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On the Semi-absoluteness of Isomorphisms between the Pro-p Arithmetic Fundamental Groups of Smooth Varieties

by

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Abstract

Let p be a prime number. In the present paper, we consider a certain pro-p analogue of the semi-absoluteness of isomorphisms between the étale fundamental groups of smooth varieties over p-adic local fields [i.e., finite extensions of the field of p-adic numbers $\mathbb{Q}_p$] obtained by Mochizuki. This research was motivated by Higashiyama's recent work on the pro-p analogue of the semi-absolute version of the Grothendieck Conjecture for configuration spaces [of dimension ≥ 2] associated to hyperbolic curves over generalized sub-p-adic fields [i.e., subfields of finitely generated extensions of the completion of the maximal unramified extension of $\mathbb{Q}_p$].

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Introduction

Let p be a prime number. For a connected Noetherian scheme S, we shall write Π_S for the étale fundamental group of S, relative to a suitable choice of basepoint. For any field F of characteristic 0 and any algebraic variety [i.e., a separated, of finite type, and geometrically integral scheme] X over F, we shall write $\overline{F}$ for the algebraic closure [determined up to isomorphisms] of F; $G_F \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{F}/F)$; $\Delta_X \stackrel{\text{def}}{=} \Pi_{X \times_F \overline{F}}$.

In anabelian geometry, the relative version of the Grothendieck Conjecture proved by Mochizuki is a central result:

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Theorem 0.1 ([17], Theorem A; [19], Theorem 4.12). *Let K be a generalized sub-p-adic field [i.e., a subfield of a finitely generated extension of the completion of the maximal unramified extension of the field of p-adic numbers $\mathbb{Q}_p$ — cf. [19], Definition 4.11]; X, X' hyperbolic curves over K. Write* $\mathrm{Isom}_K(X, X')$ *for the set of K-isomorphisms between X and X';* $\mathrm{Isom}_{G_K}(\Pi_X, \Pi_{X'})/\mathrm{Inn}(\Delta_{X'})$ *for the set of isomorphisms $\Pi_X \xrightarrow{\sim} \Pi_{X'}$ of profinite groups that lie over G_K, considered up to composition with an inner automorphism arising from $\Delta_{X'}$. Then the natural map*

$$\mathrm{Isom}_K(X, X') \longrightarrow \mathrm{Isom}_{G_K}(\Pi_X, \Pi_{X'})/\mathrm{Inn}(\Delta_{X'})$$

is bijective.

On the other hand, concerning the above theorem, we recall the following open questions:

> Question 1 (Absolute version of the Grothendieck Conjecture): Let X, X' be hyperbolic curves over p-adic local fields [i.e., finite extensions of $\mathbb{Q}_p$] K, K', respectively. Write $\mathrm{Isom}(X, X')$ for the set of isomorphisms of schemes between X and X'; $\mathrm{Isom}(\Pi_X, \Pi_{X'})/\mathrm{Inn}(\Pi_{X'})$ for the set of isomorphisms $\Pi_X \xrightarrow{\sim} \Pi_{X'}$ of profinite groups, considered up to composition with an inner automorphism arising from $\Pi_{X'}$. Then is the natural map
>
> $$\mathrm{Isom}(X, X') \longrightarrow \mathrm{Isom}(\Pi_X, \Pi_{X'})/\mathrm{Inn}(\Pi_{X'})$$
>
> bijective?
>
> Question 2 (Semi-absolute version of the Grothendieck Conjecture): Let X, X' be hyperbolic curves over p-adic local fields K, K', respectively. Write
>
> $$\mathrm{Isom}(\Pi_X/G_K,\ \Pi_{X'}/G_{K'})/\mathrm{Inn}(\Pi_{X'})$$
>
> for the set of isomorphisms $\Pi_X \xrightarrow{\sim} \Pi_{X'}$ of profinite groups that induce isomorphisms $G_K \xrightarrow{\sim} G_{K'}$ via the natural surjections $\Pi_X \twoheadrightarrow G_K$ and $\Pi_{X'} \twoheadrightarrow G_{K'}$, considered up to composition with an inner automorphism arising from $\Pi_{X'}$. Then is the natural map
>
> $$\mathrm{Isom}(X, X') \longrightarrow \mathrm{Isom}(\Pi_X/G_K,\ \Pi_{X'}/G_{K'})/\mathrm{Inn}(\Pi_{X'})$$
>
> bijective?

[Here, we note that the analogous assertions of Questions 1, 2, for hyperbolic curves over *subfields* of p-adic local fields do not hold — cf. [10], Remark 5.6.1.] With regard to Questions 1, 2, Mochizuki proved the following result, which asserts that the absolute version of the Grothendieck Conjecture and the semi-absolute version of the Grothendieck Conjecture are equivalent [cf. [21], Corollary 2.8; [6]; [30], Lemma 4.2]:

Theorem 0.2. *Let K, K' be p-adic local fields; X, X' smooth varieties [i.e., smooth, separated, of finite type, and geometrically integral schemes] over K, K', respectively;*

$$\alpha : \Pi_X \xrightarrow{\sim} \Pi_{X'}$$

an isomorphism of profinite groups. Then α induces an isomorphism $G_K \xrightarrow{\sim} G_{K'}$ that fits into a commutative diagram

$$\begin{array}{ccc} \Pi_X & \xrightarrow[\alpha]{\sim} & \Pi_{X'} \\ \downarrow & & \downarrow \\ G_K & \xrightarrow{\sim} & G_{K'}, \end{array}$$

where the vertical arrows denote the natural surjections [determined up to composition with an inner automorphism] induced by the structure morphisms of the smooth varieties X, X'.

[Note that there exists a certain generalization of this result — cf. [15], Corollary D].

Moreover, Mochizuki also proved that, if an isomorphism $\alpha : \Pi_X \xrightarrow{\sim} \Pi_{X'}$ preserves the decomposition subgroups associated to the closed points, then α is induced by a unique isomorphism $X \xrightarrow{\sim} X'$ of schemes [cf. [22], Corollary 2.9]. One of the ways[1] to reconstruct the decomposition subgroups associated to closed points is Mochizuki's Belyi cuspidalization technique for strictly Belyi type curves [cf. [22], §3]. However, due to the difficulty of verifying the preservation of the decomposition subgroups, we do not know whether or not the absolute version of the Grothendieck Conjecture has an affirmative answer in general.

On the other hand, one may pose analogous questions of Questions 1, 2, in the pro-p setting. In this pro-p setting, it appears that no analogous result of Mochizuki's results [cf. Theorem 0.2; [22], Corollary 2.9] has been obtained. In this context, Higashiyama studied a certain pro-p analogue of the semi-absolute version of the Grothendieck Conjecture for configuration spaces [of dimension ≥ 2] associated to hyperbolic curves over generalized sub-p-adic fields [i.e., subfields of finitely generated extensions of the completion of the maximal unramified extension of $\mathbb{Q}_p$] and obtained a partial result [cf. Definition 4.1; [5], Theorem 0.1].

[1] Recently, it appears that E. Lepage discovered a different way to reconstruct the decomposition subgroups associated to the closed points of hyperbolic Mumford curves based on his [highly nontrivial] result on resolution of nonsingularities.

In the present paper, inspired by Higashiyama's research, we consider a certain pro-p analogue of Theorem 0.2 for the configuration spaces associated to hyperbolic curves over p-adic local fields. Note that the proof of Theorem 0.2 depends heavily on the l-independence of a certain numerical invariant associated to Π_X and G_K, where l ranges over the prime numbers [cf. [21], Theorem 2.6, (ii), (v)]. Thus, we need to apply a different argument to obtain a pro-p analogue of Theorem 0.2.

Let F be a field of characteristic 0; X an algebraic variety over F. Then we have an exact sequence of profinite groups

$$1 \longrightarrow \Delta_X \longrightarrow \Pi_X \longrightarrow G_F \longrightarrow 1$$

[cf. [4], Exposé IX, Théorème 6.1]. We shall say that X satisfies the *p-exactness* [cf. Definition 3.1] if the above exact sequence induces an exact sequence of pro-p groups

$$1 \longrightarrow \Delta_X^p \longrightarrow \Pi_X^p \longrightarrow G_F^p \longrightarrow 1$$

[where we note that the natural sequence of pro-p groups

$$\Delta_X^p \longrightarrow \Pi_X^p \longrightarrow G_F^p \longrightarrow 1$$

is exact without imposing any assumption on X].

Then our main result is the following:

Theorem A. *Let (n, n') be a pair of positive integers; K, K' fields of characteristic 0; X, X' smooth varieties over K, K', respectively. Then the following hold:*

(i) Let

$$\alpha : \Pi_X^p \xrightarrow{\sim} \Pi_{X'}^p$$

be an isomorphism of profinite groups. Suppose that

- *K is either a Henselian discrete valuation field with infinite residues of characteristic p or a Hilbertian field [i.e., a field for which Hilbert's irreducibility theorem holds — cf. [3], Chapter 12];*
- *K' is either a Henselian discrete valuation field with residues of characteristic p or a Hilbertian field;*
- *K and K' contain a primitive p-th root of unity.*

Then α induces an isomorphism $G_K^p \xrightarrow{\sim} G_{K'}^p$ that fits into a commutative diagram

$$\begin{array}{ccc} \Pi_X^p & \xrightarrow[\alpha]{\sim} & \Pi_{X'}^p \\ \downarrow & & \downarrow \\ G_K^p & \xrightarrow{\sim} & G_{K'}^p, \end{array}$$

where the vertical arrows denote the natural surjections [determined up to composition with an inner automorphism] induced by the structure morphisms of the smooth varieties X, X'.

(ii) *Suppose that* X, X' *are hyperbolic curves over* K, K', *respectively. Write* X_n *(respectively,* $X'_{n'}$*) for the* n*-th (respectively, the* n'*-th) configuration space associated to* X *(respectively,* X'*) [cf. Definition 4.1]. Let*

$$\alpha : \Pi^p_{X_n} \xrightarrow{\sim} \Pi^p_{X'_{n'}}$$

be an isomorphism of profinite groups. Suppose, moreover, that

- K *and* K' *are either Henselian discrete valuation fields of residue characteristic* p *or Hilbertian fields;*
- X_n *and* $X'_{n'}$ *satisfy the* p*-exactness.*

Then the following hold:

- *Let* Π *be a topological group isomorphic to* $\Pi^p_{X_n}$. *Then there exists a functorial group-theoretic algorithm*

$$\Pi \quad \rightsquigarrow \quad n$$

for constructing the dimension n *from* Π. *In particular, it holds that* $n = n'$.

- α *induces an isomorphism* $G^p_K \xrightarrow{\sim} G^p_{K'}$ *that fits into a commutative diagram*

$$\begin{array}{ccc} \Pi^p_{X_n} & \xrightarrow[\alpha]{\sim} & \Pi^p_{X'_n} \\ \downarrow & & \downarrow \\ G^p_K & \xrightarrow{\sim} & G^p_{K'}, \end{array}$$

where the vertical arrows denote the natural surjections [determined up to composition with an inner automorphism] induced by the structure morphisms of the configuration spaces X_n, X'_n.

Recall that every finitely generated extension of the field of rational numbers $\mathbb{Q}$ or $\mathbb{Q}_p$ is a Hilbertian field or a Henselian discrete valuation field of residue characteristic p [cf. [3], Theorem 13.4.2]. In particular, by combining Theorem A, (ii), with Higashiyama's result [cf. [5], Theorem 0.1], we obtain the "*absolute version*" of Higashiyama's result in the case where the base fields are such fields.

Furthermore, it would be interesting to investigate to which extent the assumptions of Theorem A may be weakened. Thus, it is natural to pose the following question, which may be regarded as a generalization of the above theorem [cf. [15], Corollary D]:

Question 3: Let X, X' be smooth varieties over fields K, K' of characteristic 0, respectively;

$$\alpha : \Pi_X^p \xrightarrow{\sim} \Pi_{X'}^p$$

an isomorphism of profinite groups. Suppose that K and K' are either

- *subfields* of Henselian discrete valuation fields of residue characteristic p or
- Hilbertian fields.

Then does α induce an isomorphism $G_K^p \xrightarrow{\sim} G_{K'}^p$ via the natural surjections $\Pi_X^p \twoheadrightarrow G_K^p$ and $\Pi_{X'}^p \twoheadrightarrow G_{K'}^p$?

However, at the time of writing of the present paper, the author does not know whether the answer is affirmative or not. Moreover, we note that Theorem A, (ii), is not proved in a "*mono-anabelian*" fashion [cf. [21], Introduction; [23], Introduction], and, at the time of writing of the present paper, the author does not know whether or not such a proof exists. Since Theorem 0.2 is proved in a "*mono-anabelian*" fashion, it would be also interesting to investigate a *mono-anabelian reconstruction* of the closed subgroup $\mathrm{Ker}(\Pi_X^p \to G_K^p) \subseteq \Pi_X^p$ from [the underlying topological group structure of] Π_X^p.

Finally, we remark that there exist other researches on the semi-absoluteness of isomorphisms between the étale fundamental groups of algebraic varieties [cf. [12], Theorem; [15], Corollary D].

The present paper is organized as follows. In §1, we review some group-theoretic properties of the maximal pro-p quotients of the absolute Galois groups of p-adic local fields. In §2, we review some group-theoretic properties of the maximal pro-p quotients of the étale fundamental groups of hyperbolic curves over p-adic local fields. In §3, by applying the results reviewed in §1, §2, we give a proof of Theorem A, (ii), for hyperbolic curves over p-adic local fields. In §4, by combining the results obtained in §3 with some considerations on the geometry of configuration spaces associated to hyperbolic curves, we complete the proof of Theorem A.

Notations and Conventions

Numbers: The notation $\mathbb{N}$ will be used to denote the set of nonnegative integers. The notation $\mathbb{Z}$ will be used to denote the additive group of integers. The notation $\mathbb{Q}$ will be used to denote the field of rational numbers. If p is a prime number, then the notation $\mathbb{Q}_p$ will be used to denote the field of p-adic numbers; the notation $\mathbb{Z}_p$ will be used to denote the additive group or ring of p-adic integers. We shall refer to a finite extension field of $\mathbb{Q}_p$ as a *p-adic local field*.

Fields: Let F be a field of characteristic 0. Then the notation $\overline{F}$ will be used to denote an algebraic closure [determined up to isomorphisms] of F. The notation G_F will be used to denote the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ of F. If p is a prime number, then we shall fix a primitive p-th root of unity $\zeta_p \in \overline{F}$. Let E $(\subseteq \overline{F})$ be a finite extension field of F. Then we shall denote by $[E : F]$ the extension degree of the finite extension $F \subseteq E$.

Profinite groups: Let G be a profinite group and $H \subseteq G$ a closed subgroup of G. Then we shall denote by $Z_G(H)$ the *centralizer* of $H \subseteq G$, i.e.,

$$Z_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid ghg^{-1} = h \text{ for any } h \in H\}.$$

Let p be a prime number. Then we shall write G^p for the maximal pro-p quotient of G; G^{ab} for the abelianization of G, i.e., the quotient of G by the closure of the commutator subgroup of G; $\mathrm{cd}_p(G)$ for the cohomological p-dimension of G [cf. [27], §7.1]. If G is abelian, then we shall write $G_{\mathrm{tor}} \subseteq G$ for the maximal torsion subgroup. If G is a topologically finitely generated pro-p group, then the notation rank G will be used to denote the rank of G [cf. [26], Definition 3.5.18].

Schemes: Let K be a field; $K \subseteq L$ a field extension; X an algebraic variety [i.e., a separated, of finite type, and geometrically integral scheme] over K. Then we shall write $X_L \stackrel{\mathrm{def}}{=} X \times_K L$; $X(L)$ for the set of L-rational points of X.

Fundamental groups: For a connected Noetherian scheme S, we shall write Π_S for the étale fundamental group of S, relative to a suitable choice of basepoint. Let K be a field of characteristic 0; X an algebraic variety over K. Then we shall write $\Delta_X \stackrel{\mathrm{def}}{=} \Pi_{X_{\overline{K}}}$.

§1. The maximal pro-p quotients of the absolute Galois groups of p-adic local fields

Let p be a prime number; K a p-adic local field. In the present section, we review some group-theoretic properties of G_K^p [cf. Notations and Conventions], which will be of use in the later sections.

Definition 1.1 ([26], Definition 3.9.9). Let G be a topologically finitely generated pro-p group. Then we shall say that G is a *Demushkin group* if

$$\dim_{\mathbb{Z}/p\mathbb{Z}} H^2(G, \mathbb{Z}/p\mathbb{Z}) = 1,$$

and the cup-product

$$H^1(G,\mathbb{Z}/p\mathbb{Z}) \times H^1(G,\mathbb{Z}/p\mathbb{Z}) \to H^2(G,\mathbb{Z}/p\mathbb{Z})$$

is non-degenerate.

Remark 1.1.1. Let G be a Demushkin group. Then it follows immediately from [27], Theorem 7.7.4, that G is not a free pro-p group.

Definition 1.2 ([21], Definition 1.1, (ii)). Let G be a profinite group.

(i) We shall say that G is *slim* if $Z_G(H) = \{1\}$ [cf. Notations and Conventions] for any open subgroup H of G.
(ii) We shall say that G is *elastic* if every nontrivial topologically finitely generated normal closed subgroup of an open subgroup of G is open in G.

Proposition 1.3. *Write p^a for the cardinality of the group of p-power roots of unity $\in K$; $d \stackrel{\text{def}}{=} [K : \mathbb{Q}_p]$. Then $(G_K^p)^{\text{ab}}$ is isomorphic to $\mathbb{Z}/p^a\mathbb{Z} \oplus \mathbb{Z}_p^{\oplus d+1}$ [cf. Notations and Conventions]. In particular, $(G_K^p)^{\text{ab}}$ has a torsion element in the case where $\zeta_p \in K$.*

Proof. Proposition 1.3 follows immediately from local class field theory, together with the well-known structure of the multiplicative group of a p-adic local field [cf. [25], Chapter II, Proposition 5.7, (i); [25], Chapter V, Theorems 1.3, 1.4]. □

Theorem 1.4 ([26], Theorem 7.5.11). *Write $d \stackrel{\text{def}}{=} [K : \mathbb{Q}_p]$. Then the following hold:*

(i) Suppose that $\zeta_p \notin K$. Then G_K^p is a free pro-p group of rank $d+1$.
(ii) Suppose that $\zeta_p \in K$. Then G_K^p is a Demushkin group of rank $d+2$.

Theorem 1.5 ([21], Proposition 1.6; [21], Theorem 1.7; [26], Theorem 7.1.8). *The following hold:*

(i) G_K^p is slim.
(ii) G_K^p is elastic.

(iii) *Suppose that* $\zeta_p \in K$. *Then* $\mathrm{cd}_p(G_K^p) = 2$, *and every closed subgroup* $H \subseteq G_K^p$ *of infinite index is a free pro-p group.*

Proof. First, since the maximal pro-p quotient G_K^p is an almost maximal pro-p quotient of G_K, assertions (i), (ii) follow immediately from [21], Theorem 1.7, (ii). Assertion (iii) follows immediately from [21], Proposition 1.6, (ii), (iii); [26], Theorem 7.1.8, (i). This completes the proof of Theorem 1.5. □

Lemma 1.6. G_K^p *is a nonabelian infinite torsion-free group.*

Proof. First, we suppose that $\zeta_p \notin K$. Then G_K^p is a free pro-p group of rank ≥ 2 [cf. Theorem 1.4, (i)]. Thus, we have nothing to prove. Next, we suppose that $\zeta_p \in K$. Then it follows from Theorem 1.5, (iii), that $\mathrm{cd}_p(G_K^p) = 2 < \infty$, hence, in particular, that G_K^p is torsion-free. Thus, we conclude from Proposition 1.3 that G_K^p is a nonabelian infinite torsion-free group. This completes the proof of Lemma 1.6. □

§2. The maximal pro-p quotients of the étale fundamental groups of hyperbolic curves over p-adic local fields

Let p be a prime number; K a p-adic local field; $\overline{X}$ a proper hyperbolic curve over K. Write $\mathcal{O}_K$ for the ring of integers of K; k for the residue field of $\mathcal{O}_K$. Suppose that

$$\overline{X} \text{ has } \textit{stable reduction} \text{ over } \mathcal{O}_K.$$

Write $\mathcal{X}$ for the stable model of $\overline{X}$ over $\mathcal{O}_K$.

In the present section, following [8], we review some group-theoretic properties of $\Delta_{\overline{X}}^p$ [cf. Notations and Conventions] and its quotients.

Definition 2.1 ([8], Definition 2.3).

(i) We shall write $\mathrm{Irr}(\overline{X})$ for the set of irreducible components of $\mathcal{X} \times_{\mathcal{O}_K} \overline{k}$;

(ii) We shall write $\Delta_{\overline{X}}^{p,\text{ét}}$ for the maximal pro-p quotient of $\Pi_{\mathcal{X} \times_{\mathcal{O}_K} \overline{k}}$;

(iii) Let v be an irreducible component of $\mathcal{X} \times_{\mathcal{O}_K} \overline{k}$. Then we shall write $\mathcal{D}_v$ (respectively, $\mathcal{D}_v^p$) for the decomposition subgroup [determined up to composition with an inner automorphism] of $\Pi_{\mathcal{X} \times_{\mathcal{O}_K} \overline{k}}$ (respectively, $\Delta_{\overline{X}}^{p,\text{ét}}$) associated to v;

(iv) We shall write $\Delta_{\overline{X}}^{\mathrm{cmb}}$ (respectively, $\Delta_{\overline{X}}^{p,\mathrm{cmb}}$) for the quotient of $\Pi_{\mathcal{X}\times_{\mathcal{O}_K}\overline{k}}$ (respectively, $\Delta_{\overline{X}}^{p,\text{ét}}$) by the normal closed subgroup topologically normally generated by the closed subgroups $\{\mathcal{D}_w\}_{w\in\mathrm{Irr}(\overline{X})}$ (respectively, $\{\mathcal{D}_w^p\}_{w\in\mathrm{Irr}(\overline{X})}$).

Remark 2.1.1. The natural open immersion from $X_{\overline{K}}$ to the stable model of $X_{\overline{K}}$ over the ring of integers of $\overline{K}$ induces natural surjections

$$\Delta_{\overline{X}} \twoheadrightarrow \Pi_{\mathcal{X}\times_{\mathcal{O}_K}\overline{k}}, \quad \Delta_{\overline{X}}^{p} \twoheadrightarrow \Delta_{\overline{X}}^{p,\text{ét}}.$$

On the other hand, it follows immediately from the various definitions involved that there exist natural surjections

$$\Pi_{\mathcal{X}\times_{\mathcal{O}_K}\overline{k}} \twoheadrightarrow \Delta_{\overline{X}}^{\mathrm{cmb}}, \quad \Delta_{\overline{X}}^{p,\text{ét}} \twoheadrightarrow \Delta_{\overline{X}}^{p,\mathrm{cmb}}.$$

Next, we review some well-known group-theoretic properties of $\Delta_{\overline{X}}^{p}$ and $\Delta_{\overline{X}}^{p,\mathrm{cmb}}$.

Proposition 2.2 ([24], Remark 1.2.2; [24], Proposition 1.4; [24], Theorem 1.5; [8], Proposition 2.5; [9], Lemma 2.1).

(i) $\Delta_{\overline{X}}^{p}$ *is slim.*

(ii) $\Delta_{\overline{X}}^{p}$ *is elastic.*

(iii) $\Delta_{\overline{X}}^{p,\mathrm{cmb}}$ *is a free pro-p group.*

(iv) $\mathrm{cd}_p(\Delta_{\overline{X}}^{p}) = 2$*, and every closed subgroup* $M \subseteq \Delta_{\overline{X}}^{p}$ *of infinite index is a free pro-p group.*

Remark 2.2.1. In [9], Lemma 2.1, Hoshi imposed the condition [on M] that the closed subgroup $M \subseteq \Delta_{\overline{X}}^{p}$ is *normal* in order to assert that M is *not topologically finitely generated*. However, we do not need this assertion, and the proof of [9], Lemma 2.1, implies that every closed subgroup $M \subseteq \Delta_{\overline{X}}^{p}$ of infinite index is a free pro-p group.

Remark 2.2.2. In the remainder of the present paper, we do not apply Proposition 2.2, (ii), (iv). We reviewed these properties to observe the group-theoretic similarities between G_K^p and $\Delta_{\overline{X}}^{p}$ [cf. Theorem 1.5].

Next, we recall the following well-known [but nontrivial] fact [cf. [8], Lemma 3.2; [20], Lemma 1.1.5].

Lemma 2.3. *Let M be a free $\mathbb{Z}_p$-module equipped with the trivial G_K-action; $X \hookrightarrow \overline{X}$ an open immersion over K [so X is a hyperbolic curve over K]. Recall that G_K acts naturally on $(\Delta_X^p)^{\mathrm{ab}}$. Then every G_K-equivariant homomorphism*

$$(\Delta_X^p)^{\mathrm{ab}} \to M$$

factors through the composite of natural surjections

$$(\Delta_X^p)^{\mathrm{ab}} \twoheadrightarrow (\Delta_{\overline{X}}^p)^{\mathrm{ab}} \twoheadrightarrow (\Delta_{\overline{X}}^{p,\mathrm{cmb}})^{\mathrm{ab}}$$

[cf. Remark 2.1.1].

Proof. First, we note that the image of the p-adic cyclotomic character $G_K \to \mathbb{Z}_p^\times$ is open. On the other hand, if we replace K by a finite extension field of K, then the kernel of the natural surjection $(\Delta_X^p)^{\mathrm{ab}} \twoheadrightarrow (\Delta_{\overline{X}}^p)^{\mathrm{ab}}$ is isomorphic to a direct sum of copies of $\mathbb{Z}_p(1)$ as G_K-modules, where "(1)" denotes the Tate twist. Thus, we may assume without loss of generality that

$$X = \overline{X}.$$

Next, since M is a free $\mathbb{Z}_p$-module equipped with the trivial G_K-action, it suffices to prove that every G_K-equivariant homomorphism $\mathrm{Ker}((\Delta_{\overline{X}}^p)^{\mathrm{ab}} \twoheadrightarrow (\Delta_{\overline{X}}^{p,\mathrm{cmb}})^{\mathrm{ab}}) \to \mathbb{Z}_p$ is trivial. Recall our assumption that X has stable reduction over $\mathcal{O}_K$. Then it follows from the theory of Raynaud extension [cf. [2], Chapter III, Corollary 7.3; [14], Corollary 6.4.9] that, if we replace K by a finite extension field of K, then there exist an abelian variety A over K with *good reduction* and an exact sequence of G_K-modules

$$0 \longrightarrow \bigoplus \mathbb{Z}_p(1) \longrightarrow \mathrm{Ker}((\Delta_{\overline{X}}^p)^{\mathrm{ab}} \twoheadrightarrow (\Delta_{\overline{X}}^{p,\mathrm{cmb}})^{\mathrm{ab}}) \longrightarrow T_p(A) \longrightarrow 0,$$

where $T_p(A)$ denotes the p-adic Tate module of A.

Next, we verify the following assertion:

Claim 2.3.A: Every G_K-equivariant homomorphism $T_p(A) \to \mathbb{Z}_p$ is trivial.

Indeed, in light of the duality theory of abelian varieties, it suffices to prove that every G_K-equivariant homomorphism

$$\mathbb{Z}_p(1) \to T_p(A^\vee)$$

is trivial, where $A^\vee$ denotes the dual abelian variety of A; $T_p(A^\vee)$ denotes the p-adic Tate module of $A^\vee$. However, since $A^\vee$ has *good reduction* over K [cf. [29], §1, Corollary 2], this follows formally from [13], Theorem. This completes the proof of Claim 2.3.A.

Finally, since the image of the p-adic cyclotomic character $G_K \to \mathbb{Z}_p^\times$ is open, we conclude from Claim 2.3.A that every G_K-equivariant homomorphism $\mathrm{Ker}((\Delta^p_{\overline{X}})^{\mathrm{ab}} \twoheadrightarrow (\Delta^{p,\mathrm{cmb}}_{\overline{X}})^{\mathrm{ab}}) \to \mathbb{Z}_p$ is trivial. This completes the proof of Lemma 2.3. □

Definition 2.4. Let Y be a hyperbolic curve over K.

(i) Suppose that Y is proper over K. Recall from [1], Corollary 2.7, that there exists a finite extension $K \subseteq L$ ($\subseteq \overline{K}$) such that Y_L has stable reduction over the ring of integers of L. Fix such a finite extension $K \subseteq L$ ($\subseteq \overline{K}$). Then we shall write

$$\Delta_Y^{\mathrm{cmb}} \stackrel{\mathrm{def}}{=} \Delta_{Y_L}^{\mathrm{cmb}}, \quad \Delta_Y^{p,\mathrm{cmb}} \stackrel{\mathrm{def}}{=} \Delta_{Y_L}^{p,\mathrm{cmb}}$$

[cf. Definition 2.1, (iv)]. Here, we note that it follows immediately from the various definitions involved that

- $\Delta_{Y_L}^{\mathrm{cmb}}$ (respectively, $\Delta_{Y_L}^{p,\mathrm{cmb}}$) is independent of the choice of L, and
- if Y has stable reduction over $\mathcal{O}_K$, then the two definitions of Δ_Y^{cmb} (respectively, $\Delta_Y^{p,\mathrm{cmb}}$) coincide.

(ii) Write $\overline{Y}$ for the smooth compactification of Y over K. Suppose that $\overline{Y}$ has genus ≥ 2 [so $\overline{Y}$ is a proper hyperbolic curve over K]. Then we shall write

$$\Delta_Y^{p,w}$$

for the kernel of the natural composite

$$\Delta_Y^p \twoheadrightarrow \Delta_{\overline{Y}}^p \twoheadrightarrow \Delta_{\overline{Y}}^{p,\mathrm{cmb}},$$

where the first arrow denotes the surjection induced by the natural open immersion $Y \hookrightarrow \overline{Y}$; the second arrow denotes the natural surjection [cf. Definitions 2.1, (ii), (iv); 2.4, (i); Remark 2.1.1].

§3. Semi-absoluteness of isomorphisms between the maximal pro-p quotients of the étale fundamental groups of hyperbolic curves over p-adic local fields

Let p be a prime number. In the present section, we apply the group-theoretic properties of various pro-p groups reviewed in the previous sections to prove the semi-absoluteness of isomorphisms between the maximal pro-p quotients of the étale fundamental groups of hyperbolic curves [cf. Theorem 3.6 below; [21], Definition 2.4, (ii)].

Definition 3.1. Let K be a field of characteristic 0; X an algebraic variety over K. Then we have an exact sequence of profinite groups

$$1 \longrightarrow \Delta_X \longrightarrow \Pi_X \longrightarrow G_K \longrightarrow 1$$

[cf. [4], Exposé IX, Théorème 6.1]. We shall say that X satisfies the *p-exactness* if the above exact sequence induces an exact sequence of pro-p groups

$$1 \longrightarrow \Delta_X^p \longrightarrow \Pi_X^p \longrightarrow G_K^p \longrightarrow 1.$$

Remark 3.1.1. In the notation of Definition 3.1, it follows immediately from the various definitions involved that the natural sequence of pro-p groups

$$\Delta_X^p \longrightarrow \Pi_X^p \longrightarrow G_K^p \longrightarrow 1$$

is exact without imposing any assumption on X. In particular, X satisfies the *p-exactness* if and only if the natural homomorphism $\Delta_X^p \to \Pi_X^p$ is *injective*.

Remark 3.1.2. Let K be a field of characteristic 0; $K \subseteq L$ a field extension; X an algebraic variety over K that satisfies the *p-exactness*. Then X_L also satisfies the *p-exactness*. Indeed, this follows immediately from the facts that

- the natural homomorphism $\Delta_{X_L} \to \Delta_X$ is an isomorphism [cf. [4], Exposé X, Corollaire 1.8], which thus induces an isomorphism $\Delta_{X_L}^p \xrightarrow{\sim} \Delta_X^p$;
- the composite $\Delta_{X_L}^p \xrightarrow{\sim} \Delta_X^p \to \Pi_X^p$ factors as the composite of the natural homomorphisms $\Delta_{X_L}^p \to \Pi_{X_L}^p$ and $\Pi_{X_L}^p \to \Pi_X^p$.

Lemma 3.2. *Let K be a field of characteristic* 0*; X a hyperbolic curve over K. Suppose that X satisfies the p-exactness [cf. Definition 3.1]. Then it holds that $\zeta_p \in K$.*

Proof. First, we note that $[K(\zeta_p) : K]$ is coprime to p. Then since X satisfies the p-exactness, by replacing Π_X^p by a suitable open subgroup of Π_X^p, we may assume without loss of generality that X has genus ≥ 2. [Note that the existence of such an open subgroup follows immediately from Hurwitz's formula.] Next, we note that since X satisfies the p-exactness, the natural outer representation $G_K \to \mathrm{Out}(\Delta_X^p)$ [induced by the natural exact sequence of profinite groups $1 \to \Delta_X \to \Pi_X \to G_K \to 1$] factors through the maximal pro-p quotient $G_K \twoheadrightarrow G_K^p$. Write $\overline{X}$ for the smooth compactification of X over K. Then it follows immediately that the natural outer representation $G_K \to \mathrm{Out}(\Delta_{\overline{X}}^p)$ [induced by the natural exact sequence of profinite groups $1 \to \Delta_{\overline{X}} \to \Pi_{\overline{X}} \to G_K \to 1$] also factors through the maximal pro-p quotient $G_K \twoheadrightarrow G_K^p$. In particular, the natural action of G_K on

$$\mathrm{Hom}(H^2(\Delta_{\overline{X}}^p,\ \mathbb{Z}_p),\ \mathbb{Z}_p)$$

induced by the natural outer action $G_K \to \mathrm{Out}(\Delta_{\overline{X}}^p)$ factors through the maximal pro-p quotient $G_K \twoheadrightarrow G_K^p$. Observe that since $\overline{X}$ is a proper hyperbolic curve, it holds that $\mathrm{Hom}(H^2(\Delta_{\overline{X}}^p,\ \mathbb{Z}_p),\ \mathbb{Z}_p)$ is isomorphic to $\mathbb{Z}_p(1)$ as G_K-modules, where "(1)" denotes the Tate twist. Thus, we conclude that $\zeta_p \in K$. This completes the proof of Lemma 3.2. □

Proposition 3.3. *Let K be a p-adic local field; X a hyperbolic curve over K that has genus ≥ 2; G a free pro-p group of finite rank, or a Demushkin group isomorphic to the maximal pro-p quotient of the absolute Galois group of some p-adic local field;*

$$\phi : \Pi_X^p \to G$$

an open homomorphism. Write $i : \Delta_X^p \to \Pi_X^p$ for the natural homomorphism induced by the natural injection $\Delta_X \hookrightarrow \Pi_X$. Then

$$\phi \circ i(\Delta_X^{p,w}) = \{1\}$$

[cf. Definition 2.4, (ii)].

Proof. Note that, for each finite extension $K \subseteq L$ $(\subseteq \overline{K})$, the natural homomorphism $i : \Delta_X^p \to \Pi_X^p$ factors as the composite of the natural homomorphism $\Delta_X^p \to \Pi_{X_L}^p$ with the natural *open* homomorphism $\Pi_{X_L}^p \to \Pi_X^p$ [induced by the natural open injection $\Pi_{X_L} \hookrightarrow \Pi_X$]. Thus, by applying the well-known stable reduction theorem [cf. [1], Corollary 2.7], we may assume without loss of generality that X has stable reduction over the ring of integers of K.

Next, we observe that every open subgroup of G is also a free pro-p group of finite rank or a Demushkin group isomorphic to the maximal pro-p quotient of the absolute Galois group of some p-adic local field. Thus, we may also assume without loss of generality that ϕ is surjective.

Then since G is a pro-solvable group, to verify Proposition 3.3, it suffices to verify the following assertion:

Claim 3.3.A: Let $N \subseteq G$ be an open subgroup such that $\phi \circ i(\Delta_X^{p,w}) \subseteq N$. Then the image of $\phi \circ i(\Delta_X^{p,w})$ via the natural surjection $N \twoheadrightarrow N^{\mathrm{ab}}$ is trivial.

Indeed, by replacing Π_X^p by $\phi^{-1}(N)$, we may assume without loss of generality that $N = G$. Then we obtain a G_K-equivariant homomorphism

$$(\Delta_X^p)^{\mathrm{ab}} \to G^{\mathrm{ab}},$$

where G^{ab} is endowed with the trivial action of G_K. Thus, it follows immediately from Lemma 2.3 that the image of $\phi \circ i(\Delta_X^{p,w})$ via the composite of the natural surjections

$$f : G \twoheadrightarrow G^{\mathrm{ab}} \twoheadrightarrow G^{\mathrm{ab}}/(G^{\mathrm{ab}})_{\mathrm{tor}}$$

is trivial. In particular, since the abelianization of any free pro-p group is torsion-free, we complete the proof of Claim 3.3.A in the case where G is a free pro-p group of finite rank. Thus, we may assume without loss of generality that G is a Demushkin group that equals $G_{K'}^p$ for some p-adic local field K'. Write

- p^a for the cardinality of $(G^{\mathrm{ab}})_{\mathrm{tor}}$, i.e., the cardinality of the set of p-power roots of unity $\in K'$, where we note that $a \geq 1$ [cf. Remark 1.1.1; Proposition 1.3; Theorem 1.4, (i)];
- $K' \subseteq L'$ $(\subseteq \overline{K}')$ for the unramified extension of degree p^a.

In the remainder of the proof, we regard $G_{L'}^p$ as an open subgroup of G via the natural open injection $G_{L'}^p \hookrightarrow G$. Here, we note that since $K' \subseteq L'$ $(\subseteq \overline{K}')$ is an unramified extension, the natural quotient $G_{K'}^p \twoheadrightarrow G_{K'}^p/G_{L'}^p$ factors through the quotient of $G_{K'}^p$ by the inertia subgroup of $G_{K'}^p$, which is torsion-free and abelian. Therefore, the normal open subgroup $G_{L'}^p \subseteq G$ coincides with the pull-back of a normal open subgroup of $G^{\mathrm{ab}}/(G^{\mathrm{ab}})_{\mathrm{tor}}$ via f. Then since $f \circ \phi \circ i(\Delta_X^{p,w}) = \{1\}$, it holds that

$$\Delta_X^{p,w} \subseteq (\phi \circ i)^{-1}(G_{L'}^p) \subseteq \Delta_X^p.$$

Thus, by applying Lemma 2.3 to the open homomorphism $\phi^{-1}(G_{L'}^p) \twoheadrightarrow G_{L'}^p$, we observe that the image of $\phi \circ i(\Delta_X^{p,w})$ $(\subseteq G_{L'}^p)$ via the composite of the natural surjections

$$G_{L'}^p \twoheadrightarrow (G_{L'}^p)^{\mathrm{ab}} \twoheadrightarrow (G_{L'}^p)^{\mathrm{ab}}/((G_{L'}^p)^{\mathrm{ab}})_{\mathrm{tor}}$$

is trivial. On the other hand, it follows immediately from the functoriality of the reciprocity map [cf. [25], Chapter IV, Proposition 5.8] that the image of $((G_{L'}^p)^{\mathrm{ab}})_{\mathrm{tor}}$ via the natural homomorphism

$$(G_{L'}^p)^{\mathrm{ab}} \to (G_{K'}^p)^{\mathrm{ab}} = G^{\mathrm{ab}}$$

[induced by the inclusion $G^p_{L'} \subseteq G^p_{K'} = G$] is trivial. Thus, we conclude that the image of $\phi \circ i(\Delta_X^{p,w})$ via the natural surjection $G \twoheadrightarrow G^{\mathrm{ab}}$ is trivial. This completes the proof of Claim 3.3.A, hence of Proposition 3.3. □

Corollary 3.4. *Let K be a p-adic local field; X a hyperbolic curve over K; I a cuspidal inertia subgroup of Δ_X^p; G a free pro-p group of finite rank, or a Demushkin group isomorphic to the maximal pro-p quotient of the absolute Galois group of some p-adic local field;*

$$\phi : \Pi_X^p \to G$$

an open homomorphism. Write $i : \Delta_X^p \to \Pi_X^p$ for the natural homomorphism induced by the natural injection $\Delta_X \hookrightarrow \Pi_X$. Then

$$\phi \circ i(I) = \{1\}.$$

Proof. Let $Y \to X_L$ be a finite étale Galois covering over some finite extension $K \subseteq L$ $(\subseteq \overline{K})$ such that the hyperbolic curve Y has genus ≥ 2. [Note that the existence of such a covering follows immediately from Hurwitz's formula.] Write

$$g : \Pi_Y^p \longrightarrow \Pi_{X_L}^p \longrightarrow \Pi_X^p \xrightarrow{\phi} G$$

for the composite of the open homomorphisms, where the first and second arrow denote the open homomorphisms induced by the finite étale covering $Y \to X_L$ and the projection morphism $X_L \to X$;

$$i_Y : \Delta_Y^p \to \Pi_Y^p$$

for the natural homomorphism induced by the natural injection $\Delta_Y \hookrightarrow \Pi_Y$. Then, by applying Proposition 3.3 to the open homomorphism g, we conclude that, for each cuspidal inertia subgroup I_Y of Δ_Y^p, it holds that $g \circ i_Y(I_Y) = \{1\}$. On the other hand, it follows immediately from the various definitions involved that there exists a cuspidal inertia subgroup I_Y of Δ_Y^p whose image in Δ_X^p via the natural homomorphism $\Delta_Y^p \to \Delta_X^p$ is an open subgroup of I. Thus, we conclude that $\phi \circ i(I) \subseteq G$ is a finite subgroup. However, since G is torsion-free [cf. Lemma 1.6], it holds that $\phi \circ i(I) = \{1\}$. This completes the proof of Corollary 3.4. □

Lemma 3.5. *Let*

$$1 \longrightarrow \Delta \longrightarrow \Pi \longrightarrow G \longrightarrow 1$$

be an exact sequence of profinite groups. Write

$$\rho : G \to \mathrm{Out}(\Delta)$$

for the outer representation determined by the above exact sequence. Suppose that $\mathrm{Im}(\rho) = \{1\}$*, and* Δ *is center-free. Then there exists a unique section* $s : G \hookrightarrow \Pi$ *of the surjection* $\Pi \twoheadrightarrow G$ *such that* $s(G)$ $(\subseteq \Pi)$ *commutes with* Δ $(\subseteq \Pi)$*. In particular, the inclusion* $\Delta \subseteq \Pi$ *and the section* s *determine a direct product decomposition*

$$\Delta \times G \xrightarrow{\sim} \Pi,$$

which thus induces a splitting $\Pi \twoheadrightarrow \Delta$ *of the inclusion* $\Delta \subseteq \Pi$*.*

Proof. It suffices to prove that, for each $g \in G$, there exists a unique lifting $\tilde{g} \in \Pi$ of g that commutes with Δ $(\subseteq \Pi)$. However, the existence (respectively, the uniqueness) follows immediately from our assumption that $\mathrm{Im}(\rho) = \{1\}$ (respectively, Δ is center-free). This completes the proof of Lemma 3.5. □

Next, we prove our first main result [cf. Theorem A, (ii), for hyperbolic curves over p-adic local fields].

Theorem 3.6. *Let* K*,* K' *be* p*-adic local fields;* X*,* X' *hyperbolic curves over* K*,* K'*, respectively;*

$$\alpha : \Pi_X^p \xrightarrow{\sim} \Pi_{X'}^p$$

an isomorphism of profinite groups. Then the following hold:

(i) Write Γ *for the dual semi-graph associated to the special fiber of the stable model of* $X_{\overline{K}}$ *[over the ring of integers of* $\overline{K}$*]. Suppose that the first Betti number of* $\Gamma \leq 1$*. Then* α *induces an isomorphism* $G_K^p \xrightarrow{\sim} G_{K'}^p$ *that fits into a commutative diagram*

$$\begin{array}{ccc} \Pi_X^p & \xrightarrow[\alpha]{\sim} & \Pi_{X'}^p \\ \downarrow & & \downarrow \\ G_K^p & \xrightarrow{\sim} & G_{K'}^p, \end{array}$$

where the vertical arrows denote the natural surjections [determined up to composition with an inner automorphism] induced by the structure morphisms of the hyperbolic curves X*,* X'*.*

(ii) Suppose that

$$X \text{ and } X' \text{ satisfy the } p\text{-exactness [cf. Definition 3.1].}$$

Then α induces an isomorphism $G_K^p \overset{\sim}{\to} G_{K'}^p$ that fits into a commutative diagram

$$\begin{array}{ccc} \Pi_X^p & \xrightarrow[\alpha]{\sim} & \Pi_{X'}^p \\ \big\downarrow & & \big\downarrow \\ G_K^p & \xrightarrow{\sim} & G_{K'}^p, \end{array}$$

where the vertical arrows denote the natural surjections [determined up to composition with an inner automorphism] induced by the structure morphisms of the hyperbolic curves X, X'.

Proof. First, we verify assertion (i). Write $\overline{k}$ for the residue field of the ring of integers of $\overline{K}$; $X_{\overline{k}}$ for the special fiber of the stable model of $X_{\overline{K}}$. Let $Y_{\overline{k}} \to X_{\overline{k}}$ be an admissible covering over $\overline{k}$ [cf. [16], §2] such that

- $Y_{\overline{k}}$ has genus ≥ 2, and
- the first Betti number of $\Gamma_{Y_{\overline{k}}} \leq 1$, where $\Gamma_{Y_{\overline{k}}}$ denotes the dual semi-graph associated to $Y_{\overline{k}}$.

[Observe that, in light of our assumption that the first Betti number of $\Gamma \leq 1$, such an admissible covering may be constructed by gluing together suitable admissible coverings of the irreducible [pointed] stable curves associated to the irreducible components of $X_{\overline{k}}$.] Write $Y_{\overline{K}} \to X_{\overline{K}}$ for the connected finite étale covering over $\overline{K}$ obtained by deforming the admissible covering $Y_{\overline{k}} \to X_{\overline{k}}$ over $\overline{k}$. Let $K \subseteq L$ ($\subseteq \overline{K}$) be a finite field extension such that

- the connected finite étale covering $Y_{\overline{K}} \to X_{\overline{K}}$ over $\overline{K}$ descends to a connected finite étale covering $Y \to X_L$ over L, and
- the smooth compactification $\overline{Y}$ of Y has stable reduction over the ring of integers of L [cf. [1], Corollary 2.7].

Here, we note that $\Delta_{\overline{Y}}^{p,\mathrm{cmb}}$ is isomorphic to the pro-p completion of the topological fundamental group of $\Gamma_{Y_{\overline{k}}}$. Then since this topological fundamental group is free, it follows immediately from the various definitions involved that the first Betti number of $\Gamma_{Y_{\overline{k}}}$ coincides with rank $\Delta_{\overline{Y}}^{p,\mathrm{cmb}}$. Thus, in summary,

- Y is a hyperbolic curve over L of genus ≥ 2 whose smooth compactification $\overline{Y}$ has stable reduction over the ring of integers of L, and
- rank $\Delta_{\overline{Y}}^{p,\mathrm{cmb}} \leq 1$. [In particular, $\Delta_{\overline{Y}}^{p,\mathrm{cmb}}$ is abelian.]

Then we obtain a commutative diagram of profinite groups

$$\begin{array}{ccccccc}
\Delta_Y^p & \longrightarrow & \Pi_Y^p & \longrightarrow & G_L^p & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
\Delta_X^p & \longrightarrow & \Pi_X^p & \longrightarrow & G_K^p & \longrightarrow & 1 \\
& & \alpha \downarrow \wr & & & & \\
\Delta_{X'}^p & \longrightarrow & \Pi_{X'}^p & \longrightarrow & G_{K'}^p & \longrightarrow & 1,
\end{array}$$

where the horizontal sequences are the natural exact sequences as in Remark 3.1.1; the vertical arrows $\Delta_Y^p \to \Delta_X^p$, $\Pi_Y^p \to \Pi_X^p$, and $G_L^p \to G_K^p$ denote the natural open homomorphisms. Write

$$g : \Pi_Y^p \to \Pi_X^p \underset{\alpha}{\xrightarrow{\sim}} \Pi_{X'}^p \to G_{K'}^p$$

for the composite of the open homomorphisms that appear in the above commutative diagram;

$$g|_{\Delta_Y^p} : \Delta_Y^p \to G_{K'}^p$$

for the composite of the natural homomorphism $\Delta_Y^p \to \Pi_Y^p$ with the homomorphism g. Then it follows immediately from the various definitions involved that

- $\mathrm{Im}(g) \subseteq G_{K'}^p$ is an open subgroup;
- $\mathrm{Im}(g|_{\Delta_Y^p}) \subseteq \mathrm{Im}(g)$ is a topologically finitely generated normal closed subgroup.

Then since $G_{K'}^p$ is elastic [cf. Theorem 1.5, (ii)], it holds that $\mathrm{Im}(g|_{\Delta_Y^p})$ is trivial or an open subgroup of $G_{K'}^p$. Recall that every open subgroup of $G_{K'}^p$ is nonabelian [cf. Lemma 1.6]. Thus, since $\Delta_Y^{p,\mathrm{cmb}}$ is abelian, it follows immediately from Proposition 3.3 that $\mathrm{Im}(g|_{\Delta_Y^p})$ is trivial. Therefore, the image of the composite

$$\Delta_X^p \to \Pi_X^p \underset{\alpha}{\xrightarrow{\sim}} \Pi_{X'}^p \to G_{K'}^p$$

of the homomorphisms that appear in the above commutative diagram is a finite group. Then since $G_{K'}^p$ is torsion-free [cf. Lemma 1.6], we observe that this image is also trivial. In particular, the above commutative diagram induces a surjection $G_K^p \twoheadrightarrow G_{K'}^p$, whose kernel is topologically finitely generated. However, since G_K^p is elastic, and $G_{K'}^p$ is infinite, it holds that this surjection is an isomorphism. This completes the proof of assertion (i).

Next, we verify assertion (ii). Note that G_K^p and $G_{K'}^p$ are torsion-free [cf. Lemma 1.6]. Then since X and X' satisfy the p-exactness, by replacing Π_X^p and $\Pi_{X'}^p$ by suitable normal open subgroups, we may assume without loss of generality that X and X' have genus ≥ 2. Moreover, by applying the proof of assertion (i), we may assume without loss of generality that

$$\mathrm{rank}\ \Delta^{p,\mathrm{cmb}}_{\overline{X}} \geq 2, \quad \mathrm{rank}\ \Delta^{p,\mathrm{cmb}}_{\overline{X}'} \geq 2$$

[cf. Proposition 2.2, (iii); Definition 2.4, (i), (ii)]. In particular, $\Delta^{p,\mathrm{cmb}}_{\overline{X}}$ and $\Delta^{p,\mathrm{cmb}}_{\overline{X}'}$ are *center-free.*

Next, it follows from the well-known stable reduction theorem [cf. [1], Corollary 2.7] that there exists a finite Galois extension $K \subseteq L$ $(\subseteq \overline{K})$ (respectively, $K' \subseteq L'$ $(\subseteq \overline{K}')$) such that

- the smooth compactification of X_L (respectively, $X'_{L'}$) has stable reduction over the ring of integers of L (respectively, L');
- the natural outer action of G_L on $\Delta^{\mathrm{cmb}}_{\overline{X}}$ (respectively, $G_{L'}$ on $\Delta^{\mathrm{cmb}}_{\overline{X}'}$) is trivial;
- $X_L(L) \neq \emptyset$ (respectively, $X'_{L'}(L') \neq \emptyset$).

Fix such finite Galois extensions $K \subseteq L$ $(\subseteq \overline{K})$ and $K' \subseteq L'$ $(\subseteq \overline{K}')$. Thus, by applying Lemma 3.5, we obtain a natural surjection $\Pi_{X'_{L'}} \twoheadrightarrow \Delta^{\mathrm{cmb}}_{\overline{X}'}$ whose restriction to $\Delta_{X'}$ coincides with the natural quotient $\Delta_{X'} \twoheadrightarrow \Delta^{\mathrm{cmb}}_{\overline{X}'}$ [cf. Remark 2.1.1]. Write

- $\Pi^{w}_{X'_{L'}} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{X'_{L'}} \twoheadrightarrow \Delta^{\mathrm{cmb}}_{\overline{X}'})$, where we note that the normal closed subgroup $\Pi^{w}_{X'_{L'}} \subseteq \Pi_{X'_{L'}}$ $(\subseteq \Pi_{X'})$ is a normal closed subgroup of $\Pi_{X'}$ topologically normally generated by the normal closed subgroup $\mathrm{Ker}(\Delta_{X'} \twoheadrightarrow \Delta^{\mathrm{cmb}}_{\overline{X}'}) \subseteq \Pi_{X'}$ and the image of a section of the surjection $\Pi_{X'_{L'}} \twoheadrightarrow G_{L'}$ determined by an L'-valued point of $X'_{L'}$;
- $\Pi^{p,w}_{X'} \stackrel{\mathrm{def}}{=} \mathrm{Im}(\Pi^{w}_{X'_{L'}} \subseteq \Pi_{X'_{L'}} \subseteq \Pi_{X'} \twoheadrightarrow \Pi^{p}_{X'})$. [In particular, $\Pi^{p,w}_{X'} \subseteq \Pi^{p}_{X'}$ is a normal closed subgroup.]

Next, we verify the following assertion:

Claim 3.6.A: The homomorphism $\Delta^{p,\mathrm{cmb}}_{\overline{X}'} \to \Pi^{p}_{X'}/\Pi^{p,w}_{X'}$ induced by the natural homomorphism $\Delta^{p}_{X'} \to \Pi^{p}_{X'}$ is injective. In particular, there exists a commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta^{p}_{X'} & \longrightarrow & \Pi^{p}_{X'} & \longrightarrow & G^{p}_{K'} & \longrightarrow & 1 \\ & & \downarrow & & \psi\downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta^{p,\mathrm{cmb}}_{\overline{X}'} & \longrightarrow & \Pi^{p}_{X'}/\Pi^{p,w}_{X'} & \longrightarrow & \mathrm{Gal}(L'/K')^{p} & \longrightarrow & 1, \end{array}$$

where the vertical arrows denote the natural surjections.

Note that there exists a natural exact sequence of profinite groups

$$1 \longrightarrow \Delta_{\overline{X'}}^{\mathrm{cmb}} \longrightarrow \Pi_{X'}/\Pi_{X'_{L'}}^{w} \longrightarrow \mathrm{Gal}(L'/K') \longrightarrow 1.$$

Write

$$\rho : \mathrm{Gal}(L'/K') \to \mathrm{Out}(\Delta_{\overline{X'}}^{p,\mathrm{cmb}})$$

for the composite of the outer representation $\mathrm{Gal}(L'/K') \to \mathrm{Out}(\Delta_{\overline{X'}}^{\mathrm{cmb}})$ determined by the above exact sequence with the homomorphism $\mathrm{Out}(\Delta_{\overline{X'}}^{\mathrm{cmb}}) \to \mathrm{Out}(\Delta_{\overline{X'}}^{p,\mathrm{cmb}})$ induced by the natural surjection $\Delta_{\overline{X'}}^{\mathrm{cmb}} \twoheadrightarrow \Delta_{\overline{X'}}^{p,\mathrm{cmb}}$. Recall that $\Delta_{\overline{X'}}^{p,\mathrm{cmb}}$ is *center-free*. Thus, it suffices to prove that the outer representation ρ factors through the maximal pro-p quotient $\mathrm{Gal}(L'/K') \twoheadrightarrow \mathrm{Gal}(L'/K')^p$. Observe that since X' satisfies the p-exactness, the composite $G_{K'} \twoheadrightarrow \mathrm{Gal}(L'/K') \xrightarrow{\rho} \mathrm{Out}(\Delta_{\overline{X'}}^{p,\mathrm{cmb}})$ of the natural surjections factors through the maximal pro-p quotient $G_{K'} \twoheadrightarrow G_{K'}^p$. Thus, we obtain the desired conclusion. This completes the proof of Claim 3.6.A.

Next, we verify the following assertion:

Claim 3.6.B: $\alpha(\Delta_X^{p,w}) = \Delta_{X'}^{p,w}$ [cf. Definition 2.4, (ii)].

Indeed, by applying Proposition 3.3 to the composite $\Pi_X^p \twoheadrightarrow G_{K'}^p$ of α with the natural surjection $\Pi_{X'}^p \twoheadrightarrow G_{K'}^p$, we observe that

$$\alpha(\Delta_X^{p,w}) \subseteq \Delta_{X'}^p.$$

Then it holds that

- $(\psi \circ \alpha)^{-1}(\Delta_{\overline{X'}}^{p,\mathrm{cmb}}) \subseteq \Pi_X^p$ is a normal open subgroup [cf. Claim 3.6.A];
- $\psi \circ \alpha(\Delta_X^{p,w}) \subseteq \Delta_{\overline{X'}}^{p,\mathrm{cmb}}$ [cf. the fact that $\alpha(\Delta_X^{p,w}) \subseteq \Delta_{X'}^p$, together with Claim 3.6.A].

Therefore, by applying Proposition 3.3 to the natural surjection

$$(\psi \circ \alpha)^{-1}(\Delta_{\overline{X'}}^{p,\mathrm{cmb}}) \twoheadrightarrow \Delta_{\overline{X'}}^{p,\mathrm{cmb}}$$

induced by $\psi \circ \alpha$, we observe that

$$\psi \circ \alpha(\Delta_X^{p,w}) = \{1\}.$$

Then since $\alpha(\Delta_X^{p,w}) \subseteq \Delta_{X'}^p$, it follows from Claim 3.6.A that

$$\alpha(\Delta_X^{p,w}) \subseteq \Delta_{X'}^{p,w}.$$

On the other hand, by applying a similar argument [to the argument applied above] to α^{-1}, we also have $\alpha^{-1}(\Delta_{X'}^{p,w}) \subseteq \Delta_X^{p,w}$. Thus, we conclude that $\alpha(\Delta_X^{p,w}) = \Delta_{X'}^{p,w}$. This completes the proof of Claim 3.6.B.

Next, by applying Claim 3.6.B, we obtain a diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_{\overline{X}}^{p,\mathrm{cmb}} & \longrightarrow & \Pi_X^p/\Delta_X^{p,w} & \longrightarrow & G_K^p & \longrightarrow & 1 \\ & & & & \beta \downarrow \wr & & & & \\ 1 & \longrightarrow & \Delta_{\overline{X'}}^{p,\mathrm{cmb}} & \longrightarrow & \Pi_{X'}^p/\Delta_{X'}^{p,w} & \xrightarrow{q'} & G_{K'}^p & \longrightarrow & 1, \end{array}$$

where β denotes the isomorphism induced by α; q' denotes the surjection induced by the natural surjection $\Pi_{X'}^p \twoheadrightarrow G_{K'}^p$. Suppose that

$$q' \circ \beta(\Delta_{\overline{X}}^{p,\mathrm{cmb}}) \neq \{1\}.$$

Then since $G_{K'}^p$ is elastic, it holds that $q' \circ \beta(\Delta_{\overline{X}}^{p,\mathrm{cmb}}) \subseteq G_{K'}^p$ is a normal open subgroup. On the other hand, since $\Delta_{\overline{X}}^{p,\mathrm{cmb}}$ is *center-free*, and the natural outer action of G_L^p on $\Delta_{\overline{X}}^{p,\mathrm{cmb}}$ is trivial, it follows from Lemma 3.5 that we obtain a commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_X^p & \longrightarrow & \Pi_{X_L}^p & \longrightarrow & G_L^p & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Delta_{\overline{X}}^{p,\mathrm{cmb}} & \longrightarrow & \Delta_{\overline{X}}^{p,\mathrm{cmb}} \times G_L^p & \longrightarrow & G_L^p & \longrightarrow & 1 \\ & & \| & & \downarrow h & & \downarrow & & \\ 1 & \longrightarrow & \Delta_{\overline{X}}^{p,\mathrm{cmb}} & \longrightarrow & \Pi_X^p/\Delta_X^{p,w} & \longrightarrow & G_K^p & \longrightarrow & 1, \end{array}$$

where $\Delta_{\overline{X}}^{p,\mathrm{cmb}} \times G_L^p \to G_L^p$ denotes the second projection; $G_L^p \to G_K^p$ denotes the natural open homomorphism [induced by the natural open injection $G_L \subseteq G_K$]; h denotes the open homomorphism determined by the natural open homomorphism $\Pi_{X_L}^p \to \Pi_X^p$ [induced by the natural open injection $\Pi_{X_L} \subseteq \Pi_X$]. Write

$$s : G_L^p \hookrightarrow \Delta_{\overline{X}}^{p,\mathrm{cmb}} \times G_L^p$$

for the section of the second projection $\Delta_{\overline{X}}^{p,\mathrm{cmb}} \times G_L^p \twoheadrightarrow G_L^p$ that maps $x \in G_L^p$ to $(1, x) \in \Delta_{\overline{X}}^{p,\mathrm{cmb}} \times G_L^p$. Then since

- $\mathrm{Im}(h \circ s) \subseteq Z_{\Pi_X^p/\Delta_X^{p,w}}(\Delta_{\overline{X}}^{p,\mathrm{cmb}})$,
- $\Delta_{\overline{X}}^{p,\mathrm{cmb}}$ is *center-free*, and
- the homomorphism $G_L^p \to G_K^p$ is open,

it holds that the centralizer $Z_{\Pi_X^p/\Delta_X^{p,w}}(\Delta_{\overline{X}}^{p,\mathrm{cmb}})$ is isomorphic to an open subgroup of G_K^p. Recall from Theorem 1.4, (ii), together with Lemma 3.2, that G_K^p and $G_{K'}^p$ are Demushkin groups. In particular, the centralizer $Z_{\Pi_X^p/\Delta_X^{p,w}}(\Delta_{\overline{X}}^{p,\mathrm{cmb}})$ is a

Demushkin group. On the other hand, it follows from the slimness of $G^p_{K'}$, together with the fact that $q' \circ \beta(\Delta^{p,\mathrm{cmb}}_{\overline{X}})$ is an open subgroup of $G^p_{K'}$, that there exists an inclusion

$$\beta(Z_{\Pi^p_X/\Delta^{p,w}_X}(\Delta^{p,\mathrm{cmb}}_{\overline{X}})) = Z_{\Pi^p_{X'}/\Delta^{p,w}_{X'}}(\beta(\Delta^{p,\mathrm{cmb}}_{\overline{X}})) \subseteq \Delta^{p,\mathrm{cmb}}_{\overline{X}'}.$$

Then since $\Delta^{p,\mathrm{cmb}}_{\overline{X}'}$ is a free pro-p group [cf. Proposition 2.2, (iii)], it holds that the centralizer $Z_{\Pi^p_X/\Delta^{p,w}_X}(\Delta^{p,\mathrm{cmb}}_{\overline{X}})$ is also a *free pro-p group* [cf. [27], Corollary 7.7.5]. However, this contradicts Remark 1.1.1. Thus, we conclude that

$$q' \circ \beta(\Delta^{p,\mathrm{cmb}}_{\overline{X}}) = \{1\},$$

hence that

$$\beta(\Delta^{p,\mathrm{cmb}}_{\overline{X}}) \subseteq \Delta^{p,\mathrm{cmb}}_{\overline{X}'}.$$

Moreover, by applying a similar argument [to the argument applied above] to β^{-1}, we also have

$$\beta^{-1}(\Delta^{p,\mathrm{cmb}}_{\overline{X}'}) \subseteq \Delta^{p,\mathrm{cmb}}_{\overline{X}}.$$

In particular, it holds that $\beta(\Delta^{p,\mathrm{cmb}}_{\overline{X}}) = \Delta^{p,\mathrm{cmb}}_{\overline{X}'}$, which thus induces an isomorphism $G^p_K \xrightarrow{\sim} G^p_{K'}$. This completes the proof of assertion (ii), hence of Theorem 3.6. □

§4. Semi-absoluteness of isomorphisms between the maximal pro-*p* quotients of the étale fundamental groups of configuration spaces associated to hyperbolic curves

In the present section, we apply the results obtained in the previous sections [especially, the semi-absoluteness of isomorphisms between the maximal pro-p quotients of the étale fundamental groups of hyperbolic curves — cf. Theorem 3.6; [21], Definition 2.4, (ii)] and some facts that appear in combinatorial anabelian geometry [especially, the "*mono-anabelian*" reconstruction of the dimensions of configuration spaces associated to hyperbolic curves obtained by Hoshi-Minamide-Mochizuki — cf. [11], Theorem 1.6] to prove the analogous assertion [i.e., the semi-absoluteness] for higher dimensional configuration spaces associated to hyperbolic curves.

Let p be a prime number. First, we begin by recalling the definition of configuration spaces associated to hyperbolic curves.

Definition 4.1. Let n be a positive integer; K a field; X a hyperbolic curve over K. Write

$$X_n \overset{\text{def}}{=} X^{\times n} \setminus (\bigcup_{1\leq i<j\leq n} \Delta_{i,j}),$$

where $X^{\times n}$ denotes the fiber product of n copies of X over K; $\Delta_{i,j}$ denotes the diagonal divisor of $X^{\times n}$ associated to the i-th and j-th components. We shall refer to X_n as the *n-th configuration space* associated to X.

Remark 4.1.1. In the notation of Definition 4.1, suppose that K is of characteristic 0. Then it follows immediately from [24], Proposition 2.2, (i), that X_n satisfies the p-exactness if and only if X satisfies the p-exactness.

Proposition 4.2. *Let n be a positive integer; K a p-adic local field; X a hyperbolic curve over K. Write X_n for the n-th configuration space associated to X;*

$$t \overset{\text{def}}{=} \max\{s \in \mathbb{N} \mid \exists \text{a closed subgroup of } \Pi^p_{X_n} \text{isomorphic to } \mathbb{Z}_p^{\oplus s}\}.$$

Suppose that

$$X_n \text{ satisfies the } p\text{-exactness.}$$

Then the following hold:

(i) Suppose, moreover, that X is a proper hyperbolic curve over K. Then

- $\mathrm{cd}_p(\Pi^p_{X_n}) = n + 3$;
- $n \leq t \leq n + 1$.

(ii) Suppose, moreover, that X is an affine hyperbolic curve over K. Then

- $\mathrm{cd}_p(\Pi^p_{X_n}) = n + 2$;
- $t = n + 1$.

In particular, the following hold:

- *X is proper if and only if* $\mathrm{cd}_p(\Pi^p_{X_n}) - t \geq 2$.
- *Let Π be a topological group isomorphic to $\Pi^p_{X_n}$. Then there exists a functorial group-theoretic algorithm*

$$\Pi \quad \rightsquigarrow \quad n$$

for constructing the dimension n from Π.

Proof. Let Δ be a pro-p surface group [cf. [24], Definition 1.2 — where we take "$\mathcal{C}$" to be the family of all finite p-groups]. Recall that, if Δ is a free pro-p group (respectively, not a free pro-p group), then $\mathrm{cd}_p(\Delta) = 1$ (respectively, $\mathrm{cd}_p(\Delta) = 2$). On the other hand, since X_n satisfies the p-exactness, it follows immediately from Theorem 1.5, (iii); Lemma 3.2; Remark 4.1.1, that $\mathrm{cd}_p(G_K^p) = 2$. Thus, the assertions concerning $\mathrm{cd}_p(\Pi_{X_n}^p)$ follow immediately from [24], Proposition 2.2, (i); [27], Proposition 7.4.2, (b), (i).

Next, we verify the following assertion:

Claim 4.2.A: $n \leq t \leq n+1$.

Indeed, it follows immediately from [11], Theorem 1.6, that $n \leq t$, and there exists a closed subgroup of G_K^p isomorphic to $\mathbb{Z}_p^{\oplus t-m} \oplus T$, where m denotes an integer such that $0 \leq m \leq n$; T denotes a finite group. Suppose that $t \geq n+2$. Then there exists a closed subgroup $H \subseteq G_K^p$ such that

$$H \cong \mathbb{Z}_p^{\oplus 2}.$$

In particular, $H \subseteq G_K^p$ is an abelian closed subgroup. Moreover, since every open subgroup of G_K^p is nonabelian [cf. Lemma 1.6], it follows from Theorem 1.5, (iii), that H is a free pro-p group. This contradicts the fact that $H \cong \mathbb{Z}_p^{\oplus 2}$. Thus, we conclude that $t \leq n+1$. This completes the proof of Claim 4.2.A, hence of assertion (i).

Finally, in light of Claim 4.2.A, to complete the proof of assertion (ii), it suffices to prove that there exists a closed subgroup of $\Pi_{X_n}^p$ isomorphic to $\mathbb{Z}_p^{\oplus n+1}$. Write $X_n^{\log}$ for the n-th log configuration space associated to the hyperbolic curve X [cf. [11], §0, Curves — where we note that, in our notation, the interior of $X_n^{\log}$ may be identified with X_n]; $(\Pi_{X_n} \xrightarrow{\sim})\ \Pi_{X_n^{\log}}$ for the log étale fundamental group of $X_n^{\log}$, relative to a suitable choice of basepoint [cf. [18], Theorem B]. Write $D \subseteq \Pi_{X_n}$ for the decomposition subgroup associated to a log-full point of $X_n^{\log}$ [cf. [11], Definition 1.1], where we note that the existence of a log-full point follows from [11], Proposition 1.2, (i); [11], Proposition 1.3, (i), together with our assumption that X is affine. Then it follows immediately from a [log] scheme-theoretic consideration that there exist a finite extension $K \subseteq L$ $(\subseteq \overline{K})$ and a natural exact sequence of profinite groups

$$1 \longrightarrow \bigoplus \widehat{\mathbb{Z}}(1) \longrightarrow D \longrightarrow G_L \longrightarrow 1$$

[where "(1)" denotes the Tate twist], which induces [cf. our assumption that X_n satisfies the p-exactness] an exact sequence of pro-p groups

$$1 \longrightarrow \bigoplus \mathbb{Z}_p(1) \longrightarrow D^p \stackrel{r}{\longrightarrow} G_L^p \longrightarrow 1.$$

Let $I \subseteq G_L^p$ be a closed subgroup such that

- $I \cong \mathbb{Z}_p$;
- the image of I via the natural open homomorphism $G_L^p \to G_K^p$ [induced by the inclusion $G_L \subseteq G_K$] is also isomorphic to $\mathbb{Z}_p$;
- the image of I via the p-adic cyclotomic character $G_L^p \to \mathbb{Z}_p^\times$ is trivial [where we note that $\zeta_p \in K \subseteq L$ — cf. Lemma 3.2; Remark 4.1.1].

Write $H \subseteq \Pi_{X_n}^p$ for the image of $r^{-1}(I)$ via the natural homomorphism $D^p \to \Pi_{X_n}^p$ [induced by the inclusion $D \subseteq \Pi_{X_n}$]. Then it follows immediately from the various definitions involved that $H \cong \mathbb{Z}_p^{\oplus n+1}$. This completes the proof of Proposition 4.2. □

Remark 4.2.1. The fact that the dimension of X_n may be reconstructed, in a purely group-theoretically way, from $\Pi_{X_n}^p$ was pointed out to the author of the present paper by K. Sawada. More precisely, he explained to the author that such a result may be obtained by applying a similar argument to the argument applied in the proof of [28], Theorem 2.15. However, since the above proof [of Proposition 4.2] is a direct and easy application of the results obtained in [11], §1 [which is also a direct and easy application of log geometry], the author decided to include this proof in the present paper.

Proposition 4.3. *Let K be a field of characteristic 0 that contains ζ_p ($\in \overline{K}$). Suppose that K is either*

- *a Henselian discrete valuation field with infinite residues of characteristic p or*
- *a Hilbertian field [i.e., a field for which Hilbert's irreducibility theorem holds — cf. [3], Chapter 12].*

Then G_K^p is elastic and not topologically finitely generated.

Proof. First, it follows from [15], Theorem C, that we may assume without loss of generality that K is a Hilbertian field. Then since K contains ζ_p, it follows from [3], Corollary 16.2.7, (b), that G_K^p is not topologically finitely generated.

Next, we verify the elasticity of G_K^p. Observe that it suffices to prove that every topologically finitely generated normal closed subgroup of G_K^p is trivial [cf. [3], Corollary 12.2.3]. Let $F \subseteq G_K^p$ be a topologically finitely generated normal closed subgroup. Write $K \subseteq K^p$ ($\subseteq \overline{K}$) for the maximal pro-p extension [so $G_K^p =$

$\mathrm{Gal}(K^p/K)$]; $K_F \subseteq K^p$ for the subfield fixed by F. Here, we note that $K^p \subsetneq \overline{K}$ [cf. [3], Corollary 16.2.7, (a)].

Suppose that $K_F \subsetneq K^p$. Then since $K \subseteq K_F$ is a Galois extension, it follows from [3], Theorem 13.9.1, (b), together with [3], Corollary 16.2.7, (b), that the extension $K_F \subsetneq K^p$ is not finite. Let $K_F \subsetneq L$ be a finite extension such that $L \subsetneq K^p$. Again, by applying [3], Theorem 13.9.1, (b), we observe that L is a Hilbertian field, hence [cf. [3], Corollary 16.2.7, (b)] that $\mathrm{Gal}(K^p/L) = G_L^p$ is not topologically finitely generated. In particular, since $K_F \subsetneq L$ is a finite extension, it holds that $F = \mathrm{Gal}(K^p/K_F)$ is not topologically finitely generated. This is a contradiction. Thus, we conclude that $K_F = K^p$, hence that $F = \{1\}$. This completes the proof of Proposition 4.3. ☐

Proposition 4.4. *Let*

$$\Delta \longrightarrow \Pi \longrightarrow G \longrightarrow 1$$

be an exact sequence of profinite groups. Suppose that Δ *is topologically finitely generated, and* G *is elastic and not topologically finitely generated. Write* $\underline{\Delta}$ $(\subseteq \Pi)$ *for the image of the homomorphism* $\Delta \to \Pi$*. Then* $\underline{\Delta}$ $(\subseteq \Pi)$ *may be characterized as the maximal topologically finitely generated normal closed subgroup of* Π*. In particular, there exists a functorial group-theoretic algorithm*

$$\Pi \quad \rightsquigarrow \quad \underline{\Delta}, \quad G$$

for constructing the closed subgroup $\underline{\Delta}$ *and the quotient* G *of* Π *[where we regard* G *as the quotient of* Π *via the surjection* $\Pi \twoheadrightarrow G$*] from* Π*.*

Proof. Note that since Δ is topologically finitely generated, it holds that $\underline{\Delta} \subseteq \Pi$ is a topologically finitely generated normal closed subgroup. On the other hand, since G is elastic and not topologically finitely generated, every topologically finitely generated normal closed subgroup of G is trivial. Thus, since the homomorphism $\Pi \twoheadrightarrow G$ is surjective, we conclude that $\underline{\Delta}$ $(\subseteq \Pi)$ may be characterized as the maximal topologically finitely generated normal closed subgroup of Π. This completes the proof of Proposition 4.4. ☐

Next, we prove the following [cf. Theorem A, (i)]:

Theorem 4.5. *Let* K, K' *be fields of characteristic* 0*;* X, X' *smooth varieties over* K, K'*, respectively;*

$$\alpha : \Pi_X^p \xrightarrow{\sim} \Pi_{X'}^p$$

an isomorphism of profinite groups. Suppose that

- *K is either a Henselian discrete valuation field with infinite residues of characteristic p or a Hilbertian field;*
- *K' is either a Henselian discrete valuation field with residues of characteristic p or a Hilbertian field;*
- *$\zeta_p \in K$, $\zeta_p \in K'$.*

Then α induces an isomorphism $G_K^p \xrightarrow{\sim} G_{K'}^p$ that fits into a commutative diagram

$$\begin{array}{ccc} \Pi_X^p & \xrightarrow[\alpha]{\sim} & \Pi_{X'}^p \\ \downarrow & & \downarrow \\ G_K^p & \xrightarrow{\sim} & G_{K'}^p, \end{array}$$

where the vertical arrows denote the natural surjections [determined up to composition with an inner automorphism] induced by the structure morphisms of the smooth varieties X, X'.

Proof. First, it follows from Proposition 4.3, together with our assumptions on K, that G_K^p is elastic and not topologically finitely generated. Next, we consider a diagram of profinite groups

$$\begin{array}{ccccccc} \Delta_X^p & \longrightarrow & \Pi_X^p & \longrightarrow & G_K^p & \longrightarrow & 1 \\ & & \alpha \downarrow \wr & & & & \\ \Delta_{X'}^p & \longrightarrow & \Pi_{X'}^p & \longrightarrow & G_{K'}^p & \longrightarrow & 1, \end{array}$$

where the horizontal sequences are the natural exact sequences as in Remark 3.1.1. Then since $\Delta_{X'}^p$ is topologically finitely generated [cf. [15], Lemma 4.2], it follows immediately from Theorem 1.4, (ii); Proposition 4.3; [15], Lemma 3.1, together with our assumptions on K', that $G_{K'}^p$ is also elastic and not topologically finitely generated. Thus, since Δ_X^p and $\Delta_{X'}^p$ are topologically finitely generated [cf. [15], Lemma 4.2], it follows immediately from Proposition 4.4 that α induces an isomorphism $G_K^p \xrightarrow{\sim} G_{K'}^p$. This completes the proof of Theorem 4.5. □

Proposition 4.6. *Let n be a positive integer; K a p-adic local field; X a hyperbolic curve over K. Write X_n for the n-th configuration space associated to X; $(\Pi_X^p)^{\times n}$ for the fiber product of n copies of Π_X^p over G_K^p;*

$$f : \Pi_{X_n}^p \twoheadrightarrow (\Pi_X^p)^{\times n}$$

for the natural surjection induced by the natural open immersion $X_n \hookrightarrow X^{\times n}$ over K. Let G be a free pro-p group of finite rank, or a Demushkin group isomorphic to *the maximal pro-p quotient of the absolute Galois group of some p-adic local field;*

$$\phi : \Pi^p_{X_n} \to G$$

an open homomorphism. Then ϕ factors as the composite of f with an open homomorphism $(\Pi^p_X)^{\times n} \to G$.

Proof. Write

$$h : \Delta^p_{X_n} \to \Pi^p_{X_n}$$

for the natural homomorphism induced by the natural injection $\Delta_{X_n} \hookrightarrow \Pi_{X_n}$. For each positive integer j $(\leq n)$, write

$$p_j : \Pi^p_{X_n} \twoheadrightarrow \Pi^p_{X_{n-1}}$$

for the surjection that lies over G^p_K [determined up to composition with an inner automorphism] induced by the natural projection morphism $X_n \to X_{n-1}$ obtained by forgetting the j-th factor. For each pair of distinct positive integers i, j such that $1 \leq i, j \leq n$, let

$$I_{i,j} \subseteq \Delta^p_{X_n}$$

be an inertia subgroup associated to the diagonal divisor $\Delta_{i,j}$ [cf. Definition 4.1].

To verify Proposition 4.6, it suffices to prove that $\phi \circ h(I_{i,j}) = \{1\}$ for each pair of distinct positive integers i, j such that $1 \leq i, j \leq n$. Let $K \subseteq L$ $(\subseteq \overline{K})$ be a finite field extension such that the cardinality of $X(L) \geq n-1$; $x_1, \cdots, x_{n-1} \in X(L)$ distinct L-rational points of X. Write $Z \subseteq X_L$ for the open subscheme obtained by forming the complement of the closed subset $\{x_1, \cdots, x_{n-1}\} \subseteq X_L$. [In particular, Z is a hyperbolic curve over L.] Note that the distinct L-rational points $x_1, \cdots, x_{n-1} \in X(L)$ determine a morphism Spec $L \to X_{n-1}$ over K. Note also that, for each positive integer j $(\leq n)$, there exists a natural isomorphism $Z \xrightarrow{\sim} X_n \times_{X_{n-1}} \text{Spec } L$, where the fiber product is determined by the morphism Spec $L \to X_{n-1}$ and the natural projection morphism $X_n \to X_{n-1}$ obtained by forgetting the j-th factor. Write $c_j : Z \to X_n$ for the composite morphism [over K] of the isomorphism $Z \xrightarrow{\sim} X_n \times_{X_{n-1}} \text{Spec } L$ with the projection morphism $X_n \times_{X_{n-1}} \text{Spec } L \to X_n$. Then there exists a commutative diagram of profinite groups

$$\begin{array}{ccccccc}
\Delta^p_Z & \longrightarrow & \Pi^p_Z & \longrightarrow & G^p_L & \longrightarrow & 1 \\
\| & & \downarrow & & \downarrow & & \\
\Delta^p_Z & \longrightarrow & \Pi^p_{X_n} & \xrightarrow{p_j} & \Pi^p_{X_{n-1}} & \longrightarrow & 1,
\end{array}$$

where the upper horizontal sequence denotes the exact sequence induced by the structure morphism $Z \to \text{Spec } L$; the right-hand vertical arrow denotes the homomorphism that lies over G_K^p [determined up to composition with an inner automorphism] induced by the morphism Spec $L \to X_{n-1}$ over K; the middle vertical arrow denotes the homomorphism that lies over G_K^p [determined up to composition with an inner automorphism] induced by the morphism $c_j : Z \to X_n$ over K. Note that $h(I_{i,j})$ ($\subseteq \Pi_{X_n}^p$) coincides with the image of a cuspidal inertia subgroup of Δ_Z^p via the homomorphism $\Delta_Z^p \to \Pi_{X_n}^p$ that appears in the above commutative diagram.

Write $\phi_Z : \Pi_Z^p \to G$ (respectively, $h_Z : \Delta_Z^p \to G$) for the composite of the homomorphism $\Pi_Z^p \to \Pi_{X_n}^p$ (respectively, $\Delta_Z^p \to \Pi_{X_n}^p$) [that appears in the above commutative diagram] with ϕ. If $\mathrm{Im}(h_Z) = \{1\}$, then we have nothing to prove. If $\mathrm{Im}(h_Z) \neq \{1\}$, then since G is elastic, and $\mathrm{Im}(h_Z)$ ($\subseteq G$) is a topologically finitely generated normal closed subgroup of an open subgroup of G, it holds that $\mathrm{Im}(h_Z) \subseteq G$ is an open subgroup. In particular, ϕ_Z is an open homomorphism. Thus, by applying Corollary 3.4 to ϕ_Z, we conclude that $h(I_{i,j}) = \{1\}$. This completes the proof of Proposition 4.6. $\square$

Before proceeding, we recall the definition of fiber subgroups, which will be of use in the proof of Theorem 4.8.

Definition 4.7 ([24], Definition 2.3, (iii)). Let n be a positive integer ≥ 2; i a positive integer $\leq n$; K an algebraically closed field of characteristic 0; X a hyperbolic curve over K. Write

- X_m for the m-th configuration space associated to X for each positive integer m;
- $p_i : \Pi_{X_n}^p \twoheadrightarrow \Pi_{X_{n-1}}^p$ for the outer surjection induced by the projection morphism $X_n \to X_{n-1}$ obtained by forgetting the i-th factor;
- $q_i : \Pi_{X_n}^p \twoheadrightarrow \Pi_X^p$ for the outer surjection induced by the projection morphism $X_n \to X$ associated to the i-th factor.

Then we shall refer to $\mathrm{Ker}(p_i)$ (respectively, $\mathrm{Ker}(q_i)$) as the *fiber subgroup* of $\Pi_{X_n}^p$ of length 1 (respectively, co-length 1) associated to i.

Finally, we prove the following [cf. Theorem A, (ii)]:

Theorem 4.8. *Let (n, n') be a pair of positive integers; K, K' fields of characteristic 0; X, X' hyperbolic curves over K, K', respectively. Write X_n (respectively, $X'_{n'}$) for the n-th (respectively, the n'-th) configuration space associated to X (respectively, X'). Let*

$$\alpha : \Pi_{X_n}^p \xrightarrow{\sim} \Pi_{X'_{n'}}^p$$

be an isomorphism of profinite groups. Suppose that

- *K and K' are either Henselian discrete valuation fields of residue characteristic p or Hilbertian fields;*
- *X_n and $X'_{n'}$ satisfy the p-exactness.*

Then the following hold:

(i) Let Π be a topological group isomorphic to $\Pi^p_{X_n}$. Then there exists a functorial group-theoretic algorithm

$$\Pi \quad \rightsquigarrow \quad n$$

for constructing the dimension n from Π. In particular, it holds that $n = n'$.

(ii) α induces an isomorphism $G^p_K \xrightarrow{\sim} G^p_{K'}$ that fits into a commutative diagram

$$\begin{array}{ccc} \Pi^p_{X_n} & \xrightarrow[\alpha]{\sim} & \Pi^p_{X'_n} \\ \downarrow & & \downarrow \\ G^p_K & \xrightarrow{\sim} & G^p_{K'}, \end{array}$$

where the vertical arrows denote the natural surjections [determined up to composition with an inner automorphism] induced by the structure morphisms of the configuration spaces X_n, X'_n.

Proof. First, we verify assertion (i). Recall that $\Delta^p_{X_n}$ is topologically finitely generated [cf. [15], Lemma 4.2]. On the other hand, since X_n satisfies the p-exactness, it follows immediately from Lemma 3.2; Remark 4.1.1, that K contains ζ_p. Thus, we conclude from Proposition 4.3, together with our assumption on K, that Π is topologically finitely generated if and only if K is a Henselian discrete valuation field with finite residues of characteristic p [cf. Theorem 1.4, (ii); [15], Lemma 3.1].

Suppose that Π is topologically finitely generated. Then it follows from Proposition 4.2 that there exists a functorial group-theoretic algorithm $\Pi \rightsquigarrow n$ for constructing the dimension n from Π [cf. [15], Lemma 3.1]. Next, suppose that Π is not topologically finitely generated. Then it holds that G^p_K is elastic and not topologically finitely generated [cf. Proposition 4.3]. In particular, since $\Delta^p_{X_n}$ is topologically finitely generated, by applying the functorial group-theoretic algorithm that appears in Proposition 4.4 to Π, we obtain a closed subgroup $\Delta(\Pi)$ of Π isomorphic to $\Delta^p_{X_n}$. Thus, by applying [the functorial group-theoretic algorithm that appears implicitly in] [11], Theorem 1.6, to $\Delta(\Pi)$, we obtain a functorial group-theoretic algorithm $\Pi \rightsquigarrow n$ for constructing the dimension n from Π. In conclusion, since the condition that a topological group is topological finitely generated is group-theoretic, we obtain a desired functorial group-theoretic algorithm. This completes the proof of assertion (i).

Next, we verify assertion (ii). Note that since X_n and X'_n satisfy the p-exactness, it holds that K and K' contain ζ_p [cf. Lemma 3.2, Remark 4.1.1]. Then it follows immediately from Theorem 4.5 that we may assume without loss of generality that

K and K' are p-adic local fields that contain ζ_p

[cf. [15], Lemma 3.1].

Next, it follows from Theorem 3.6 that we may assume without loss of generality that $n \geq 2$. Write $\phi : \Delta^p_{X_n} \to G^p_{K'}$ (respectively, $\psi : \Delta^p_{X'_n} \to G^p_K$) for the composite

$$\Delta^p_{X_n} \longrightarrow \Pi^p_{X_n} \underset{\alpha}{\overset{\sim}{\to}} \Pi^p_{X'_n} \longrightarrow G^p_{K'}$$

(respectively,

$$\Delta^p_{X'_n} \longrightarrow \Pi^p_{X'_n} \underset{\alpha^{-1}}{\overset{\sim}{\to}} \Pi^p_{X_n} \longrightarrow G^p_K),$$

where the first arrow denotes the injection [determined up to composition with an inner automorphism] induced by the projection morphism $(X_n)_{\overline{K}} \to X_n$ (respectively, $(X'_n)_{\overline{K'}} \to X'_n$); the final arrow denotes the surjection [determined up to composition with an inner automorphism] induced by the structure morphism $X'_n \to \operatorname{Spec} K'$ (respectively, $X_n \to \operatorname{Spec} K$).

Next, we verify the following assertion:

> Claim 4.8.A: Let (i, j) be a pair of integers such that $1 \leq i, j \leq n$. Write F_i, F_j for the fiber subgroups of $\Delta^p_{X_n}$ (respectively, $\Delta^p_{X'_n}$) of length 1 associated to i, j, respectively. Suppose that $\phi(F_i) \neq \{1\}$, and $\phi(F_j) \neq \{1\}$ (respectively, $\psi(F_i) \neq \{1\}$, and $\psi(F_j) \neq \{1\}$). Then $F_i = F_j$.

Since the proof of the non-resp'd case is similar to the proof of the resp'd case, we verify the non-resp'd case only. Note that $\phi(F_i)$ and $\phi(F_j)$ are nontrivial topologically finitely generated normal closed subgroup of $G^p_{K'}$ [cf. [24], Proposition 2.2, (i)]. Then since $G^p_{K'}$ is elastic [cf. Theorem 1.5, (ii)], $\phi(F_i)$ and $\phi(F_j)$ are open subgroups of $G^p_{K'}$. Suppose that

$$F_i \neq F_j.$$

Write $(\Delta^p_X)^{\times n}$ for the direct product of n copies of Δ^p_X. Then it follows immediately from Proposition 4.6 that ϕ factors as the composite of the natural surjection $\Delta^p_{X_n} \twoheadrightarrow (\Delta^p_X)^{\times n}$ [induced by the natural open immersion $(X_n)_{\overline{K}} \hookrightarrow (X_{\overline{K}})^{\times n}$ over $\overline{K}$] with a homomorphism $(\Delta^p_X)^{\times n} \to G^p_{K'}$. In particular, it holds that $\phi(F_i)$ commutes with $\phi(F_j)$. Then since $\phi(F_i)$ and $\phi(F_j)$ are open subgroups of $G^p_{K'}$, there exists an abelian open subgroup of $G^p_{K'}$. This contradicts Lemma 1.6. Thus, we conclude that $F_i = F_j$. This completes the proof of Claim 4.8.A.

Next, we verify the following assertion:

Claim 4.8.B: There exists a fiber subgroup $F \subseteq \Delta^p_{X_n}$ (respectively, $G \subseteq \Delta^p_{X'_n}$) of co-length 1 associated to some positive integer $\leq n$ such that $\phi(F) = \{1\}$ (respectively, $\psi(G) = \{1\}$).

Indeed, Claim 4.8.B follows immediately from Claim 4.8.A, together with [24], Proposition 2.4, (vi).

Let $F \subseteq \Delta^p_{X_n}$ be a fiber subgroup of co-length 1 such that $\phi(F) = \{1\}$ [cf. Claim 4.8.B]. In the remainder of the proof, for each pair of distinct positive integers i, j such that $1 \leq i, j \leq n$, we shall write

- $\mathrm{pr}_i : \Delta^p_{X'_n} \to \Delta^p_{X'}$ for the surjection [determined up to composition with an inner automorphism] induced by the projection morphism $X'_n \to X'$ associated to the i-th factor;
- $G_i \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\mathrm{pr}_i)$;
- $\mathrm{pr}_{i,j} : \Delta^p_{X'_n} \to \Delta^p_{X'_2}$ for the surjection [determined up to composition with an inner automorphism] induced by the projection morphism $X'_n \to X'_2$ associated to the i-th and j-th factors.

Next, we verify the following assertion:

Claim 4.8.C: Suppose that there exists a positive integer i ($\leq n$) such that $\alpha(F) \subseteq G_i$. Then, for each positive integer j such that $i \neq j \leq n$, it holds that $\alpha(F) \not\subseteq G_j$.

Indeed, suppose that $\alpha(F) \subseteq G_i \cap G_j$. Note that it follows immediately from [24], Proposition 2.2, (i), together with the various definitions involved, that

- $\mathrm{pr}_{i,j}(\alpha(F)) \subseteq \mathrm{pr}_{i,j}(G_i)$ is a topologically finitely generated normal closed subgroup;
- $\mathrm{pr}_{i,j}(\alpha(F)) \subseteq \mathrm{pr}_{i,j}(G_i \cap G_j) \subseteq \mathrm{pr}_{i,j}(G_i)$;
- the closed subgroup $\mathrm{pr}_{i,j}(G_i \cap G_j) \subseteq \mathrm{pr}_{i,j}(G_i)$ is of infinite index [so the closed subgroup $\mathrm{pr}_{i,j}(\alpha(F)) \subseteq \mathrm{pr}_{i,j}(G_i)$ is of infinite index];
- $\mathrm{pr}_{i,j}(G_i)$ is elastic [cf. [24], Theorem 1.5].

Then these facts imply that $\mathrm{pr}_{i,j}(\alpha(F)) = \{1\}$. In particular, $\alpha(F)$ is contained in the maximal pro-p quotient of the étale fundamental group of an $n-2$ dimensional configuration space associated to a hyperbolic curve over an algebraically closed field of characteristic 0 [cf. [24], Proposition 2.4, (i)]. This contradicts [11], Theorem 1.6. Thus, we conclude that $\alpha(F) \not\subseteq G_j$. This completes the proof of Claim 4.8.C.

Next, we verify the following assertion:

Claim 4.8.D: Suppose that there exists a positive integer i ($\leq n$) such that $\alpha(F) \subseteq G_i$. Then $\alpha(F) = G_i$.

Indeed, for each positive integer j such that $i \neq j \leq n$, it follows from Claim 4.8.C that $\mathrm{pr}_j(\alpha(F))$ is a nontrivial topologically finitely generated normal closed subgroup of $\Delta^p_{X'}$, hence an open subgroup of $\Delta^p_{X'}$ [cf. [24], Theorem 1.5]. Thus, G_j and $\alpha(F)$ generate topologically an open subgroup $M_j \subseteq \Delta^p_{X'_n}$. Let l ($\leq n$) be a positive integer such that $\psi(G_l) = \{1\}$ [cf. Claim 4.8.B].

If $l = i$, then it holds that

$$F \subseteq \alpha^{-1}(G_i) \subseteq \Delta^p_{X_n}.$$

If $l \neq i$, then it holds that

- $M_l \subseteq \Delta^p_{X'_n}$ is an open subgroup;
- $\psi(M_l) = \{1\}$.

Thus, since G^p_K is torsion-free [cf. Lemma 1.6], we conclude that $\psi(\Delta^p_{X'_n}) = \{1\}$, hence that $\Delta^p_{X'_n} \subseteq \alpha(\Delta^p_{X_n})$. In particular,

$$F \subseteq \alpha^{-1}(G_i) \subseteq \Delta^p_{X_n}.$$

Note that $\alpha^{-1}(G_i) \subseteq \Delta^p_{X_n}$ is a topologically finitely generated normal closed subgroup of infinite index [cf. [11], Theorem 1.6; [24], Proposition 2.4, (i)], and $\Delta^p_{X_n}/F \xrightarrow{\sim} \Delta^p_X$. Thus, by applying [24], Theorem 1.5, we conclude that $F = \alpha^{-1}(G_i)$. This completes the proof of Claim 4.8.D.

Next, we verify the following assertion:

Claim 4.8.E: Suppose that, for each positive integer i ($\leq n$), $\alpha(F) \not\subseteq G_i$. Then $\Delta^p_{X'_n} = \alpha(\Delta^p_{X_n})$.

Indeed, we note that $\alpha(\Delta^p_{X_n}) \subseteq \Pi^p_{X'_n}$ is a topologically finitely generated normal closed subgroup of infinite index [cf. Lemma 1.6]. Thus, since $G^p_{K'}$ is elastic [cf. Theorem 1.5, (ii)], it suffices to prove that $\Delta^p_{X'_n} \subseteq \alpha(\Delta^p_{X_n})$.

Let l ($\leq n$) be a positive integer such that $\psi(G_l) = \{1\}$ [cf. Claim 4.8.B]. On the other hand, since $\Delta^p_{X'}$ is elastic [cf. [24], Theorem 1.5], it follows from our assumption that $\alpha(F) \not\subseteq G_i$ that $\mathrm{pr}_i(\alpha(F))$ is an open subgroup of $\Delta^p_{X'}$ for each positive integer i ($\leq n$). Then the closed subgroups $G_l \subseteq \Delta^p_{X'_n}$ and $\alpha(F) \subseteq \Delta^p_{X'_n}$ generate topologically an open subgroup $N_l \subseteq \Delta^p_{X'_n}$ such that $\psi(N_l) = \{1\}$. Thus, since G^p_K is torsion-free [cf. Lemma 1.6], we conclude that $\psi(\Delta^p_{X'_n}) = \{1\}$, hence that $\Delta^p_{X'_n} \subseteq \alpha(\Delta^p_{X_n})$. This completes the proof of Claim 4.8.E.

Finally, it follows from Claims 4.8.D, 4.8.E, together with Theorem 3.6, (ii); Remark 4.1.1, that $\Delta^p_{X'_n} = \alpha(\Delta^p_{X_n})$, hence, in particular, that α induces an isomorphism $G_K^p \xrightarrow{\sim} G_{K'}^p$. This completes the proof of assertion (ii), hence of Theorem 4.8. □

Remark 4.8.1. In light of Theorems 4.5, 4.8; [5], Theorem 0.1, it is natural to pose the following question:

Question: Let K, K' be fields of characteristic 0; X, X' smooth varieties over K, K', respectively;

$$\alpha : \Pi_X^p \xrightarrow{\sim} \Pi_{X'}^p$$

an isomorphism of profinite groups. Suppose that K and K' are either

- *subfields* of Henselian discrete valuation fields of residue characteristic p or
- Hilbertian fields.

Then does α induce an isomorphism $G_K^p \xrightarrow{\sim} G_{K'}^p$ via the natural surjections $\Pi_X^p \twoheadrightarrow G_K^p$ and $\Pi_{X'}^p \twoheadrightarrow G_{K'}^p$?

However, at the time of writing of the present paper, the author does not even know

whether or not the analogous assertions of Theorem 4.8 for hyperbolic polycurves hold

[cf. [7], Definition 2.1, (ii)].

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