{D}(J)$, we conclude that the natural conjugation action of $\Pi_X \overset{\text{out}}{\rtimes} J$ on itself induces a natural action of $\Pi_X \overset{\text{out}}{\rtimes} J$ on $\mathbb{D}(J)$. This completes the proof of assertion (i). Assertion (ii) follows immediately from assertion (i). Assertion (iii) follows immediately from the various definitions involved. This completes the proof of Corollary 1.6. □

Corollary 1.7. *In the notation of Corollary 1.6, there exist a natural surjection* $D(\mathrm{GT}) \twoheadrightarrow \overline{\mathbb{Q}} \cup \{\infty\}$ *and a natural bijection* $D(G_{\mathbb{Q}}) \xrightarrow{\sim} \overline{\mathbb{Q}} \cup \{\infty\}$.

Proof. The usual theory of Belyi cuspidalization [cf. [AbsTopIII], Theorem 1.9, (a)] yields a natural bijection $D(G_{\mathbb{Q}}) \xrightarrow{\sim} \overline{\mathbb{Q}} \cup \{\infty\}$. Next, by applying the natural inclusion $G_{\mathbb{Q}} \subseteq \mathrm{GT}$ [cf. the discussion at the beginning of the Introduction], we obtain a natural surjection $D(\mathrm{GT}) \twoheadrightarrow D(G_{\mathbb{Q}})$ [cf. Corollary 1.6, (iii)]. Thus, by considering the composite $D(\mathrm{GT}) \twoheadrightarrow D(G_{\mathbb{Q}}) \xrightarrow{\sim} \overline{\mathbb{Q}} \cup \{\infty\}$, we obtain a natural surjection $D(\mathrm{GT}) \twoheadrightarrow \overline{\mathbb{Q}} \cup \{\infty\}$. This completes the proof of Corollary 1.7. □

Remark 1.7.1. The author does not know, at the time of writing, whether or not the *surjection*

$$D(\mathrm{GT}) \twoheadrightarrow \overline{\mathbb{Q}} \cup \{\infty\}$$

in Corollary 1.7 is *bijective*.

Remark 1.7.2. It follows immediately from the various definitions involved that the inverse image of ∞ via the *surjection*

$$D(\mathrm{GT}) \twoheadrightarrow \overline{\mathbb{Q}} \cup \{\infty\}$$

in Corollary 1.7 consists of a *unique* element determined by the normalizer in $\Pi_X \overset{\text{out}}{\rtimes} \mathrm{GT}$ of a cuspidal inertia subgroup of Π_X associated to ∞.

§2. Construction of an action of $\mathrm{GT}_p^{\mathrm{tp}}$ on the field $\overline{\mathbb{Q}}$

In this section, we construct [cf. Corollary 2.4] a certain natural action of $\mathrm{GT}_p^{\mathrm{tp}}$ on the field $\overline{\mathbb{Q}}$, where $\mathrm{GT}_p^{\mathrm{tp}}$ denotes [cf. Definition 2.1] a certain subgroup of GT that contains the p-adic version of the Grothendieck-Teichmüller group GT_p defined by Y. André [cf. [André], Definition 8.6.3] by using the theory of tempered fundamental groups [cf. [André], §4, for the definition and basic properties of tempered fundamental groups]. First, we define $\mathrm{GT}_p^{\mathrm{tp}}$.

Definition 2.1. Let p be a prime number, $\overline{\mathbb{Q}}_p$ an algebraic closure of $\mathbb{Q}_p$ [cf. Notations and Conventions]. Write

- $X \stackrel{\mathrm{def}}{=} \mathbb{P}^1_{\mathbb{C}_p} \backslash \{0, 1, \infty\}$, where $\mathbb{C}_p$ denotes the p-adic completion of $\overline{\mathbb{Q}}_p$;
- Π_X^{tp} for the tempered fundamental group of X, relative to a suitable choice of basepoint.

We shall denote by $\mathrm{GT}_p^{\mathrm{tp}}$ the intersection of GT and $\mathrm{Out}(\Pi_X^{\mathrm{tp}})$ in $\mathrm{Out}(\Pi_X)$ [cf. Remark 2.1.1].

Remark 2.1.1. Observe that [for suitable choices of basepoints] Π_X may be regarded as the profinite completion of Π_X^{tp}, and Π_X^{tp} may be regarded as a subgroup of Π_X [cf. [André], §4.5]. Then the operation of passing to the profinite completion induces a natural homomorphism

$$\mathrm{Out}(\Pi_X^{\mathrm{tp}}) \to \mathrm{Out}(\Pi_X).$$

It follows immediately from the normal terminality of Π_X^{tp} in Π_X, i.e., $N_{\Pi_X}(\Pi_X^{\mathrm{tp}}) = \Pi_X^{\mathrm{tp}}$ [cf. [André], Corollary 6.2.2; [SemiAn], Lemma 6.1, (ii)], that this natural homomorphism is *injective*. Thus, we shall use this natural injection to regard $\mathrm{Out}(\Pi_X^{\mathrm{tp}})$ as a subgroup of $\mathrm{Out}(\Pi_X)$.

Remark 2.1.2. Various p-adic versions of the Grothendieck-Teichmüller group appear in the literature. It follows immediately from [André], Definition 8.6.3; [CbTpIII], Theorem B, (ii); [CbTpIII], Theorem D, (i); [CbTpIII], Theorem E; [CbTpIII], Proposition 3.6, (i), (ii); [CbTpIII], Definition 3.7, (i); [CbTpIII], Remark 3.13.1, (i); [CbTpIII], Remark 3.19.2; [CbTpIII], Remark 3.20.1, that

$$\begin{array}{ccccccccc} G_{\mathbb{Q}_p} & \subseteq & \mathrm{GT}^{\mathrm{M}} & \subseteq & \mathrm{GT}^{\mathrm{G}} & \subseteq & \mathrm{GT} \cap \mathrm{Out}^{\mathrm{G}}(\Pi_X) & = & \mathrm{GT}_p^{\mathrm{tp}} \\ \| & & \| & & & & \| & & \| \\ G_{\mathbb{Q}_p} & \subseteq & \mathrm{GT}^{\mathrm{M}} & \subseteq & \mathrm{GT}_p & \subseteq & \mathrm{GT} \cap \mathrm{Out}^{\mathrm{G}}(\Pi_X) & = & \mathrm{GT}_p^{\mathrm{tp}}. \end{array}$$

Remark 2.1.3. It follows immediately from the fact that the subgroup "$\mathrm{Out}^{\mathrm{G}}(\Pi_1) \subseteq \mathrm{Out}(\Pi_1)$" [cf. [CbTpIII], Proposition 3.6, (i), (ii); [CbTpIII], Definition 3.7, (i); [CbTpIII], Remark 3.13.1, (i)] is *closed* [cf. [CbTpIII], Theorem 3.17, (iv)] that $\mathrm{GT}_p^{\mathrm{tp}}$ is a *closed* subgroup of GT.

Next, we construct a natural action of $\mathrm{GT}_p^{\mathrm{tp}}$ on the set $\overline{\mathbb{Q}}$. The following theorem plays a central role in this construction. We prove this theorem by applying various "resolution of nonsingularities" results [cf. [Tama2], Theorem 0.2, (v); [Lpg], Theorem 2.7], as well as the reconstruction theorem of the dual semi-graph from the tempered fundamental group of a pointed stable curve [cf. [SemiAn], Corollary 3.11].

Theorem 2.2. *In the notation of Definition 2.1, let $\phi : Y \to X$ be a connected finite étale covering of X; y, y' elements of $Y(\mathbb{C}_p)$. Write Y_y (respectively, $Y_{y'}$) for $Y \setminus \{y\}$ (respectively, $Y \setminus \{y'\}$); Π_Y^{tp} (respectively, $\Pi_{Y_y}^{\mathrm{tp}}$, $\Pi_{Y_{y'}}^{\mathrm{tp}}$) for the tempered fundamental group of Y (respectively, Y_y, $Y_{y'}$), relative to a suitable choice of basepoint. Suppose that there exists an isomorphism $\Pi_{Y_y}^{\mathrm{tp}} \xrightarrow{\sim} \Pi_{Y_{y'}}^{\mathrm{tp}}$ that fits into a commutative diagram*

$$\begin{array}{ccc} \Pi_{Y_y}^{\mathrm{tp}} & \xrightarrow{\sim} & \Pi_{Y_{y'}}^{\mathrm{tp}} \\ \downarrow & & \downarrow \\ \Pi_Y^{\mathrm{tp}} & = & \Pi_Y^{\mathrm{tp}}, \end{array}$$

where the vertical arrows are the surjections [determined up to composition with an inner automorphism] induced by the natural open immersions $Y_y \hookrightarrow Y$, $Y_{y'} \hookrightarrow Y$ of hyperbolic curves. Then $y = y'$.

Proof. Suppose that $y \neq y'$. Write

- $\mathcal{O}_{\mathbb{C}_p}$ for the ring of integers of $\mathbb{C}_p$;
- Y^{cpt} for the smooth compactification of Y (over $\mathbb{C}_p$);
- S for $Y^{\mathrm{cpt}} \setminus Y$;
- $\mathcal{Y}_{y,y'}$ for the stable model over $\mathcal{O}_{\mathbb{C}_p}$ of the pointed stable curve $(Y^{\mathrm{cpt}}, S \cup \{y, y'\})$;
- $\mathcal{Y}$ for the semi-stable model over $\mathcal{O}_{\mathbb{C}_p}$ of the pointed stable curve (Y^{cpt}, S) obtained by forgetting the data of the horizontal divisors of $\mathcal{Y}_{y,y'}$ determined by y, y';
- $\overline{y}$ (respectively, $\overline{y}'$) for the closed point of $\mathcal{Y}$ determined by y (respectively, y').

Let

- $\tilde{\mathcal{Y}}$ be a proper normal model of Y^{cpt} over $\mathcal{O}_{\mathbb{C}_p}$ that dominates $\mathcal{Y}$, and whose special fiber contains an irreducible component $\tilde{y}$ (respectively, $\tilde{y}'$) that maps to $\overline{y}$ (respectively, $\overline{y}'$) in $\mathcal{Y}$;
- $\hat{y}$ (respectively, $\hat{y}'$) the valuation of the function field of $\mathcal{Y}$ determined by $\tilde{y}$ (respectively, $\tilde{y}'$).

Then, by applying [Lpg], Theorem 2.7 [cf. also the discussion at the beginning of [Lpg], §1; the discussion immediately preceding [Lpg], Definition 2.1; the discussion immediately preceding [Lpg], Corollary 2.9], to Y, we conclude that there exists a finite étale Galois covering

$$\phi : Z \to Y$$

such that, if we write

- $Y^{\mathrm{an}}_{(2)}$ for the set of type 2 points of the Berkovich space Y^{an} associated to Y [so that, by a slight abuse of notation, we may regard $\hat{y}$, $\hat{y}'$ as points of $Y^{\mathrm{an}}_{(2)}$];
- $V(\mathcal{Y})$ for the set of type 2 points of Y^{an} corresponding to the irreducible components of the special fiber of $\mathcal{Y}$;
- Z^{cpt} for the smooth compactification of Z (over $\mathbb{C}_p$);
- $\mathcal{Z}$ for the stable model of the pointed stable curve $(Z^{\mathrm{cpt}}, \phi^{-1}(S))$;
- $V(\mathcal{Z})$ for the set of type 2 points of the Berkovich space Z^{an} associated to Z corresponding to the irreducible components of the special fiber of $\mathcal{Z}$;
- $\mathrm{Im}(V(\mathcal{Z})) \subseteq Y^{\mathrm{an}}_{(2)}$ for the image of $V(\mathcal{Z})$ by the natural map $Z^{\mathrm{an}} \to Y^{\mathrm{an}}$ induced by ϕ,

then

$$\{\hat{y}, \hat{y}'\} \cup V(\mathcal{Y}) \ \subseteq\ \mathrm{Im}(V(\mathcal{Z})) \ \subseteq\ Y^{\mathrm{an}}_{(2)}.$$

Since $\mathcal{Y}$ is normal, it follows immediately, via a well-known argument [involving the closure in $\mathcal{Z} \times_{\mathcal{O}_{\mathbb{C}_p}} \mathcal{Y}$ of the graph of ϕ], from Zariski's Main Theorem, together with the first inclusion of the above display, that ϕ determines a morphism $f : \mathcal{Z} \to \mathcal{Y}$ such that

- the morphism f induces ϕ on the generic fiber;
- the image in the special fiber of $\mathcal{Y}$ of the vertical components of the special fiber of $\mathcal{Z}$ [i.e., the irreducible components of this special fiber that map to a point in the special fiber of $\mathcal{Y}$] contains $\overline{y}$ and $\overline{y}'$.

Fix a vertical component v in the special fiber of $\mathcal{Z}$ such that $f(v) = \overline{y}$. Write $\tilde{\mathcal{Y}}$ for the normalization of $\mathcal{Y}$ in the function field of Z; $\tilde{f} : \mathcal{Z} \to \tilde{\mathcal{Y}}$ for the morphism induced by the universal property of the normalization morphism $h : \tilde{\mathcal{Y}} \to \mathcal{Y}$. Since

h is finite, $\tilde{f}(v)$ is a closed point of $\tilde{\mathcal{Y}}$. By Zariski's Main Theorem, $\tilde{f}^{-1}(\tilde{f}(v))$ is connected. In particular, every irreducible component of $\tilde{f}^{-1}(\tilde{f}(v))$ is of dimension 1. Let $z \in Z(\mathbb{C}_p)$ be such that

- $f(z) = y$;
- $\overline{z} \in \tilde{f}^{-1}(\tilde{f}(v))$, where $\overline{z}$ denotes the closed point of $\mathcal{Z}$ determined by z.

Observe that the set C_z of irreducible components of the special fiber of $\mathcal{Z}$ that contain $\overline{z}$ is nonempty and of cardinality ≤ 2. Write $C_z \stackrel{\text{def}}{=} \{v_z, w_z\}$, where we note that it may or may not be the case that $v_z = w_z$. Without loss of generality, we may assume that $\overline{z} \in v_z \subseteq \tilde{f}^{-1}(\tilde{f}(v))$.

By [SemiAn], Corollary 3.11, any isomorphism of tempered fundamental groups preserves cuspidal inertia subgroups. Thus, the given commutative diagram of tempered fundamental groups

$$\begin{array}{ccc} \Pi^{\text{tp}}_{Y_y} & \xrightarrow{\sim} & \Pi^{\text{tp}}_{Y_{y'}} \\ \downarrow & & \downarrow \\ \Pi^{\text{tp}}_{Y} & = & \Pi^{\text{tp}}_{Y}, \end{array}$$

implies the existence of a $\mathbb{C}_p$-valued point z' of Z such that $\phi(z') = y'$, together with a commutative diagram of tempered fundamental groups

$$\begin{array}{ccc} \Pi^{\text{tp}}_{Z_z} & \xrightarrow{\sim} & \Pi^{\text{tp}}_{Z_{z'}} \\ \downarrow & & \downarrow \\ \Pi^{\text{tp}}_{Z} & = & \Pi^{\text{tp}}_{Z}, \end{array}$$

where $Z_z \stackrel{\text{def}}{=} Z \setminus \{z\}$; $Z_{z'} \stackrel{\text{def}}{=} Z \setminus \{z'\}$; Π^{tp}_Z (respectively, $\Pi^{\text{tp}}_{Z_z}$, $\Pi^{\text{tp}}_{Z_{z'}}$) denotes the tempered fundamental group of Z (respectively, Z_z, $Z_{z'}$), relative to a suitable choice of basepoint; the vertical arrows are the surjections [determined up to composition with an inner automorphism] induced by the natural open immersions $Z_z \hookrightarrow Z$ and $Z_{z'} \hookrightarrow Z$ of hyperbolic curves.

Write

- $\overline{z}'$ for the closed point of $\mathcal{Z}$ determined by z';
- $\mathcal{Z}_z$ for the stable model of the pointed stable curve $(Z^{\text{cpt}}, \phi^{-1}(S) \cup \{z\})$;
- $\mathcal{Z}_{z'}$ for the stable model of the pointed stable curve $(Z^{\text{cpt}}, \phi^{-1}(S) \cup \{z'\})$;
- v_z^* (respectively, w_z^*) for the unique irreducible component of the special fiber of $\mathcal{Z}_z$ that maps surjectively [via the natural morphism $\mathcal{Z}_z \to \mathcal{Z}$] onto v_z (respectively, w_z);
- Γ for the dual semi-graph of the special fiber of $\mathcal{Z}$;

- Γ_z for the dual semi-graph of the special fiber of $\mathcal{Z}_z$;
- $\Gamma_{z'}$ for the dual semi-graph of the special fiber of $\mathcal{Z}_{z'}$.

Since, by [SemiAn], Corollary 3.11 [and its proof], the isomorphism $\Pi^{\mathrm{tp}}_{Z_z} \xrightarrow{\sim} \Pi^{\mathrm{tp}}_{Z_{z'}}$ induces an isomorphism between the dual semi-graphs of special fibers of the respective stable models, the preceding commutative diagram of tempered fundamental groups induces a commutative diagram of *"generalized morphisms"* of dual semi-graphs

$$\begin{array}{ccc} \Gamma_z & \xrightarrow{\sim} & \Gamma_{z'} \\ \downarrow & & \downarrow \\ \Gamma & = & \Gamma, \end{array}$$

where the term "generalized morphism" refers to a *functor* between the respective *categories* "Cat(−)" associated to the semi-graphs in the domain and codomain [cf. the discussion immediately preceding [SemiAn], Definition 2.11].

Write

- $v^*_{z'}$ (respectively, $w^*_{z'}$) for the irreducible component of the special fiber of $\mathcal{Z}_{z'}$ corresponding to v^*_z (respectively, w^*_z) via the isomorphism $\Gamma_z \xrightarrow{\sim} \Gamma_{z'}$;
- $v_{z'}$ (respectively, $w_{z'}$) for the irreducible component of the special fiber of $\mathcal{Z}$ obtained by mapping $v^*_{z'}$ (respectively, $w^*_{z'}$) via the generalized morphism $\Gamma_{z'} \to \Gamma$.

Then the commutativity of the above diagram of generalized morphisms of dual semi-graphs implies that $\{v_z, w_z\} = \{v_{z'}, w_{z'}\}$. On the other hand, it follows from the definitions of the various objects involved that $\overline{z} \in v_z \cap w_z = v_{z'} \cap w_{z'} \ni \overline{z}'$. Thus, [if, by a slight abuse of notation, we regard closed points as closed subschemes, then] we conclude that

$$\tilde{f}(\overline{z}') \subseteq \tilde{f}(v_{z'} \cap w_{z'}) = \tilde{f}(v_z \cap w_z) \subseteq \tilde{f}(v_z) = \tilde{f}(v),$$

hence that

$$\overline{y}' = f(\overline{z}') = h(\tilde{f}(\overline{z}')) = h(\tilde{f}(v)) = f(v) = \overline{y}.$$

However, this contradicts our assumption that $\overline{y} \neq \overline{y}'$. This completes the proof of Theorem 2.2. ☐

Our goal in this section is to prove the following corollaries of Theorem 2.2.

Corollary 2.3. $\mathrm{GT}^{\mathrm{tp}}_p$ *acts naturally on the set of algebraic numbers* $\overline{\mathbb{Q}}$.

Proof. Write $X \stackrel{\mathrm{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, 1, \infty\}$, where we think of "$\overline{\mathbb{Q}}$" as the subfield of $\mathbb{C}_p$ consisting of the elements algebraic over $\mathbb{Q}$. [Thus, we have a *natural embedding*

$\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$.] In the following discussion, we shall identify $X(\overline{\mathbb{Q}})$ with $\overline{\mathbb{Q}} \setminus \{0,1\}$. We take the "natural action" in the statement of Corollary 2.3 on $\{0,1\} \subseteq \overline{\mathbb{Q}}$ to be the trivial action. Let $x \in X(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}} \setminus \{0,1\}$; $\sigma \in \mathrm{GT}_p^{\mathrm{tp}}$; $\mathbb{B}$ a Belyi diagram

$$\begin{array}{ccc} \Pi_U & \longrightarrow & \Pi_X \\ \big\downarrow & & \\ \Pi_X & & \end{array}$$

such that $x \notin U(\overline{\mathbb{Q}})$, where we identify U with the image scheme of the open immersion $U \hookrightarrow X$. Thus, we obtain an element $x_{\mathbb{B}} \in D(\mathrm{GT})$ [cf. Definitions 1.4, 1.5; Corollary 1.6, (ii)] such that $x_{\mathbb{B}} \mapsto x \in \overline{\mathbb{Q}}$ via the surjection $D(\mathrm{GT}) \twoheadrightarrow \overline{\mathbb{Q}} \cup \{\infty\}$ of Corollary 1.7. Write $(x_{\mathbb{B}})^\sigma \in \overline{\mathbb{Q}} \cup \{\infty\}$ for the image of the composite

$$D(\mathrm{GT}) \xrightarrow{\sim} D(\mathrm{GT}) \twoheadrightarrow \overline{\mathbb{Q}} \cup \{\infty\},$$

where the first arrow denotes the bijection induced by σ [cf. Corollary 1.6, (ii), in the case where $J = \mathrm{GT}$]; the second arrow denotes the surjection of Corollary 1.7. Since $x \in \overline{\mathbb{Q}}$, and the outer action of GT on Π_X preserves the cuspidal inertia subgroups of Π_X associated to ∞, it follows from Remark 1.7.2 that $(x_{\mathbb{B}})^\sigma \in \overline{\mathbb{Q}}$. Thus, to complete the proof of Corollary 2.3, it suffices to show that

> the natural action of σ on $D(\mathrm{GT})$ [cf. Corollary 1.6, (ii)] *descends* to a *natural action* of σ on the *quotient* $D(\mathrm{GT}) \twoheadrightarrow \overline{\mathbb{Q}} \cup \{\infty\}$ of Corollary 1.7,

i.e., that

$$(x_{\mathbb{B}})^\sigma = (x_{\mathbb{B}^\dagger})^\sigma \in \overline{\mathbb{Q}}$$

for any Belyi diagram $\mathbb{B}^\dagger$

$$\begin{array}{ccc} \Pi_{U^\dagger} & \longrightarrow & \Pi_X \\ \big\downarrow & & \\ \Pi_X & & \end{array}$$

such that $x \notin U^\dagger(\overline{\mathbb{Q}})$ [where we identify $U^\dagger$ with the image scheme of the open immersion $U^\dagger \hookrightarrow X$], and $x_{\mathbb{B}^\dagger} \mapsto x \in \overline{\mathbb{Q}}$ via the surjection $D(\mathrm{GT}) \twoheadrightarrow \overline{\mathbb{Q}}$ of Corollary 1.7. Write

- $X_x \stackrel{\mathrm{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, x, \infty\}$;
- $X_{(x_{\mathbb{B}})^\sigma} \stackrel{\mathrm{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, (x_{\mathbb{B}})^\sigma, \infty\}$;
- $X_{(x_{\mathbb{B}^\dagger})^\sigma} \stackrel{\mathrm{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, (x_{\mathbb{B}^\dagger})^\sigma, \infty\}$.

Next, by recalling the [right-hand square in the final display of the] proof of Corollary 1.6, (i), in the case where $J = \mathrm{GT}$, we obtain a commutative diagram of outer homomorphisms

$$\begin{array}{ccccc}
\Pi_{X_{(x_{\mathbb{B}})^{\sigma}}} & \stackrel{\sim}{\longleftarrow} & \Pi_{X_x} & \stackrel{\sim}{\longrightarrow} & \Pi_{X_{(x_{\mathbb{B}^\dagger})^{\sigma}}} \\
\downarrow & & \downarrow & & \downarrow \\
\Pi_X & \underset{\sigma}{\stackrel{\sim}{\longleftarrow}} & \Pi_X & \underset{\sigma}{\stackrel{\sim}{\longrightarrow}} & \Pi_X,
\end{array}$$

where the vertical arrows are the outer surjections induced by the natural open immersions $X_x \hookrightarrow X$, $X_{(x_{\mathbb{B}})^{\sigma}} \hookrightarrow X$, $X_{(x_{\mathbb{B}^\dagger})^{\sigma}} \hookrightarrow X$ of hyperbolic curves; the horizontal arrows are outer isomorphisms of topological groups. Since $\sigma \in \mathrm{GT}_p^{\mathrm{tp}}$, by recalling the [construction of the diagram in the final display of the] proof of Corollary 1.6, (i), in the case where $J = \mathrm{GT}$, we conclude that the above commutative diagram is induced by the following tempered version of the above commutative diagram

$$\begin{array}{ccccc}
\Pi^{\mathrm{tp}}_{X_{(x_{\mathbb{B}})^{\sigma}}} & \stackrel{\sim}{\longleftarrow} & \Pi^{\mathrm{tp}}_{X_x} & \stackrel{\sim}{\longrightarrow} & \Pi^{\mathrm{tp}}_{X_{(x_{\mathbb{B}^\dagger})^{\sigma}}} \\
\downarrow & & \downarrow & & \downarrow \\
\Pi^{\mathrm{tp}}_X & \underset{\sigma}{\stackrel{\sim}{\longleftarrow}} & \Pi^{\mathrm{tp}}_X & \underset{\sigma}{\stackrel{\sim}{\longrightarrow}} & \Pi^{\mathrm{tp}}_X,
\end{array}$$

where Π^{tp}_X (respectively, $\Pi^{\mathrm{tp}}_{X_{(x_{\mathbb{B}})^{\sigma}}}$, $\Pi^{\mathrm{tp}}_{X_{(x_{\mathbb{B}^\dagger})^{\sigma}}}$) denotes the tempered fundamental group of the base extension of X_x (respectively, $X_{(x_{\mathbb{B}})^{\sigma}}$, $X_{(x_{\mathbb{B}^\dagger})^{\sigma}}$) by the embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$; the vertical arrows are the outer surjections induced by the natural open immersions $X_x \hookrightarrow X$, $X_{(x_{\mathbb{B}})^{\sigma}} \hookrightarrow X$, $X_{(x_{\mathbb{B}^\dagger})^{\sigma}} \hookrightarrow X$ of hyperbolic curves; the horizontal arrows are outer isomorphisms of topological groups. Note, moreover, that it follows from the surjectivity [cf. [André], the discussion of §4.5] of the vertical arrows in the diagram of the preceding display that the inner automorphism indeterminacies in this diagram may be eliminated in a consistent fashion. Thus, by applying Theorem 2.2 [in the case where "ϕ" is taken to be the identity morphism], we conclude that $(x_{\mathbb{B}})^{\sigma} = (x_{\mathbb{B}^\dagger})^{\sigma} \in \overline{\mathbb{Q}}$. This completes the proof of Corollary 2.3. □

Corollary 2.4. *One may construct a surjection* $\mathrm{GT}_p^{\mathrm{tp}} \twoheadrightarrow G_{\mathbb{Q}_p}$ *whose restriction to* $G_{\mathbb{Q}_p}$ *[cf. Remark 2.1.2] is the identity automorphism.*

Proof. We continue to use the notation $X = \mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$, $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ of the proof of Corollary 2.3. Write $Y \stackrel{\mathrm{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}}$. [Thus, $X \subseteq Y$ is an open subscheme of Y.] It suffices to show that the action of $\mathrm{GT}_p^{\mathrm{tp}}$ on the set $\overline{\mathbb{Q}}$ ($\subseteq \overline{\mathbb{Q}} \cup \{\infty\} = Y(\overline{\mathbb{Q}})$) [cf. Corollary 2.3] is compatible with the field structure of $\overline{\mathbb{Q}}$ and the p-adic topology of $\overline{\mathbb{Q}}$ induced by the embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Fix $\sigma \in \mathrm{GT}_p^{\mathrm{tp}} \subseteq \mathrm{GT}$.

First, we verify the compatibility with the field structure of $\overline{\mathbb{Q}}$. We begin by verifying the following assertion:

> Claim 2.4.A: The action of $\mathrm{GT}_p^{\mathrm{tp}}$ on the set $Y(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}} \cup \{\infty\}$ induced by the action of $\mathrm{GT}_p^{\mathrm{tp}}$ on the set $\overline{\mathbb{Q}}$ commutes with the natural action of $\mathrm{Aut}_{\overline{\mathbb{Q}}}(X)$ [i.e., the group of scheme-theoretic automorphisms of X over $\overline{\mathbb{Q}}$] on the set $Y(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}} \cup \{\infty\}$.

Recall that every element of $\mathrm{GT}_p^{\mathrm{tp}}$ commutes with the outomorphisms of Π_X induced by elements of $\mathrm{Aut}_{\overline{\mathbb{Q}}}(X)$ [cf. [CmbCsp], Definition 1.11, (i); [CmbCsp], Remark 1.11.1]. Thus, Claim 2.4.A follows immediately from the definition of the action of $\mathrm{GT}_p^{\mathrm{tp}}$ on $\overline{\mathbb{Q}}$ in the proof of Corollary 2.3 via the action discussed in the proof of Corollary 1.6, (i), (ii) [cf., especially, the right-hand vertical isomorphism in the final display of the proof of Corollary 1.6, (i)].

Next, we verify the following assertion:

> Claim 2.4.B: Suppose that
>
> (∗) the action of $\mathrm{GT}_p^{\mathrm{tp}}$ on the set $\overline{\mathbb{Q}}^\times \stackrel{\mathrm{def}}{=} \overline{\mathbb{Q}} \setminus \{0\}$ is compatible with the multiplicative group structure of $\overline{\mathbb{Q}}^\times$.
>
> Then the action of $\mathrm{GT}_p^{\mathrm{tp}}$ on the set $\overline{\mathbb{Q}}$ is compatible with the field structure of $\overline{\mathbb{Q}}$.

Indeed, suppose that (∗) holds. Since $-1 \in \overline{\mathbb{Q}}$ may be characterized as the unique element $x \in \overline{\mathbb{Q}} \setminus \{1\}$ such that $x^2 = 1$, we conclude that σ preserves $-1 \in \overline{\mathbb{Q}}$. Let $a, b \in \overline{\mathbb{Q}}^\times$. Then $a + b = a \cdot (1 - ((-1) \cdot a^{-1} \cdot b))$. Since the action of σ commutes with the action of the automorphism of X over $\overline{\mathbb{Q}}$ given [relative to the standard coordinate "t" on $Y = \mathbb{P}^1_{\overline{\mathbb{Q}}}$] by $t \mapsto 1 - t$ [cf. Claim 2.4.A], we obtain the desired conclusion. This completes the proof of Claim 2.4.B.

Thus, by Claim 2.4.B, it suffices to show that (∗) holds. Let $x, y \in \overline{\mathbb{Q}}^\times \setminus \{1\}$; $\mathbb{B}^{\rtimes}$ an arithmetic Belyi diagram [in the case where N is a normal open subgroup of $J = \mathrm{GT}$]

$$\begin{array}{ccc} \Pi_U \overset{\mathrm{out}}{\rtimes} N & \longrightarrow & \Pi_X \overset{\mathrm{out}}{\rtimes} N \\ \big\downarrow & & \\ \Pi_X \overset{\mathrm{out}}{\rtimes} N & & \end{array}$$

such that $x^{-1}, y \notin U(\overline{\mathbb{Q}})$, where we identify U with the image scheme of the open immersion $U \hookrightarrow X$. Write

$$U_x \subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, x, \infty\} \subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, x, \infty\}$$

for the image scheme of the composite of the open immersion $U \hookrightarrow X$ with the isomorphism $X \xrightarrow{\sim} \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, x, \infty\}$ induced by multiplication by x. Thus, we obtain an arithmetic Belyi diagram $\mathbb{B}_x^{\rtimes}$

$$\begin{array}{ccc} \Pi_{U_x} \overset{\text{out}}{\rtimes} N & \longrightarrow & \Pi_X \overset{\text{out}}{\rtimes} N \\ \Big\downarrow & & \\ \Pi_X \overset{\text{out}}{\rtimes} N, & & \end{array}$$

where the horizontal arrow $\Pi_{U_x} \overset{\text{out}}{\rtimes} N \to \Pi_X \overset{\text{out}}{\rtimes} N$ denotes the Π_X-outer homomorphism induced by the composite of inclusions

$$U_x \subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, 1, x, \infty\} \subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, 1, \infty\} = X;$$

the vertical arrow $\Pi_{U_x} \overset{\text{out}}{\rtimes} N \to \Pi_X \overset{\text{out}}{\rtimes} N$ denotes the composite of the vertical arrow

$$\Pi_U \overset{\text{out}}{\rtimes} N \to \Pi_X \overset{\text{out}}{\rtimes} N$$

in the arithmetic Belyi diagram $\mathbb{B}^{\rtimes}$ with an isomorphism

$$\mu_{x^{-1}} : \Pi_{U_x} \overset{\text{out}}{\rtimes} N \xrightarrow{\sim} \Pi_U \overset{\text{out}}{\rtimes} N$$

over N induced by the natural scheme-theoretic isomorphism $U_x \xrightarrow{\sim} U$.

Next, by recalling the right-hand square in the final display of the proof of Corollary 1.6, (i), in the case where $N = M \subseteq J = \mathrm{GT}$, we obtain commutative diagrams of outer homomorphisms of profinite groups

$$\begin{array}{ccc} \Pi_U \overset{\text{out}}{\rtimes} N & \longrightarrow & \Pi_X \overset{\text{out}}{\rtimes} N \\ \sigma \Big\downarrow \wr & & \sigma \Big\downarrow \wr \\ \Pi_{U^\sigma} \overset{\text{out}}{\rtimes} N & \longrightarrow & \Pi_X \overset{\text{out}}{\rtimes} N, \end{array}$$

$$\begin{array}{ccc} \Pi_{U_x} \overset{\text{out}}{\rtimes} N & \longrightarrow & \Pi_X \overset{\text{out}}{\rtimes} N \\ \sigma \Big\downarrow \wr & & \sigma \Big\downarrow \wr \\ \Pi_{(U_x)^\sigma} \overset{\text{out}}{\rtimes} N & \longrightarrow & \Pi_X \overset{\text{out}}{\rtimes} N. \end{array}$$

Write

$$(U_x)^\sigma_{(x^\sigma)^{-1}} \subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, 1, (x^\sigma)^{-1}, \infty\} \subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, (x^\sigma)^{-1}, \infty\}$$

for the image scheme of the composite of the open immersion $(U_x)^\sigma \hookrightarrow X$ [cf. the proof of Corollary 1.6, (i)] with the isomorphism $X \xrightarrow{\sim} \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, (x^\sigma)^{-1}, \infty\}$

induced by multiplication by $(x^\sigma)^{-1}$. Note that there exists a natural $\Pi_{(U_x)^\sigma}$-outer isomorphism

$$\mu_{x^\sigma} : \Pi_{(U_x)^\sigma_{(x^\sigma)^{-1}}} \overset{\text{out}}{\rtimes} N \xrightarrow{\sim} \Pi_{(U_x)^\sigma} \overset{\text{out}}{\rtimes} N$$

over N induced by the natural scheme-theoretic isomorphism $(U_x)^\sigma_{(x^\sigma)^{-1}} \xrightarrow{\sim} (U_x)^\sigma$.

Thus, by taking the *composite of the* $\Pi_{(-)}$*-outer isomorphisms*

- $\mu_{x^\sigma} : \Pi_{(U_x)^\sigma_{(x^\sigma)^{-1}}} \overset{\text{out}}{\rtimes} N \xrightarrow{\sim} \Pi_{(U_x)^\sigma} \overset{\text{out}}{\rtimes} N$,
- the inverse of $\Pi_{U_x} \overset{\text{out}}{\rtimes} N \xrightarrow{\sim} \Pi_{(U_x)^\sigma} \overset{\text{out}}{\rtimes} N$ [cf. the second of the above two commutative diagrams],
- $\mu_{x^{-1}} : \Pi_{U_x} \overset{\text{out}}{\rtimes} N \xrightarrow{\sim} \Pi_U \overset{\text{out}}{\rtimes} N$, and
- $\Pi_U \overset{\text{out}}{\rtimes} N \xrightarrow{\sim} \Pi_{U^\sigma} \overset{\text{out}}{\rtimes} N$ [cf. the first of the above two commutative diagrams],

we obtain a Π_{U^σ}-outer isomorphism

$$\Pi_{(U_x)^\sigma_{(x^\sigma)^{-1}}} \overset{\text{out}}{\rtimes} N \xrightarrow{\sim} \Pi_{U^\sigma} \overset{\text{out}}{\rtimes} N$$

over N. Note that the conjugacy class of cuspidal inertia subgroups of $\Pi_{(U_x)^\sigma_{(x^\sigma)^{-1}}}$ associated to

$$0 \text{ (respectively, } 1,\ (x^\sigma)^{-1},\ (x^\sigma)^{-1}(xy)^\sigma,\ \infty)$$

maps, via the above *composite of* $\Pi_{(-)}$*-outer isomorphisms*, to the conjugacy classes of cuspidal inertia subgroups of $\Pi_{(-)}$ given as follows:

$$\begin{aligned} &\rightsquigarrow \quad 0 \text{ (respectively, } x^\sigma,\ 1,\ (xy)^\sigma,\ \infty) \\ &\rightsquigarrow \quad 0 \text{ (respectively, } x,\ 1,\ xy,\ \infty) \\ &\rightsquigarrow \quad 0 \text{ (respectively, } 1,\ x^{-1},\ y,\ \infty) \\ &\rightsquigarrow \quad 0 \text{ (respectively, } 1,\ (x^{-1})^\sigma,\ y^\sigma,\ \infty). \end{aligned}$$

Thus, by restricting to $G_{\mathbb{Q}} \subseteq \mathrm{GT} = J$ [cf. Corollary 1.7], we conclude that

$$(x^\sigma)^{-1}(xy)^\sigma = y^\sigma \ (\Leftrightarrow (xy)^\sigma = x^\sigma y^\sigma).$$

This completes the proof of $(*)$ and hence of the compatibility of the action of σ with the field structure of $\overline{\mathbb{Q}}$.

Next, we verify the compatibility with the p-adic topology of $\overline{\mathbb{Q}}$. Write

- X_x (respectively, X_{x^σ}) for $\mathbb{P}^1_{\mathbb{C}_p} \setminus \{0, 1, x, \infty\}$ (respectively, $\mathbb{P}^1_{\mathbb{C}_p} \setminus \{0, 1, x^\sigma, \infty\}$);
- $\Pi^{\mathrm{tp}}_{X_x}$ (respectively, $\Pi^{\mathrm{tp}}_{X_{x^\sigma}}$) for the tempered fundamental group of X_x (respectively, X_{x^σ}), relative to a suitable choice of basepoint;

- Γ_x (respectively, Γ_{x^σ}) for the dual semi-graph of the special fiber of the stable model of X_x (respectively, X_{x^σ});
- $V_x(y)$ (respectively, $V_{x^\sigma}(y)$) for the vertex of Γ_x (respectively, Γ_{x^σ}) to which the open edge determined by a cusp y of X_x (respectively, X_{x^σ}) abuts;
- $v_p : \overline{\mathbb{Q}}^\times \to \mathbb{Q}$ for the p-adic valuation normalized so that $v_p(p) = 1$.

Recall [cf. the upper horizontal isomorphisms in the final display of the proof of Corollary 2.3] that there exists an isomorphism of topological groups

$$\Pi^{\mathrm{tp}}_{X_x} \xrightarrow{\sim} \Pi^{\mathrm{tp}}_{X_{x^\sigma}}$$

such that the conjugacy class of cuspidal inertia subgroups associated to 0 (respectively, 1, x, ∞) maps to the conjugacy class of cuspidal inertia subgroups associated to 0 (respectively, 1, x^σ, ∞). Thus, by applying [SemiAn], Corollary 3.11, we conclude that the isomorphism of topological groups of the above display induces an isomorphism of semi-graphs $\Gamma_x \xrightarrow{\sim} \Gamma_{x^\sigma}$, and hence that

$$\begin{aligned} v_p(x) > 0 &\Leftrightarrow V_x(x) = V_x(0) \neq V_x(1) \\ &\Leftrightarrow V_{x^\sigma}(x^\sigma) = V_{x^\sigma}(0) \neq V_{x^\sigma}(1) \\ &\Leftrightarrow v_p(x^\sigma) > 0. \end{aligned}$$

This completes the proof of the compatibility of the action of σ with the p-adic topology of $\overline{\mathbb{Q}}$ and hence of Corollary 2.4. □

§3. Analogous results for stably $\times\mu$-indivisible fields

Write $\mathbb{Q}^{\mathrm{ab}} \subseteq \overline{\mathbb{Q}}$ [cf. Notations and Conventions] for the maximal abelian extension field of $\mathbb{Q}$, i.e., the subfield generated by the roots of unity $\in \overline{\mathbb{Q}}$. In this section, we begin by proving the injectivity portion of the Section Conjecture for abelian varieties over finite extensions of $\mathbb{Q}^{\mathrm{ab}}$ [cf. Theorem 3.1]. As a corollary, we obtain the injectivity portion of the Section Conjecture for hyperbolic curves over finite extensions of $\mathbb{Q}^{\mathrm{ab}}$ [cf. Corollary 3.2]. On the other hand, if we restrict to the case of the hyperbolic curves of genus 0, then we may prove [cf. Corollary 3.7] the injectivity portion of the Section Conjecture over a *stably p-$\times\mu$-indivisible* field [cf. Definition 3.3, (viii)] K by means of different techniques. Here, we note that the class of stably p-$\times\mu$-indivisible fields is much larger than the class of the finite extensions of $\mathbb{Q}^{\mathrm{ab}}$ [cf. Lemma 3.4]. Finally, we construct [cf. Corollary 3.9] a natural action of $C_{\mathrm{GT}}(G_K)$ [cf. Notations and Conventions] on the field of algebraic numbers. This construction is obtained as a consequence of Corollary 3.7.

Theorem 3.1. *Let $K \subseteq \overline{\mathbb{Q}}$ be a number field, i.e., a finite extension of $\mathbb{Q}$; A an abelian variety over K. Write $K^{\mathrm{cycl}} = K \cdot \mathbb{Q}^{\mathrm{ab}}$; $G_{K^{\mathrm{cycl}}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/K^{\mathrm{cycl}})$; $A(K^{\mathrm{cycl}})$ for the group of K^{cycl}-valued points of A; $A_{K^{\mathrm{cycl}}} \stackrel{\mathrm{def}}{=} A \times_K K^{\mathrm{cycl}}$; $A_{\overline{\mathbb{Q}}} \stackrel{\mathrm{def}}{=} A \times_K \overline{\mathbb{Q}}$. Then the natural map*

$$A(K^{\mathrm{cycl}}) \to H^1(G_{K^{\mathrm{cycl}}}, \Pi_{A_{\overline{\mathbb{Q}}}})$$

— i.e., obtained by taking the difference between the two sections of $\Pi_{A_{K^{\mathrm{cycl}}}} \twoheadrightarrow G_{K^{\mathrm{cycl}}}$ [each of which is well-defined up to composition with an inner automorphism induced by an element of $\Pi_{A_{\overline{\mathbb{Q}}}}$] induced by an element of $A(K^{\mathrm{cycl}})$ and the origin — is injective.

Proof. By considering the Kummer exact sequence for $A(K^{\mathrm{cycl}})$, we obtain natural maps

$$A(K^{\mathrm{cycl}}) \to \varprojlim_{n} A(K^{\mathrm{cycl}})/n \cdot A(K^{\mathrm{cycl}}) \hookrightarrow H^1(G_{K^{\mathrm{cycl}}}, \Pi_{A_{\overline{\mathbb{Q}}}}),$$

where the first map is the natural homomorphism; the second map is injective; the inverse limit is indexed by the positive integers, regarded multiplicatively. By a well-known general nonsense argument [cf., e.g., the proof of [Cusp], Proposition 2.2, (i)], it follows that the composite map of the above display coincides with the natural map in the statement of Theorem 3.1. Thus, it suffices to show that $A(K^{\mathrm{cycl}})$ has no divisible elements. But this follows immediately from [KLR], Appendix, Theorem 1, and [Moon], Proposition 7. This completes the proof of Theorem 3.1. □

Corollary 3.2. *Let $K \subseteq \overline{\mathbb{Q}}$ be a number field, i.e., a finite extension of $\mathbb{Q}$; Y a hyperbolic curve over K. Write $K^{\mathrm{cycl}} = K \cdot \mathbb{Q}^{\mathrm{ab}}$; $Y_{K^{\mathrm{cycl}}} \stackrel{\mathrm{def}}{=} Y \times_K K^{\mathrm{cycl}}$; $G_{K^{\mathrm{cycl}}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/K^{\mathrm{cycl}})$; $Y(K^{\mathrm{cycl}})$ for the set of K^{cycl}-valued points of Y; $Y_{\overline{\mathbb{Q}}} \stackrel{\mathrm{def}}{=} Y \times_K \overline{\mathbb{Q}}$; $\mathrm{Sect}(\Pi_{Y_{K^{\mathrm{cycl}}}} \twoheadrightarrow G_{K^{\mathrm{cycl}}})$ for the set of equivalence classes of sections of the natural surjection $\Pi_{Y_{K^{\mathrm{cycl}}}} \twoheadrightarrow G_{K^{\mathrm{cycl}}}$, where we consider two such sections to be equivalent if they differ by composition with an inner automorphism induced by an element of $\Pi_{Y_{\overline{\mathbb{Q}}}}$. Then the natural map*

$$Y(K^{\mathrm{cycl}}) \to \mathrm{Sect}(\Pi_{Y_{K^{\mathrm{cycl}}}} \twoheadrightarrow G_{K^{\mathrm{cycl}}})$$

is injective.

Proof. One verifies immediately that, by replacing Y by a suitable finite étale covering of Y, we may assume without loss of generality Y is of genus ≥ 1. Then the desired injectivity follows immediately from Theorem 3.1 by considering the Albanese embedding of Y. □

Remark 3.2.1. [Stix] discusses various results in the anabelian geometry of hyperbolic curves of genus 0 over the maximal cyclotomic extension of a number field. Note that, if we only consider hyperbolic curves of genus 0, then the injectivity portion of the Section Conjecture discussed in Corollary 3.2 follows immediately from [Stix], Theorem 63. On the other hand, it appears that the argument in the final paragraph [i.e., the paragraph in which Belyi's theorem [cf. [Belyi]] is applied] of the proof of [Stix], Theorem 63, is *incomplete*. In this final paragraph, Stix asserts that a contradiction could be derived by taking suitable connected finite étale coverings $U' \to U$ and $V' \to V$ whose existence follows from Belyi's theorem and considering open immersions $U' \hookrightarrow U''$ and $V' \hookrightarrow V''$ into hyperbolic curves U'' and V'' of type $(0, 4)$. However, even if one shows that U'' is isomorphic to V'', one cannot derive any conclusions concerning the relationship between U and V in the absence of *more detailed information* concerning the coverings $U' \to U$ and $V' \to V$. In the final paragraph of the proof of Theorem 3.5 below, we show how this problem may be resolved, under more general hypotheses than those of [Stix], Theorem 63, at least in the cases where one assumes [in the notation of *loc. cit.*] either *condition (A')* or *conditions (B) and (D)*.

Definition 3.3. Let p be a prime number, K a field, $f \in K$. Then:

(i) We shall say that f is *p-divisible* (respectively, *divisible*) if $f = 0$ or $f \in K^{\times p^\infty}$ (respectively, $f = 0$ or $f \in K^{\times\infty}$).

(ii) We shall say that f is *p-indivisible* (respectively, *indivisible*) if f is not p-divisible (respectively, not divisible).

(iii) We shall say that K is p-$\times$ (respectively, $\times$)-*indivisible* if $K^{\times p^\infty} = \{1\}$ (respectively, $K^{\times\infty} = \{1\}$).

(iv) We shall say that K is p-$\times\mu$ (respectively, $\times\mu$)-*indivisible* if $K^{\times p^\infty} \subseteq \mu(K)$ (respectively, $K^{\times\infty} \subseteq \mu(K)$).

(v) Let $\square \in \{p\text{-}\times, p\text{-}\times\mu, \times, \times\mu\}$. Then we shall say that K is *stably* $\square$-*indivisible* if, for every finite extension L of K, L is $\square$-indivisible.

(vi) We shall say that K is μ_{p^∞} (respectively, μ)-*finite* if $\mu_{p^\infty}(K)$ (respectively, $\mu(K)$) is a finite group.

(vii) We shall say that K is *stably* μ_{p^∞} (respectively, *stably* μ)-*finite* if, for every finite extension $K^\dagger$ of K, $\mu_{p^\infty}(K^\dagger)$ (respectively, $\mu(K^\dagger)$) is a finite group.

Remark 3.3.1. Let K be a field. Then K is stably $\times$-indivisible if and only if K is *torally Kummer-faithful*, in the sense of [AbsTopIII], Definition 1.5.

In the following, we fix a prime number p.

Lemma 3.4. *Let K be a field of characteristic $\neq p$.*

(i) If K is p-$\times$ (respectively, $\times$)-indivisible, then K is p-$\times\mu$ (respectively, $\times\mu$)-indivisible. Let $\Box \in \{\times\mu, \times\}$. If K is p-$\Box$-indivisible, then K is $\Box$-indivisible.

(ii) Let $\Box \in \{p\text{-}\times, p\text{-}\times\mu, \times, \times\mu\}$; L an extension field of K. Then if L is $\Box$-indivisible, then K is $\Box$-indivisible.

(iii) Suppose that K is a generalized sub-p-adic field (respectively, sub-p-adic field) [for example, a finite extension of $\mathbb{Q}$ or $\mathbb{Q}_p$ — cf. [AnabTop], Definition 4.11 (respectively, [LocAn], Definition 15.4, (i))]. Then K is stably p-$\times\mu$-indivisible (respectively, stably p-$\times\mu$-indivisible and stably $\times$-indivisible) and stably μ_{p^∞} (respectively, stably μ)-finite.

(iv) Suppose that K is stably μ_{p^∞} (respectively, stably μ)-finite. Let L be an (algebraic) abelian extension of K. Then if K is stably p-$\times\mu$ (respectively, stably $\times\mu$)-indivisible, then L is stably p-$\times\mu$ (respectively, stably $\times\mu$)-indivisible.

(v) Let L be a(n) (algebraic) Galois extension of K. Suppose that L is stably μ_{p^∞} (respectively, stably μ)-finite. Then if K is stably p-$\times\mu$ (respectively, stably $\times\mu$)-indivisible, then L is stably p-$\times\mu$ (respectively, stably $\times\mu$)-indivisible.

(vi) Let L be a(n) (algebraic) pro-prime-to-p Galois extension of K. Then if K is stably p-$\times\mu$-indivisible, then L is stably p-$\times\mu$-indivisible.

Proof. Assertions (i), (ii) follow immediately from the various definitions involved.

Next, we verify assertion (iii). First, we recall that every finite extension of a generalized sub-p-adic field (respectively, sub-p-adic field) is generalized sub-p-adic (respectively, sub-p-adic). Suppose that K is a generalized sub-p-adic (respectively, sub-p-adic) field. Then one verifies immediately, by using well-known properties of valuations on function fields that arise from geometric divisors, that we may assume without loss of generality that K is a finite extension of the quotient field F of the ring of Witt vectors associated to the algebraic closure of a finite field (respectively, to a finite field). Thus, there exists an embedding of topological fields $K \hookrightarrow \mathbb{C}_p$. Then it follows immediately, by considering the p-adic logarithm on the group of units of the ring of integers of $\mathbb{C}_p$ [cf. [Kobl], p.81], together with the fact that the *ramification index* of K over F is *finite* [which implies that the image of the p-adic logarithm on the group of units of the ring of integers of K is *bounded*], that K is *p-$\times\mu$-indivisible.* Moreover, it follows immediately, by considering well-known ramification properties of cyclotomic extensions [cf. [Neu], Chapter I, Lemma 10.1] (respectively, the well-known structure of the multiplicative group of a finite extension of $\mathbb{Q}_p$ [cf. [Neu], Chapter II, Proposition 5.7, (i)]), that K is μ_{p^∞} (respectively, μ)-finite, and $K^{\times\infty} = \{1\}$. This completes the proof of assertion (iii).

In the remainder of the proof, we *fix* an algebraic closure $\overline{K}$ of K. Next, we verify assertion (iv). By replacing K by a suitable finite extension of K, we conclude that it suffices to verify that L is *p-×μ-indivisible* (respectively, *×μ-indivisible*). Then it follows immediately from assertion (ii) that we may assume without loss of generality that

$$\mu(L) = \mu(\overline{K}), \quad L \subseteq \overline{K}.$$

Let

$$f \in L^{\times p^\infty} \text{ (respectively, } f \in L^{\times\infty}).$$

Then, by replacing K by a suitable finite extension of K, we may assume without loss of generality that

$$f \in K.$$

Write

- $M \stackrel{\text{def}}{=} K(f^{\frac{1}{p^\infty}}) \subseteq L$ (respectively, $M \stackrel{\text{def}}{=} K(f^{\frac{1}{\infty}}) \subseteq L$) for the subfield generated over K by the set of all p-power roots (respectively, all roots) of f [so L and M are *abelian* extensions of K, $\mu_{p^\infty}(M) = \mu_{p^\infty}(L) = \mu_{p^\infty}(\overline{K})$ (respectively, $\mu_\infty(M) = \mu_\infty(L) = \mu_\infty(\overline{K})$)];
- $G_K \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{K}/K)$, $G \stackrel{\text{def}}{=} \mathrm{Gal}(M/K)$;
- $\Lambda \stackrel{\text{def}}{=} \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \mu(L))$ (respectively, $\Lambda \stackrel{\text{def}}{=} \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mu(L))$) [so G acts naturally on Λ ($\cong \mathbb{Z}_p$ (respectively, $\widehat{\mathbb{Z}}$))];
- $\kappa : K^\times \to H^1(G_K, \Lambda)$ for the Kummer map;
- $G_\Lambda \subseteq \mathrm{Aut}(\Lambda)$ for the image of the natural homomorphism $G \to \mathrm{Aut}(\Lambda)$.

Consider the profinite étale covering

$$\mathrm{Spec}\ \mathbb{Q}[T^{\frac{1}{p^\infty}}] \to \mathrm{Spec}\ \mathbb{Q}[T] \quad \text{(respectively, } \mathrm{Spec}\ \mathbb{Q}[T^{\frac{1}{\infty}}] \to \mathrm{Spec}\ \mathbb{Q}[T]),$$

where T denotes an indeterminate element, and $T^{\frac{1}{p^\infty}}$ (respectively, $T^{\frac{1}{\infty}}$) denotes the set of all p-power roots (respectively, all roots) of T in some algebraic closure of the fraction field of $\mathbb{Q}[T]$. Then since Spec L is isomorphic, over Spec K, to a connected component of the pull-back of this profinite étale covering via the morphism Spec $K \to$ Spec $\mathbb{Q}[T]$ that maps $T \mapsto f$, we conclude that there exists a natural [outer] injection

$$\xi : G \hookrightarrow \Lambda \rtimes G_\Lambda,$$

whose image we denote by G_ξ. Write $N \stackrel{\text{def}}{=} G_\xi \cap \Lambda \subseteq \Lambda \rtimes G_\Lambda$. Thus, we obtain an exact sequence of profinite groups

$$1 \longrightarrow N \longrightarrow G \longrightarrow G_\Lambda \longrightarrow 1.$$

If $N \neq \{1\}$, then it follows immediately from the definition of G_Λ, together with the assumption that K is μ_{p^∞} (respectively, μ)-finite, that G is non-abelian. Since G is abelian, we thus conclude that $N = \{1\}$, hence that $G \xrightarrow{\sim} G_\Lambda$. Next, we observe that $\kappa(f)$ is contained in the image of the natural restriction map

$$(H^1(G_\Lambda, \Lambda) \xrightarrow{\sim})\ H^1(G, \Lambda) \to H^1(G_K, \Lambda).$$

Moreover, one verifies easily that our assumption that K is μ_{p^∞} (respectively, μ)-finite implies that the first cohomology group $H^1(G_\Lambda, \Lambda)$ is isomorphic to a finite quotient of Λ. Thus, we conclude that some positive power of f is contained in

$$\mathrm{Ker}(\kappa) = K^{\times p^\infty} \quad (\text{respectively, } \mathrm{Ker}(\kappa) = K^{\times\infty}).$$

On the other hand, our assumption that K is *p-$\times\mu$-indivisible* (respectively, *$\times\mu$-indivisible*) then implies that $f \in \mu(K) \subseteq \mu(L)$. This completes the proof of assertion (iv).

Next, we verify assertion (v). By replacing K by a suitable finite extension of K, we conclude that it suffices to verify that L is *p-$\times\mu$-indivisible* (respectively, *$\times\mu$-indivisible*). Let

$$f \in L^{\times p^\infty} \quad (\text{respectively, } f \in L^{\times\infty}).$$

Then, by replacing K by a suitable finite extension of K, we may assume without loss of generality that

$$f \in K, \quad L \subseteq \overline{K}.$$

Write

- $K^\infty \stackrel{\mathrm{def}}{=} K(\mu_{p^\infty}(\overline{K}))$ (respectively, $K^\infty \stackrel{\mathrm{def}}{=} K(\mu(\overline{K}))$;
- $L^\infty \stackrel{\mathrm{def}}{=} K^\infty \cdot L$;
- $f^{\frac{1}{p^\infty}} \subseteq L^\infty$ (respectively, $f^{\frac{1}{\infty}} \subseteq L^\infty$) for the set of all p-power roots (respectively, all roots) of f;
- $\Lambda \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \mu(L^\infty))$ (respectively, $\Lambda \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mu(L^\infty)))$ [so $\mathrm{Gal}(L^\infty/K)$ acts naturally on Λ ($\cong \mathbb{Z}_p$ (respectively, $\widehat{\mathbb{Z}}$))];
- $G_\Lambda \subseteq \mathrm{Aut}(\Lambda)$ for the image of the natural homomorphism $\mathrm{Gal}(L^\infty/K) \to \mathrm{Aut}(\Lambda)$.

Since K is μ_{p^∞} (respectively, μ)*-finite*, and K^∞ is an *abelian* extension of K, by applying assertion (iv), we conclude that K^∞ is *stably* p-$\times\mu$ (respectively, *stably* $\times\mu$)*-indivisible*. In particular, by assertion (ii), $K^\infty \cap L$ is *stably* p-$\times\mu$ (respectively, *stably* $\times\mu$)*-indivisible*. Thus, by replacing K by $K^\infty \cap L$, we may assume without loss of generality that

$$K = K^\infty \cap L.$$

In particular, we obtain a natural direct product decomposition

$$\mathrm{Gal}(L^{\infty}/K) = \mathrm{Gal}(L^{\infty}/K^{\infty}) \times \mathrm{Gal}(L^{\infty}/L).$$

On the other hand, by a similar argument to the argument given in the proof of assertion (iv), we conclude that the natural action of $\mathrm{Gal}(L^{\infty}/K)$ on $f^{\frac{1}{p^{\infty}}} \subseteq L^{\infty}$ (respectively, $f^{\frac{1}{\infty}} \subseteq L^{\infty}$) determines a natural [outer] homomorphism

$$\xi : \mathrm{Gal}(L^{\infty}/K) \to \Lambda \rtimes G_{\Lambda}$$

such that $H \stackrel{\mathrm{def}}{=} \xi(\mathrm{Gal}(L^{\infty}/K^{\infty})) \subseteq \Lambda \subseteq \Lambda \rtimes G_{\Lambda}$. Write $J \stackrel{\mathrm{def}}{=} \xi(\mathrm{Gal}(L^{\infty}/L))$. Note that the fact that L is *stably* $\mu_{p^{\infty}}$ (respectively, *stably* μ)*-finite* implies that $Z_{\Lambda \rtimes G_{\Lambda}}(J) \cap \Lambda = \{1\}$, hence that $H \subseteq Z_{\Lambda \rtimes G_{\Lambda}}(J) \cap \Lambda = \{1\}$, i.e., [cf. the definition of H and ξ] that

$$f^{\frac{1}{p^{\infty}}} \subseteq K^{\infty} \text{ (respectively, } f^{\frac{1}{\infty}} \subseteq K^{\infty}).$$

Thus, since K^{∞} is *stably* p-$\times\mu$ (respectively, *stably* $\times\mu$)*-indivisible*, we conclude that $f \in \mu(K^{\infty}) \cap K = \mu(K) \subseteq \mu(L)$. This completes the proof of assertion (v).

Finally, we verify assertion (vi). By applying assertion (iv), we may assume without loss of generality that

$$\mu_{p^{\infty}}(K) = \mu_{p^{\infty}}(\overline{K}), \quad L \subseteq \overline{K}.$$

Moreover, by replacing K by a suitable finite extension of K, we conclude that it suffices to verify that L is p-$\times\mu$*-indivisible*. Let

$$f \in L^{\times p^{\infty}}.$$

Then we may assume without loss of generality that

$$f \in K.$$

Write

$$M \stackrel{\mathrm{def}}{=} K(f^{\frac{1}{p^{\infty}}}) \subseteq L$$

for the subfield generated over K by the set of all p-power roots of f. Since $\mu_{p^{\infty}}(K) = \mu_{p^{\infty}}(\overline{K})$, L and M are *pro-prime-to-p Galois* extensions of K. On the other hand, since M, by definition, is a *pro-p Galois* extension of K, we thus conclude that $K = M$, hence that $f \in K^{\times p^{\infty}}$. Thus, our assumption that K is p-$\times\mu$*-indivisible* implies that $f \in \mu(K) \subseteq \mu(L)$. This completes the proof of assertion (vi), hence of Lemma 3.4. □

Remark 3.4.1. Let K_0 be a generalized sub-p-adic field [for example, a finite extension of $\mathbb{Q}$ or $\mathbb{Q}_p$]; n a positive integer ≥ 2;

$$K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n$$

field extensions of K_0. Suppose that

- for each $i = 1, \ldots, n-2$, K_i is a *Galois* extension of K_{i-1};
- K_{n-2} is *stably* μ_{p^∞}*-finite*;
- K_{n-1} is an *abelian* extension of K_{n-2};
- K_n is a *pro-prime-to-p Galois* extension of K_{n-1}.

Then it follows immediately from Lemma 3.4, (i), (iii), (iv), (v), (vi), that the field K_n is *stably p-*$\times\mu$*-indivisible*, hence also *stably* $\times\mu$*-indivisible*.

Theorem 3.5. *Let K be a stably p-$\times\mu$ (respectively, $\times\mu$)-indivisible field of characteristic* 0*; $\overline{K}$ an algebraic closure of K. Write $G_K \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{K}/K)$. Let U and V be hyperbolic curves of genus* 0 *over K;*

$$\phi : \Pi_U \xrightarrow{\sim} \Pi_V$$

an isomorphism of profinite groups such that ϕ lies over the identity automorphism on G_K. We consider the following conditions:

(a) ϕ induces a bijection between the cuspidal inertia subgroups of Π_U and the cuspidal inertia subgroups of Π_V.

(b) Let $I \subseteq \Pi_U$ be a cuspidal inertia subgroup of Π_U. Consider the natural composite

$$\widehat{\mathbb{Z}}(1) \xrightarrow{\sim} I \xrightarrow{\sim} \phi(I) \xleftarrow{\sim} \widehat{\mathbb{Z}}(1)$$

— where "(1)" denotes the Tate twist; the first and final isomorphisms are the natural isomorphisms [obtained by considering the action of each cuspidal inertia subgroup on the roots of a uniformizer of the local ring of the compactified curve at the cusp under consideration]; the middle isomorphism is the isomorphism induced by ϕ. Then this natural composite is the identity automorphism.

Suppose that condition (a) holds (respectively, conditions (a), (b) hold). Then there exists an isomorphism of K-schemes

$$U \xrightarrow{\sim} V$$

that induces a bijection between the cusps of U and V which is compatible with the bijection between cuspidal inertia groups of Π_U and Π_V induced by ϕ.

Proof. First, we observe that the fact U and V are curves of genus 0 implies that, if $K^\dagger$ is a finite Galois extension of K over which the cusps of U and V become rational, then any isomorphism of $K^\dagger$-schemes $U \times_K K^\dagger \xrightarrow{\sim} V \times_K K^\dagger$ descends to an isomorphism of K-schemes $U \xrightarrow{\sim} V$ if and only if it is equivariant with respect to the respective $\mathrm{Gal}(K^\dagger/K)$-actions on the cusps of $U \times_K K^\dagger$ and $V \times_K K^\dagger$. In particular, we may assume without loss of generality that all cusps of U and V are K-rational. Thus, since ϕ preserves the cuspidal inertia subgroups, it follows immediately, by considering the quotients of Π_U and Π_V by the closed normal subgroups topologically generated by suitable collections of cuspidal inertia subgroups, that we may also assume without loss of generality that

- $U = \mathbb{P}^1_K \setminus \{0, 1, \lambda, \infty\}$, where $\lambda \in K \setminus \{0, 1\}$;
- $V = \mathbb{P}^1_K \setminus \{0, 1, \mu, \infty\}$, where $\mu \in K \setminus \{0, 1\}$;
- ϕ maps the cuspidal inertia subgroups of Π_U associated to $* \in \{0, 1, \infty\}$ to the cuspidal inertia subgroups of Π_V associated to $*$. [Note that this implies that ϕ maps the cuspidal inertia subgroups of Π_U associated to λ to the cuspidal inertia subgroups of Π_V associated to μ.]

Then our goal is to prove that

$$\lambda = \mu.$$

Write t for the standard coordinate [i.e., rational function] on $\mathbb{P}^1_K$;

$$\Delta_U \stackrel{\text{def}}{=} \Pi_{U \times_K \overline{K}}, \quad \Delta_V \stackrel{\text{def}}{=} \Pi_{V \times_K \overline{K}}.$$

Next, we verify the following assertion:

Claim 3.5.A: Let $* \in \{0, 1, \lambda, \infty\}$; $I_* \subseteq \Pi_U$ a cuspidal inertia subgroup associated to $*$. Consider the natural composite

$$h_* : \mathbb{Z}_p(1) \xrightarrow{\sim} I_*^p \xrightarrow{\sim} \phi(I_*)^p \xleftarrow{\sim} \mathbb{Z}_p(1)$$

— where $(-)^p$ denotes the maximal pro-p quotient of $(-)$; "(1)" denotes the Tate twist; the first and final isomorphisms are the natural isomorphisms [obtained by considering the action of each cuspidal inertia subgroup on the roots of a uniformizer of the local ring of the compactified curve at the cusp under consideration]; the middle isomorphism is the isomorphism induced by ϕ. Then h_* is the identity automorphism.

First, we note that, under condition (b), Claim 3.5.A is immediate. Thus, we may assume without loss of generality that K is *stably p-$\times\mu$-indivisible*. Since ϕ preserves the cuspidal inertia subgroups, it follows immediately, by considering suitable quotients of the abelianizations of Δ_U and Δ_V, that $h_0 = h_1 = h_\lambda = h_\infty$. Thus, it suffices to consider the case where $* = 1$. Write

- $(\mathbb{P}^1_K \supseteq)\ \ddot{U} \to U\ (\subseteq \mathbb{P}^1_K)$ for the connected finite étale covering of U of degree 2 determined by $t \mapsto (1-t)^2$.
- $(\mathbb{P}^1_K \supseteq)\ \ddot{V} \to V\ (\subseteq \mathbb{P}^1_K)$ for the connected finite étale covering of V of degree 2 determined by $t \mapsto (1-t)^2$.

Note that the open subgroup $\Delta_{\ddot{U}} \subseteq \Delta_U$ determined by the covering $\ddot{U} \to U$ may be characterized as the unique open subgroup of index 2 such that

$$I_1 \subseteq \Delta_{\ddot{U}}, \quad I_\lambda \subseteq \Delta_{\ddot{U}}.$$

The open subgroup $\Delta_{\ddot{V}} \subseteq \Delta_V$ determined by the covering $\ddot{V} \to V$ admits a similar characterization. Thus, since ϕ is compatible with these characterizations, we conclude that, after possibly replacing K by a suitable finite extension of K and ϕ by the composite of ϕ with the inner automorphism of Π_V determined by some element $\in \Delta_V$, we obtain an isomorphism of profinite groups

$$\ddot{\psi} : \Pi_{\ddot{U}} \xrightarrow{\sim} \Pi_{\ddot{V}}$$

such that

- $\ddot{\psi}$ induces the identity automorphism on G_K,
- $\ddot{\psi}$ maps the cuspidal inertia subgroups of $\Pi_{\ddot{U}}$ associated to $\ddot{*} \in \{0, 1, 2, \infty\}$ to the cuspidal inertia subgroups of $\Pi_{\ddot{V}}$ associated to $\ddot{*}$.

Let $\ddot{I}_2$ be a cuspidal inertia subgroup of $\Pi_{\ddot{U}}$ associated to 2. Thus, since the cusp 2 of $\ddot{U}$ maps to the cusp 1 of U, we may assume without loss of generality that $\ddot{I}_2 = I_1 \subseteq \Pi_U$. In particular, it suffices to prove that the natural composite

$$\mathbb{Z}_p(1) \xrightarrow{\sim} \ddot{I}_2^p \xrightarrow{\sim} \ddot{\psi}(\ddot{I}_2)^p \xleftarrow{\sim} \mathbb{Z}_p(1)$$

is the identity automorphism. Write

- $\ddot{\epsilon} \in \mathbb{Z}_p^\times$ for the element determined by this automorphism;
- $\kappa : K^\times \twoheadrightarrow K^\times / K^{\times p^\infty} \hookrightarrow H^1(G_K, \mathbb{Z}_p(1))$ for the Kummer map;
- $Y \stackrel{\text{def}}{=} \mathbb{P}^1_K \setminus \{0, \infty\}$, $\Delta_Y \stackrel{\text{def}}{=} \Pi_{Y \times_K \overline{K}}$.

Recall that by a well-known general nonsense argument [cf., e.g., the proof of [Cusp], Proposition 2.2, (i)], κ coincides with the composite

$$K^\times = Y(K) \to H^1(G_K, \Delta_Y) \to H^1(G_K, \mathbb{Z}_p(1))$$

— where the first map is obtained by taking the difference between the two sections of $\Pi_Y \twoheadrightarrow G_K$ [each of which is well-defined up to composition with an inner automorphism induced by an element of Δ_Y] induced by an element of $Y(K)$ and $1 \in Y(K)$; the final map is induced by the natural surjection $\Delta_Y \twoheadrightarrow \Delta_Y^p \xrightarrow{\sim} \mathbb{Z}_p(1)$.

Here, we recall that the image of such a section of $\Pi_Y \twoheadrightarrow G_K$ arising from an element of $Y(K)$ may also be thought of as the decomposition group in Π_Y of this element of $Y(K)$.

Next, let $\ddot{*} \in \{1, 2\}$; $\ddot{I}_{\ddot{*}}$ a cuspidal inertia subgroup of $\Delta_{\ddot{U}}$ associated to $\ddot{*}$. Recall that, since $\ddot{I}_{\ddot{*}}$ is normally terminal in $\Delta_{\ddot{U}}$ [cf. [CmbGC], Proposition 1.2, (ii)], the normalizer $N_{\Pi_{\ddot{U}}}(\ddot{I}_{\ddot{*}})$ is a decomposition subgroup $\subseteq \Pi_{\ddot{U}}$ associated to $\ddot{*}$. Similarly, since $\ddot{\psi}(\ddot{I}_{\ddot{*}})$ is normally terminal in $\Delta_{\ddot{V}}$, the normalizer $N_{\Pi_{\ddot{V}}}(\ddot{\psi}(\ddot{I}_{\ddot{*}}))$ is a decomposition subgroup $\subseteq \Pi_{\ddot{V}}$ associated to $\ddot{*}$.

Thus, since $\ddot{\psi}$ maps the cuspidal inertia subgroups of $\Pi_{\ddot{U}}$ associated to $\ddot{*}$ to the cuspidal inertia subgroups of $\Pi_{\ddot{V}}$ associated to $\ddot{*}$, we conclude [by thinking of $\ddot{U}$ and $\ddot{V}$ as open subschemes of Y] that

$$\ddot{\epsilon} \cdot \kappa(2) = \kappa(2).$$

On the other hand, our assumption that K is *stably* p-$\times\mu$-*indivisible* implies that the torsion subgroup of $K^\times/K^{\times p^\infty}$ coincides with the subgroup $\mu(K)/K^{\times p^\infty}$. Thus, we conclude that $\kappa(2)$ is not a torsion element, hence that $\mathbb{Z}_p \cdot \kappa(2) \xrightarrow{\sim} \mathbb{Z}_p$, which implies that $\ddot{\epsilon} = 1$. This completes the proof of Claim 3.5.A.

Next, we suppose that

$$\lambda \neq \mu.$$

Then it follows immediately, in light of Claim 3.5.A (respectively, condition (b)), by considering the Kummer classes of λ, μ, $1-\lambda$, and $1-\mu$, together with our assumption that K is *stably* p-$\times\mu$ (respectively, *stably* $\times\mu$)-*indivisible*, that there exist $a, b \in \mu(K)$ such that

$$\mu = a \cdot \lambda, \quad 1 - \mu = b \cdot (1 - \lambda).$$

Since $\lambda \neq \mu$, it follows immediately that $a \neq 1$, $b \neq 1$, and $a \neq b$. In particular,

$$\lambda = \frac{1-b}{a-b} \in \mathbb{Q}^\infty,$$

where $\mathbb{Q}^\infty \stackrel{\text{def}}{=} \mathbb{Q}(\mu(\overline{K})) \subseteq \overline{K}$. [Here, we recall that the characteristic of K is 0.] Since the characteristic of K is 0, if λ is a root of unity, then, by replacing λ by $1-\lambda$, we may assume without loss of generality that $\lambda \notin \mu(\overline{K})$. Thus, by applying Lemma 3.4, (iii), (iv), we conclude that $\lambda \notin (\mathbb{Q}^\infty)^{\times\infty}$. Let n be a positive integer such that some n-th root of $\lambda \notin \mathbb{Q}^\infty$. Fix such an element

$$\lambda^{\frac{1}{n}} \notin \mathbb{Q}^\infty.$$

Write

- $(\mathbb{P}^1_K \supseteq)\ U' \to U\ (\subseteq \mathbb{P}^1_K)$ for the connected finite étale covering of U of degree n determined by $t \mapsto t^n$.
- $(\mathbb{P}^1_K \supseteq)\ V' \to V\ (\subseteq \mathbb{P}^1_K)$ for the connected finite étale covering of V of degree n determined by $t \mapsto t^n$.

Note that the open subgroup $\Delta_{U'} \subseteq \Delta_U$ determined by the covering $U' \to U$ may be characterized as the unique normal open subgroup of index n such that

$$I_1 \subseteq \Delta_{U'}, \quad I_\lambda \subseteq \Delta_{U'}.$$

The open subgroup $\Delta_{V'} \subseteq \Delta_V$ determined by the covering $V' \to V$ admits a similar characterization. Thus, since ϕ is compatible with these characterizations, we conclude that, after possibly replacing K by a suitable finite extension of K and ϕ by the composite of ϕ with the inner automorphism of Π_V determined by some element $\in \Delta_V$, we obtain an isomorphism of profinite groups

$$\phi_n : \Pi_{U'} \xrightarrow{\sim} \Pi_{V'}$$

such that

- ϕ_n induces the identity automorphism on G_K,
- ϕ_n maps the cuspidal inertia subgroups of $\Pi_{U'}$ associated to $*' \in \{0, 1, \infty\}$ to the cuspidal inertia subgroups of $\Pi_{V'}$ associated to $*'$,
- ϕ_n maps the cuspidal inertia subgroups of $\Pi_{U'}$ associated to $\lambda^{\frac{1}{n}}$ to the cuspidal inertia subgroups of $\Pi_{V'}$ associated to some n-th root $\mu^{\frac{1}{n}}$ of μ.

Let $L \subseteq \overline{K}$ be a finite extension of K such that $\lambda^{\frac{1}{n}}, \mu^{\frac{1}{n}} \in L$. Write

- $U'' \stackrel{\mathrm{def}}{=} \mathbb{P}^1_L \backslash \{0, 1, \lambda^{\frac{1}{n}}, \infty\}$;
- $V'' \stackrel{\mathrm{def}}{=} \mathbb{P}^1_L \backslash \{0, 1, \mu^{\frac{1}{n}}, \infty\}$.

Since $\lambda^{\frac{1}{n}} \neq \mu^{\frac{1}{n}}$ [by our assumption that $\lambda \neq \mu$], it follows, by considering the isomorphism

$$\Pi_{U''} \xrightarrow{\sim} \Pi_{V''}$$

induced by ϕ_n and applying a similar argument to the argument applied above to λ and μ, that

$$\lambda^{\frac{1}{n}} \in \mathbb{Q}^{\infty}.$$

This contradicts our choice of $\lambda^{\frac{1}{n}}$. Thus, we conclude that $\lambda = \mu$. This completes the proof of Theorem 3.5. $\square$

Remark 3.5.1. In the notation of Theorem 3.5, at the time of writing of the present paper, the author does not know *whether or not ϕ induces a bijection between the cuspidal inertia subgroups of Π_U and the cuspidal inertia subgroups of Π_V.*

However, an affirmative answer is known in the following cases:

(i) K is a subfield of a finite extension of the maximal pro-prime-to-p extension of $\mathbb{Q}^{\mathrm{ab}}$ [cf. [Stix], Lemma 27; [Stix], Theorem 30]. [Moreover, we note that in this case, K is a *stably p-$\times\mu$-indivisible* field [cf. Lemma 3.4, (ii), (iii), (iv), (vi)].]

(ii) There exists a prime number l such that the image of the l-adic cyclotomic character

$$G_K \to \mathbb{Z}_l^\times$$

is *open* [cf. [CmbGC], Corollary 2.7, (i)]. [The following example satisfies this condition:

> Let $F \subseteq \overline{\mathbb{Q}}_p$ be a p-adic local field; n an integer ≥ 0. Write $G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$; $G_F^n \subseteq G_F$ for the higher ramification group of index n, relative to the *upper numbering*; $F_n \subseteq \overline{\mathbb{Q}}_p$ for the subfield fixed by G_F^n. Then if K is a subfield of a finite extension of F_n, then the image of the p-adic cyclotomic character $G_K \to \mathbb{Z}_p^\times$ is open [cf. Lemma 3.6, (ii) below]. Moreover, we note that in this case, K is a *stably p-$\times\mu$-indivisible* field [cf. Lemma 3.4, (ii), (iii), (v); Lemma 3.6, (ii)].]

(iii) The isomorphism of profinite groups induced by ϕ

$$\phi_\Delta : \Delta_U \xrightarrow{\sim} \Delta_V$$

is PF-*cuspidalizable* [cf. the notation of the proof of Theorem 3.5; [CbTpI], Definition 1.4, (iv); [CbTpI], Lemma 1.6].

Lemma 3.6. *Let $F \subseteq \overline{\mathbb{Q}}_p$ be a p-adic local field. For each integer $n \geq 0$, write*

- *$G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$, G_F^{ab} for the abelianization of G_F;*
- *$G_F^n \subseteq G_F$ for the higher ramification group of index n, relative to the upper numbering [cf. [Serre], Chapter IV, §3];*
- *$H^n \subseteq G_F^{\mathrm{ab}}$ for the image of G_F^n via the natural quotient $G_F \twoheadrightarrow G_F^{\mathrm{ab}}$;*
- *$F_n \subseteq \overline{\mathbb{Q}}_p$ for the subfield fixed by G_F^n;*
- *$\rho_n : G_F^n \to \mathbb{Z}_p^\times$ for the p-adic cyclotomic character.*

Then, for each integer $n \geq 0$:

(i) H^n *is open in* H^0*.*

(ii) The image of ρ_n *is open.*

Proof. Assertion (i) is well-known [cf. [Serre], Chapter IV, §2, Proposition 6, (a), (b); [Serre], Chapter XV, §2, Theorem 2 and the following Remark]. Next, let us recall that F_0 is the maximal unramified extension of F [cf. [Serre], Chapter IV, §1, Proposition 1; [Serre], Chapter IV, §3, Proposition 13, (b)], hence that the image of ρ_0 is *open* [cf. [Neu], Chapter I, Lemma 10.1]. Thus, since ρ_n factors through the natural composite

$$G_F^n \subseteq G_F \twoheadrightarrow G_F^{\mathrm{ab}},$$

assertion (ii) follows immediately from assertion (i). □

Corollary 3.7. *Let* K *be a stably* $\times\mu$*-indivisible field of characteristic* 0*;* $\overline{K}$ *an algebraic closure of* K*. Write* $G_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{K}/K)$*. Let* Y *be a hyperbolic curve of genus* 0 *over* K*. Write* $Y(K)$ *for the set of* K*-valued points of* Y*;* $Y_{\overline{K}} \stackrel{\mathrm{def}}{=} Y \times_K \overline{K}$*;* $\mathrm{Sect}(\Pi_Y \twoheadrightarrow G_K)$ *for the set of equivalence classes of sections of the natural surjection* $\Pi_Y \twoheadrightarrow G_K$*, where we consider two such sections to be equivalent if they differ by composition with an inner automorphism induced by an element of* $\Pi_{Y_{\overline{K}}}$*. Then the natural map*

$$Y(K) \to \mathrm{Sect}(\Pi_Y \twoheadrightarrow G_K)$$

is injective.

Proof. Write

- Y_2 for the second configuration space of Y over K [cf. [MT], Definition 2.1, (i)];
- $\Delta_Y \stackrel{\mathrm{def}}{=} \Pi_{Y \times_K \overline{K}}$, $\Delta_{Y_2} \stackrel{\mathrm{def}}{=} \Pi_{Y_2 \times_K \overline{K}}$;
- $p_1 : \Pi_{Y_2} \twoheadrightarrow \Pi_Y$ for the natural surjection [determined up to composition with an inner automorphism of Π_Y] induced by the first projection.

Let $y_1, y_2 \in Y(K)$ be such that y_1 and y_2 determine the same equivalence class $\in \mathrm{Sect}(\Pi_Y \twoheadrightarrow G_K)$; $s_1 : G_K \hookrightarrow \Pi_Y$, $s_2 : G_K \hookrightarrow \Pi_Y$ sections of the natural surjection $\Pi_Y \twoheadrightarrow G_K$ induced, respectively, by y_1, y_2. Since s_1 and s_2 are only well-defined up to composition with an inner automorphism induced by an element of Δ_Y, we may assume without loss of generality that $s_1 = s_2$. Thus, we obtain a commutative diagram of profinite groups

$$\begin{array}{ccccc} \Pi_{Y\setminus\{y_1\}} & \longrightarrow & \Pi_{Y_2} & \longleftarrow & \Pi_{Y\setminus\{y_2\}} \\ \big\downarrow & & \big\downarrow p_1 & & \big\downarrow \\ G_K & \underset{s_1}{\longrightarrow} & \Pi_Y & \underset{s_2}{\longleftarrow} & G_K, \end{array}$$

where the left-hand and right-hand squares are cartesian. Since $s_1 = s_2$, this commutative diagram determines an isomorphism of profinite groups

$$\phi : \Pi_{Y\setminus\{y_1\}} \xrightarrow{\sim} \Pi_{Y\setminus\{y_2\}}$$

such that

- ϕ lies over the identity automorphism on G_K;
- ϕ induces a bijection between the cuspidal inertia subgroups of $\Pi_{Y\setminus\{y_1\}}$ associated to y_1 and the cuspidal inertia subgroups of $\Pi_{Y\setminus\{y_2\}}$ associated to y_2;
- for each cusp y of Y [where we observe that y may be regarded as a cusp of $Y\setminus\{y_1\}$ or $Y\setminus\{y_2\}$ by means of the natural inclusions $Y\setminus\{y_1\} \hookrightarrow Y$, $Y\setminus\{y_2\} \hookrightarrow Y$], ϕ induces a bijection between the cuspidal inertia subgroups of $\Pi_{Y\setminus\{y_1\}}$ associated to y and the cuspidal inertia subgroups of $\Pi_{Y\setminus\{y_2\}}$ associated to y;
- ϕ satisfies condition (b) in the statement of Theorem 3.5 [where we take "U" and "V" to be $Y \setminus \{y_1\}$ and $Y \setminus \{y_2\}$ respectively].

[Indeed, these properties follow immediately from the construction of ϕ from the above commutative diagram.] Thus, it follows from Theorem 3.5 that $y_1 = y_2$. This completes the proof of Corollary 3.7. □

Corollary 3.8. *Let K be a stably $\times\mu$-indivisible field of characteristic 0; $\overline{K}$ an algebraic closure of K. Write $G_K \stackrel{\text{def}}{=} \text{Gal}(\overline{K}/K)$. Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{K}$. In the following, we shall use this embedding to regard $\overline{\mathbb{Q}}$ as a subfield of $\overline{K}$. Thus, we obtain a homomorphism $G_K \to G_{\mathbb{Q}}$ $(\subseteq \text{GT})$ [cf. the discussion at the beginning of the Introduction]. Suppose that this homomorphism $G_K \to G_{\mathbb{Q}}$ is injective. In the following, we shall use this injection $G_K \hookrightarrow G_{\mathbb{Q}}$ to regard G_K as a subgroup of $G_{\mathbb{Q}}$, hence also as a subgroup of GT. Then $C_{\text{GT}}(G_K)$ acts naturally on the set of algebraic numbers $\overline{\mathbb{Q}}$.*

Proof. Let $\sigma \in C_{\text{GT}}(G_K)$. Then it suffices to show that

> the natural action of σ on $D(\text{GT})$ [cf. Corollary 1.6, (ii)] *descends* to a *natural action* of σ on the *quotient* $D(\text{GT}) \twoheadrightarrow \overline{\mathbb{Q}} \cup \{\infty\}$ of Corollary 1.7.

Since $\sigma \in C_{\mathrm{GT}}(G_K)$, there exists a finite extension $L \subseteq \overline{K}$ of K such that

$$\sigma G_L \sigma^{-1} \subseteq G_K,$$

where we write $G_L \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{K}/L) \subseteq G_K$. Fix such a finite extension L. Write $L^\sigma \subseteq \overline{K}$ for the finite extension of K such that $G_{L^\sigma} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{K}/L^\sigma) = \sigma G_L \sigma^{-1} \subseteq G_K$.

Then it follows immediately from Corollary 1.6, (ii), in the case where $J = \mathrm{GT}$, that we have a commutative diagram

$$\begin{array}{ccccccc} D(\mathrm{GT}) & \longrightarrow & D(G_{\mathbb{Q}}) & \longrightarrow & D(G_K) & \longrightarrow & D(G_L) \\ \wr\big\downarrow\sigma & & & & & & \wr\big\downarrow\sigma \\ D(\mathrm{GT}) & \longrightarrow & D(G_{\mathbb{Q}}) & \longrightarrow & D(G_K) & \longrightarrow & D(G_{L^\sigma}), \end{array}$$

where the vertical arrows are the bijections induced by σ; the horizontal arrows are the natural surjections of Corollary 1.6, (iii). Next, we observe that it follows immediately from Corollary 3.7, together with the various definitions involved, that the *surjections* $D(G_{\mathbb{Q}}) \to D(G_K)$, $D(G_K) \to D(G_L)$, and $D(G_K) \to D(G_{L^\sigma})$ of the above diagram are *bijections*. Thus, we conclude that there exists a commutative diagram

$$\begin{array}{ccccc} D(\mathrm{GT}) & \longrightarrow & D(G_{\mathbb{Q}}) & \stackrel{\sim}{\longrightarrow} & \overline{\mathbb{Q}} \cup \{\infty\} \\ \wr\big\downarrow\sigma & & \wr\big\downarrow\sigma & & \wr\big\downarrow\sigma \\ D(\mathrm{GT}) & \longrightarrow & D(G_{\mathbb{Q}}) & \stackrel{\sim}{\longrightarrow} & \overline{\mathbb{Q}} \cup \{\infty\}, \end{array}$$

where the left-hand vertical arrow and the horizontal arrows $D(\mathrm{GT}) \to D(G_{\mathbb{Q}})$ are the arrows of the previous diagram; the horizontal arrows $D(G_{\mathbb{Q}}) \to \overline{\mathbb{Q}} \cup \{\infty\}$ are the bijections of Corollary 1.7; the middle and right-hand vertical arrows are the *unique bijections* that make the above diagram commute. Finally, since the outer action of GT on Π_X preserves the cuspidal inertia subgroups of Π_X associated to ∞, it follows immediately from Remark 1.7.2 that the bijection $\overline{\mathbb{Q}} \cup \{\infty\} \stackrel{\sim}{\to} \overline{\mathbb{Q}} \cup \{\infty\}$ in the above diagram fixes ∞. This completes the proof Corollary 3.8. $\square$

Corollary 3.9. *Let K be a stably $\times\mu$-indivisible field of characteristic* 0*; $\overline{K}$ an algebraic closure of K. Write $G_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{K}/K)$. Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{K}$. In the following, we shall use this embedding to regard $\overline{\mathbb{Q}}$ as a subfield of $\overline{K}$. Thus, we obtain a homomorphism $G_K \to G_{\mathbb{Q}}$ $(\subseteq \mathrm{GT})$ [cf. the discussion at the beginning of the Introduction]. Suppose that this homomorphism $G_K \to G_{\mathbb{Q}}$ is injective. In the following, we shall use this injection $G_K \hookrightarrow G_{\mathbb{Q}}$ to regard G_K as a subgroup of $G_{\mathbb{Q}}$, hence also as a subgroup of* GT*. Then one may construct a natural homomorphism*

$$C_{\mathrm{GT}}(G_K) \to G_{\mathbb{Q}}$$

whose restriction to $C_{G_{\mathbb{Q}}}(G_K)$ *is the natural inclusion* $C_{G_{\mathbb{Q}}}(G_K) \subseteq G_{\mathbb{Q}}$*. In particular, we obtain a natural surjection*

$$C_{\mathrm{GT}}(G_K) \twoheadrightarrow C_{G_{\mathbb{Q}}}(G_K) \ (\subseteq G_{\mathbb{Q}}).$$

whose restriction to $C_{G_{\mathbb{Q}}}(G_K)$ *is the identity automorphism.*

Proof. It follows immediately from a similar argument to the argument given in the proof of Corollary 2.4 that the natural action of $C_{\mathrm{GT}}(G_K)$ on the set $\overline{\mathbb{Q}}$ [cf. Corollary 3.8] is compatible with the field structure of $\overline{\mathbb{Q}}$. Thus, we obtain the desired conclusion. This completes the proof Corollary 3.9. $\square$

Remark 3.9.1. In the notation of Remark 3.4.1, suppose that K_0 is a number field or a p-adic local field. Then it follows immediately from Remark 3.4.1 that K_n satisfies the assumptions in Corollary 3.9.

Lemma 3.10. *In the notation of Corollary 3.9, suppose that*

$$G_K \subseteq G_{\mathbb{Q}_p} \subseteq G_{\mathbb{Q}},$$

where we think of "$G_{\mathbb{Q}_p}$*" as the decomposition group of a valuation of* $\overline{\mathbb{Q}}$ *that divides* p*. Then*

$$C_{G_{\mathbb{Q}_p}}(G_K) = C_{G_{\mathbb{Q}}}(G_K) \ (\subseteq G_{\mathbb{Q}_p}).$$

Proof. First, we observe that the inclusion $C_{G_{\mathbb{Q}_p}}(G_K) \subseteq C_{G_{\mathbb{Q}}}(G_K)$ is immediate. Suppose that

$$C_{G_{\mathbb{Q}}}(G_K) \not\subseteq G_{\mathbb{Q}_p}.$$

Let $\sigma \in C_{G_{\mathbb{Q}}}(G_K) \setminus G_{\mathbb{Q}_p}$. Then there exists a finite index subgroup H of G_K such that

$$H \subseteq G_{\mathbb{Q}_p} \ \cap \ \sigma G_{\mathbb{Q}_p} \sigma^{-1} \subseteq G_{\mathbb{Q}}.$$

Thus, since $G_{\mathbb{Q}_p} \ \cap \ \sigma G_{\mathbb{Q}_p} \sigma^{-1} = \{1\}$ [cf. [NSW], Corollary 12.1.3], we conclude that $H = \{1\}$, hence that G_K $(\subseteq G_{\mathbb{Q}_p})$ is finite. Recall that $G_{\mathbb{Q}_p}$ is torsion-free [cf. [NSW], Corollary 12.1.3; [NSW], Theorem 12.1.7]. This implies that $G_K = \{1\}$. Thus, in particular, K is an algebraically closed field of characteristic 0. However, this contradicts the fact that no algebraically closed field of characteristic 0 is $\times\mu$-*indivisible*. Thus, we conclude that $C_{G_{\mathbb{Q}}}(G_K) \subseteq G_{\mathbb{Q}_p}$, hence that $C_{G_{\mathbb{Q}_p}}(G_K) = C_{G_{\mathbb{Q}}}(G_K)$. This completes the proof of Lemma 3.10. $\square$

Corollary 3.11. *In the notation of Lemma 3.10, one may construct a natural surjection*

$$C_{\mathrm{GT}}(G_K) \twoheadrightarrow C_{G_{\mathbb{Q}_p}}(G_K)\ (\subseteq G_{\mathbb{Q}_p})$$

whose restriction to $C_{G_{\mathbb{Q}_p}}(G_K)$ *is the identity automorphism.*

Proof. Corollary 3.11 follows immediately from Corollary 3.9 and Lemma 3.10. □

Acknowledgements

The author was supported by JSPS KAKENHI Grant Number JP18J10260. This research was also supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University. The author would like to thank Professor Yuichiro Hoshi and Professor Shinichi Mochizuki for suggesting the topic [especially, Theorem 1.3], and for many helpful discussions, as well as for their warm encouragement. Moreover, the author deeply appreciates Professor Shinichi Mochizuki for taking an enormous amount of time to fix his master's thesis and the present paper. The author also deeply appreciates Professor Yuichiro Hoshi for answering his questions concerning general algebraic geometry and anabelian geometry in detail. Finally, the author also would like to thank Yu Yang and Arata Minamide for many stimulating discussions concerning various topics in anabelian geometry.

References

[André] Y. André, On a geometric description of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and a p-adic avatar of $\widehat{GT}$, *Duke Math. J.* **119** (2003), pp. 1–39.

[Belyi] G. V. Belyi, On Galois extensions of a maximal cyclotomic field, *Izv. Akad. Nauk SSSR Ser. Mat.* **43:2** (1979), pp. 269-276; English transl. in *Math. USSR-Izv.* **14** (1980), pp. 247–256.

[NodNon] Y. Hoshi and S. Mochizuki, On the combinatorial anabelian geometry of nodally nondegenerate outer representations, *Hiroshima Math. J.* **41** (2011), pp. 275–342.

[CbTpI] Y. Hoshi and S. Mochizuki, Topics surrounding the combinatorial anabelian geometry of hyperbolic curves I: Inertia groups and profinite Dehn twists, *Galois-Teichmüller Theory and Arithmetic Geometry*, *Adv. Stud. Pure Math.* **63**, *Math. Soc. Japan*, 2012, pp. 659–811.

[CbTpII] Y. Hoshi and S. Mochizuki, *Topics surrounding the combinatorial anabelian geometry of hyperbolic curves II: Tripods and combinatorial cuspidalization*, RIMS Preprint **1762** (November 2012).

[CbTpIII] Y. Hoshi and S. Mochizuki, *Topics surrounding the combinatorial anabelian geometry of hyperbolic curves III: Tripods and Tempered fundamental groups*, RIMS Preprint **1763** (November 2012).

[KLR] N. Katz, S. Lang, Finiteness theorems in geometric class field theory, with an appendix by Kenneth A. Ribet, *Enseign. Math.* (2) **27** (1981), pp. 285–319.

[Kobl] N. Koblitz, *p-adic numbers, p-adic analysis, and zeta-functions*, *Graduate Texts in Mathematics* **58**, Springer-Verlag (1984).

[Lpg] E. Lepage, Resolution of non-singularities for Mumford curves, *Publ. Res. Inst. Math. Sci.* **49** (2013), pp. 861–891.

[LocAn] S. Mochizuki, The local pro-p anabelian geometry of curves, *Invent. Math.* **138** (1999), pp. 319–423.

[AnabTop] S. Mochizuki, Topics surrounding the anabelian geometry of hyperbolic curves, *Galois groups and fundamental groups*, *Math. Sci. Res. Inst. Publ.* **41**, Cambridge Univ. Press. (2003), pp. 119–165.

[SemiAn] S. Mochizuki, Semi-graphs of anabelioids, *Publ. Res. Inst. Math. Sci.* **42** (2006), pp. 221–322.

[Cusp] S. Mochizuki, Absolute anabelian cuspidalizations of proper hyperbolic curves, *J. Math. Kyoto Univ.* **47** (2007), pp. 451–539.

[CmbGC] S. Mochizuki, A combinatorial version of the Grothendieck conjecture, *Tohoku Math. J.* **59** (2007), pp. 455–479.

[CmbCsp] S. Mochizuki, On the combinatorial cuspidalization of hyperbolic curves, *Osaka J. Math.* **47** (2010), pp. 651–715.

[AbsTopII] S. Mochizuki, Topics in absolute anabelian geometry II: Decomposition groups and endomorphisms, *J. Math. Sci. Univ. Tokyo* **20** (2013), pp. 171–269.

[AbsTopIII] S. Mochizuki, Topics in absolute anabelian aeometry III: Global reconstruction algorithms, *J. Math. Sci. Univ. Tokyo* **22** (2015), pp. 939–1156.

[MT] S. Mochizuki and A. Tamagawa, The algebraic and anabelian geometry of configuration spaces, *Hokkaido Math. J.* **37** (2008), pp. 75–131.

[Moon] H. Moon, On the Mordell-Weil groups of jacobians of hyperelliptic curves over certain elementary abelian 2-extensions, *Kyungpook Math. J.* **49** (2009), 419–424.

[Neu] J. Neukirch, *Algebraic number theory*, *Grundlehren der Mathematischen Wissenschaften* **322**, Springer-Verlag (1999).

[NSW] J. Neukirch, A. Schmidt, K. Wingberg, *Cohomology of number fields*, *Grundlehren der Mathematischen Wissenschaften* **323**, Springer-Verlag (2000).

[Serre] J.-P. Serre, *Local fields*, Graduate Texts in Mathematics **67**, Springer-Verlag (1979).

[Stix] J. Stix, On cuspidal sections of algebraic fundamental groups, *Galois-Teichmüller Theory and Arithmetic Geometry*, *Adv. Stud. Pure Math.* **63**, Math. Soc. Japan (2012), pp. 519—563.

[Tama1] A.Tamagawa, The Grothendieck conjecture for affine curves, *Compositio Math.* **109** No. 2 (1997), pp. 135–194.

[Tama2] A. Tamagawa, Resolution of nonsingularities of families of curves, *Publ. Res. Inst. Math. Sci.* **40** (2004), pp. 1291–1336.