Multiflow Theory in Combinatorial Optimization

Hiroshi Hirai

RIMS, Kyoto Univ.

Kyoto, July 2010

Contents

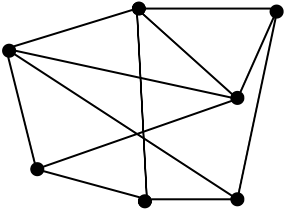
Introduction to multicommodity flow theory

I. From single-commodity flow to multicommodity flow

II. Multiflow-metric duality and beyond

III. Multiflows as LP-relaxations of NP-hard problems

Notation: undirected graph $G = (V, E)$

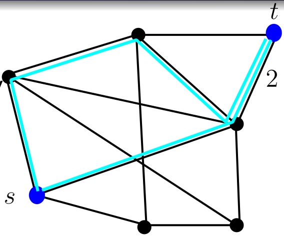


Part I: From single-commodity flow to multicommodity flow

1. Max-flow min-cut theorem (Ford-Fulkerson 56)
2. Two-commodity flow: max-biflow min-cut theorem (Hu 63)
3. Free multiflow: Lovász-Cherkassky theorem (Lovász 76, Cherkassky 77)
4. Splitting-off method
5. Fractionality

Maximum Flow Problem

(G, c): undirected network
$G = (V, E)$, $c : E \to \mathbf{Z}_+$ edge-capacity
$s, t \in V$: sink-source pair



Definition: (s,t)-flow $f = (\mathcal{P}, \lambda)$

$\overset{\text{def}}{\Longleftrightarrow}$ $\mathcal{P}$: a set of (s,t)-paths, $\lambda : \mathcal{P} \to \mathbf{R}_+$: flow-value function s.t.

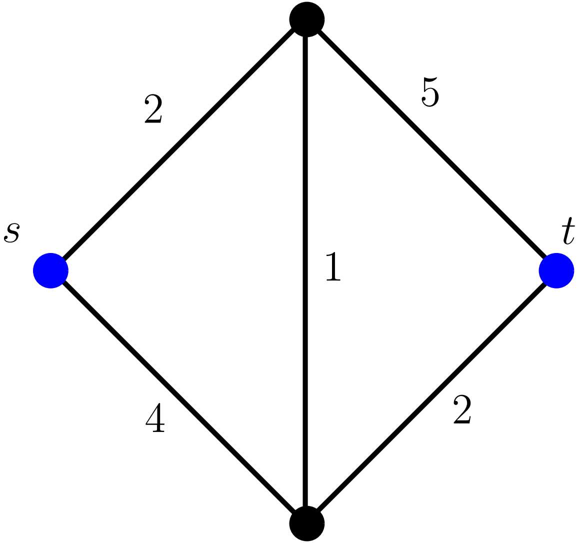
$$f(e) := \sum \{\lambda(P) \mid P \in \mathcal{P} : e \in P\} \leq c(e) \quad (e \in E).$$

Total flow-value $\|f\| := \sum \{\lambda(P) \mid P \in \mathcal{P}\}$

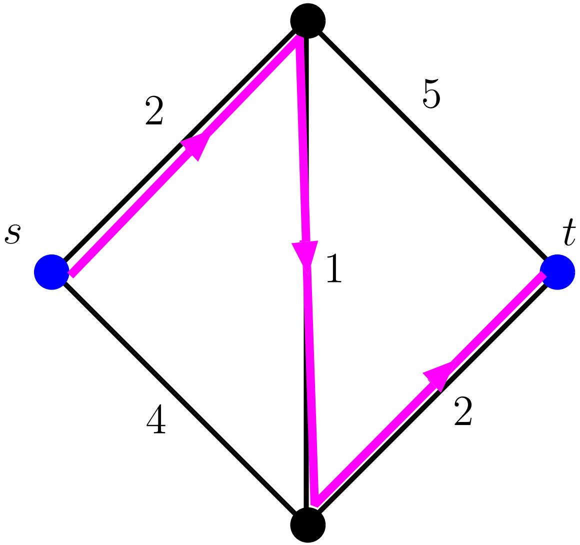
Maximum Flow Problem

Maximize $\|f\|$ over all (s,t)-flows f in (G, c).

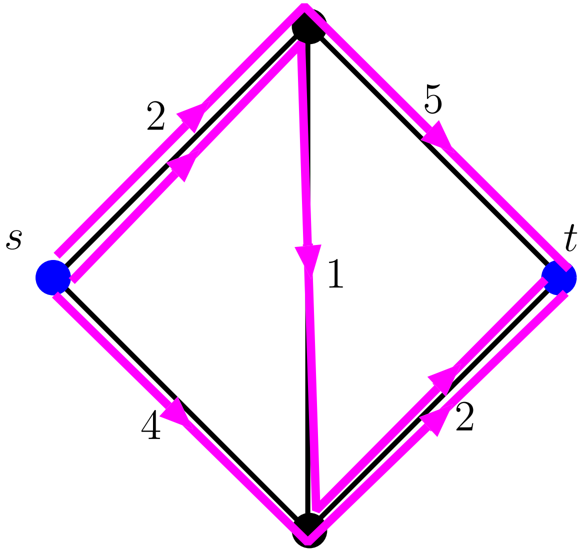
Example



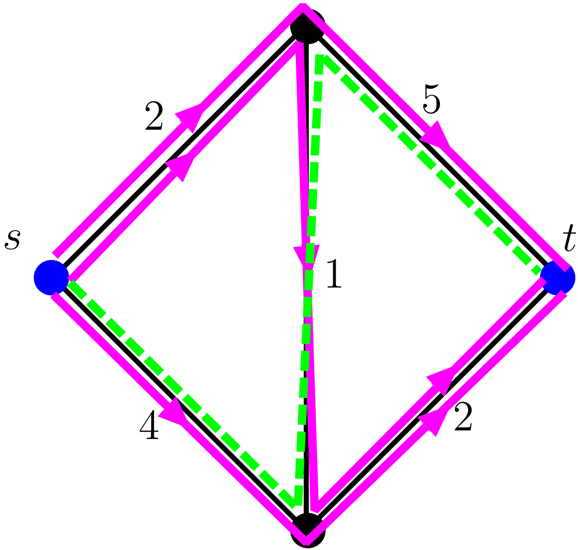
Example



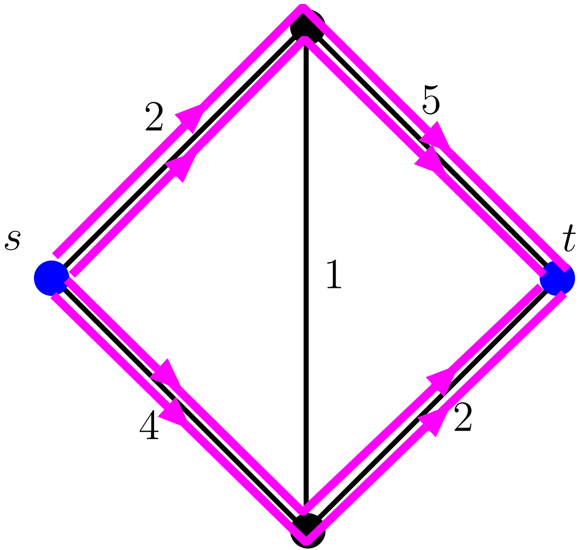
Example



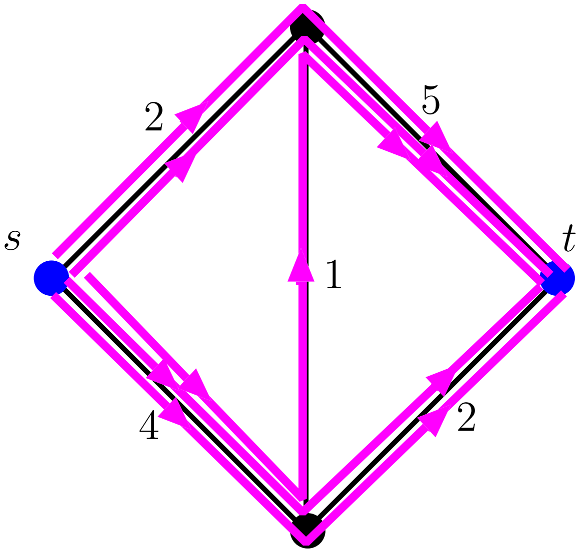
Example



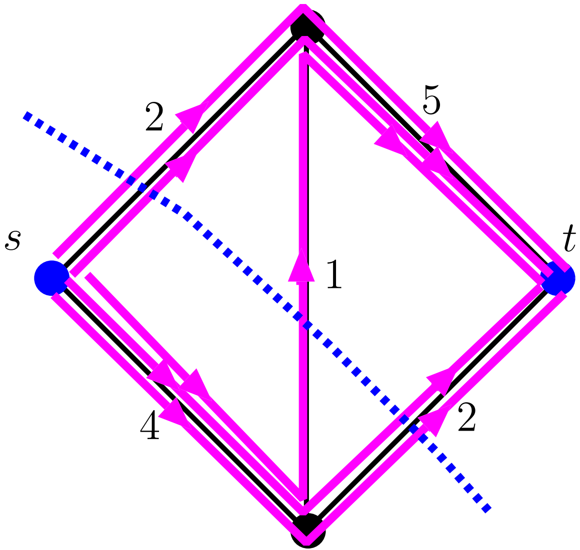
Example



Example

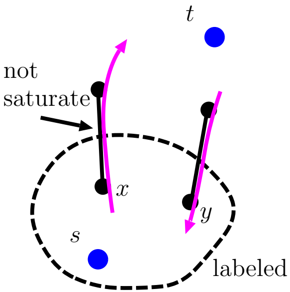


Example

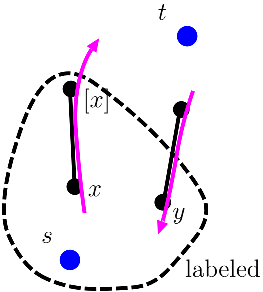


Labeling method (Ford-Fulkerson 56)

0. $\mathcal{P} = \emptyset$.
1. Orient all paths in $\mathcal{P}$ as $s \to t$.
2. Label nodes from s as

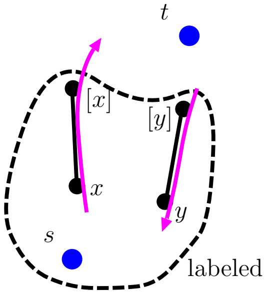


0. $\mathcal{P} = \emptyset$.
1. Orient all paths in $\mathcal{P}$ as $s \to t$.
2. Label nodes from s as

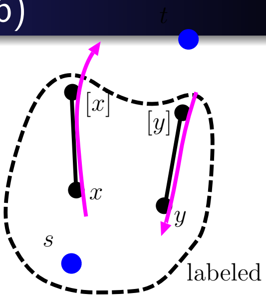


Labeling method (Ford-Fulkerson 56)

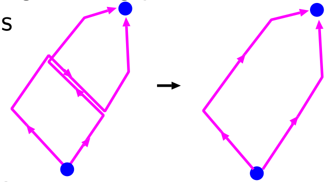
0. $\mathcal{P} = \emptyset$.
1. Orient all paths in $\mathcal{P}$ as $s \to t$.
2. Label nodes from s as



Labeling method (Ford-Fulkerson 56)



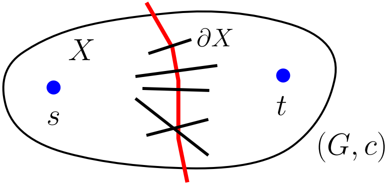
0. $\mathcal{P} = \emptyset$.
1. Orient all paths in $\mathcal{P}$ as $s \to t$.
2. Label nodes from s as
3. If t is labeled, then we get an augmenting path P and let $\mathcal{P} \leftarrow \mathcal{P} + P$, do cancellations as



   and go to 1.
4. If t is unlabeled, then $\mathcal{P}$ is maximum, and stop (labeled nodes give min-cut).

Max-Flow Min-Cut Theorem (Ford-Fulkerson 56)

∂X: edge set between X and $V \setminus X$.
$c(\partial X) = \sum_{e \in \partial X} c(e)$.



Max-Flow Min-Cut Theorem (Ford-Fulkerson 56)

$$\max_f \|f\| = \min\{c(\partial X) \mid s \in X \not\ni t\}.$$

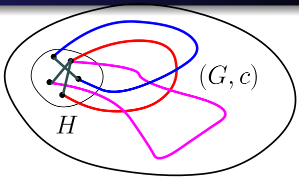
Moreover the maximum is attained by an *integral* flow.

combinatorial optimization, algorithmic proof, min-max theorem, LP-relaxation, polyhedral combinatorics, Menger's theorem, bipartite matching, multicommodity flows, . . .

Multicommodity Flows

$(G = (V, E), c)$: undirected network

$H \subseteq \binom{V}{2}$ *commodity graph*



Multiflow $f = \{(s,t)\text{-flow } f_{st}\}_{st \in H}$: $\sum_{st \in H} f_{st}(e) \leq c(e)$ $(e \in E)$

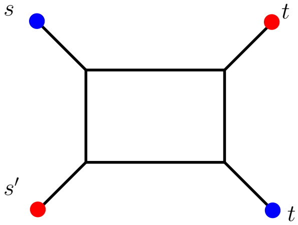
Maximum Multiflow Problem

Maximize $\sum_{st \in H} \|f_{st}\|$ over all multiflows $f = \{f_{st}\}_{st \in H}$ in (G, c).

- Many other formulations, e.g., feasibility, concurrent flows, ...
- Polynomially solvable by LP-solver (ellipsoid or interior point), but no combinatorial polynomial time algorithm is known.
- Integer version is NP-hard for almost H.
- *Half-integrality phenomena*.

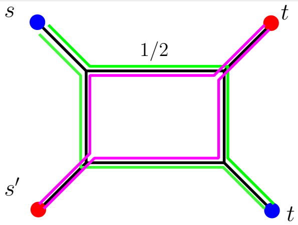
Two-Commodity Flows

$H = \{st, s't'\}$



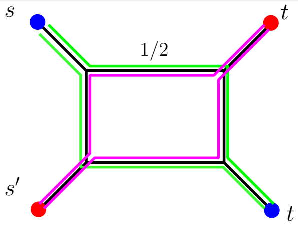
Two-Commodity Flows

$H = \{st, s't'\}$



Two-Commodity Flows

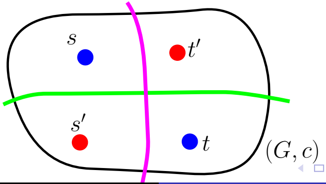
$H = \{st, s't'\}$



Max-biflow Min-Cut Theorem (Hu 63)

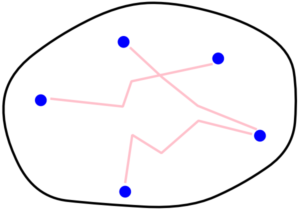
$$\max \|f_{st}\| + \|f_{s't'}\| = \min\{(ss', tt')\text{-mincut}, (st', ts')\text{-mincut}\}$$

Moreover the maximum is attained by a *half-integral* flow.

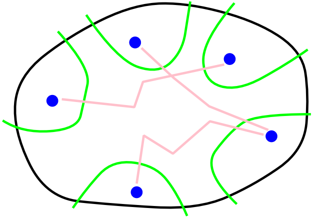


Free Multiflows

$H = \binom{S}{2}$ for terminal set $S \subseteq V$



$H = \binom{S}{2}$ for terminal set $S \subseteq V$

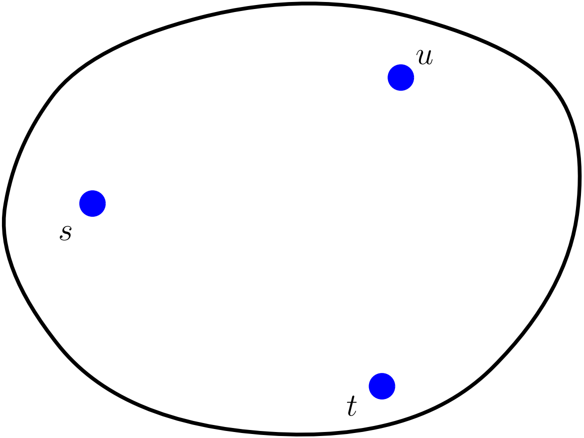


Theorem (Lovász 76, Cherkassky 77)

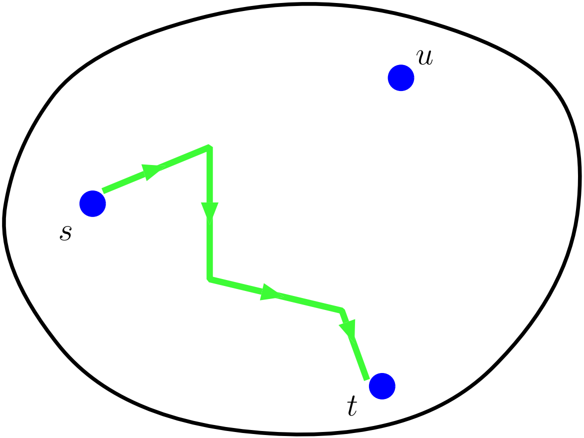
$$\max \sum_{st \in \binom{S}{2}} \|f_{st}\| = \frac{1}{2} \sum_{t \in S} (t, S \setminus t)\text{-mincut}$$

Moreover the maximum is attained by a *half-integral* flow.

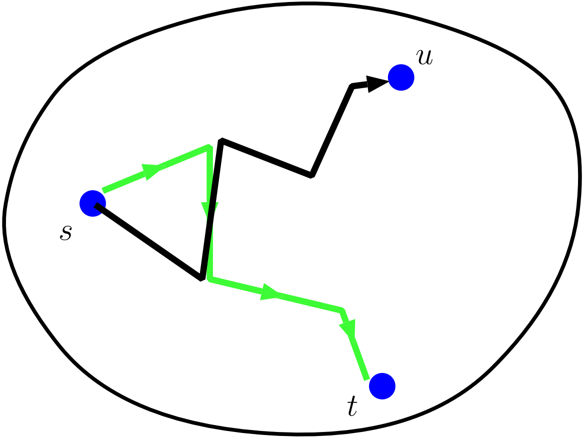
Cherkassky's algorithmic proof



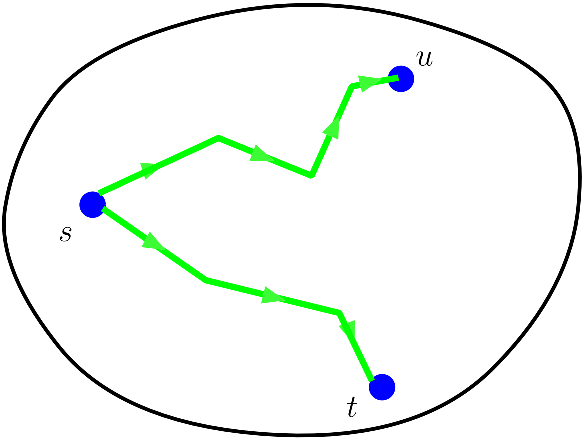
Cherkassky's algorithmic proof



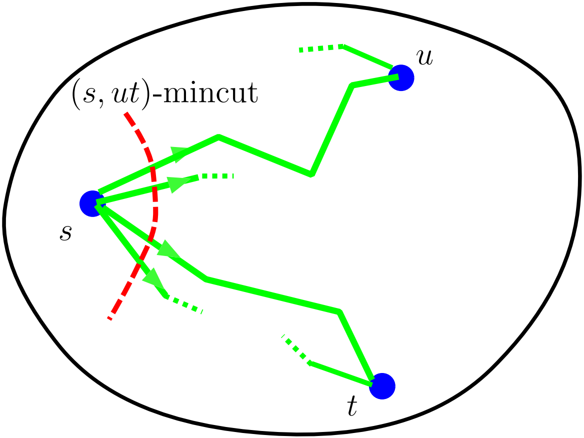
Cherkassky's algorithmic proof



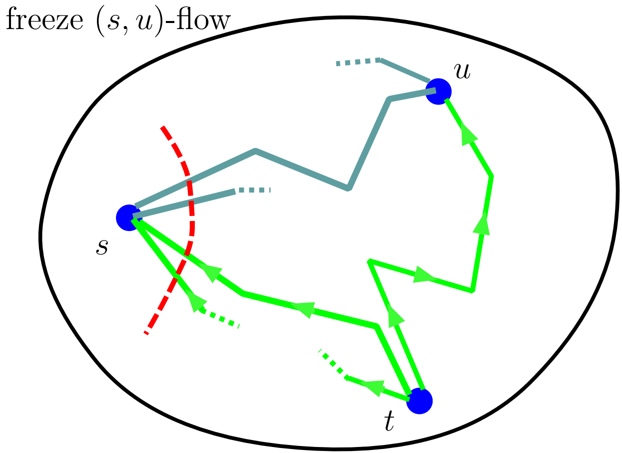
Cherkassky's algorithmic proof



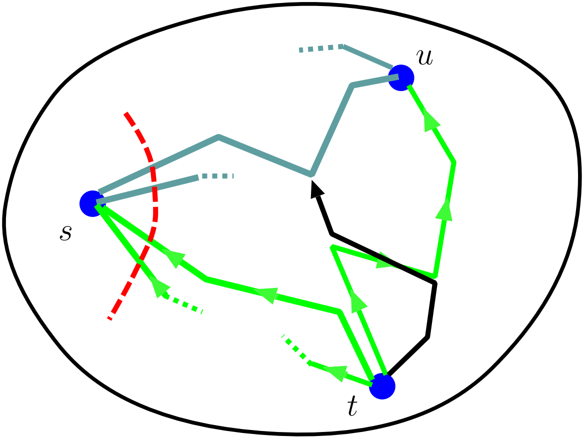
Cherkassky's algorithmic proof



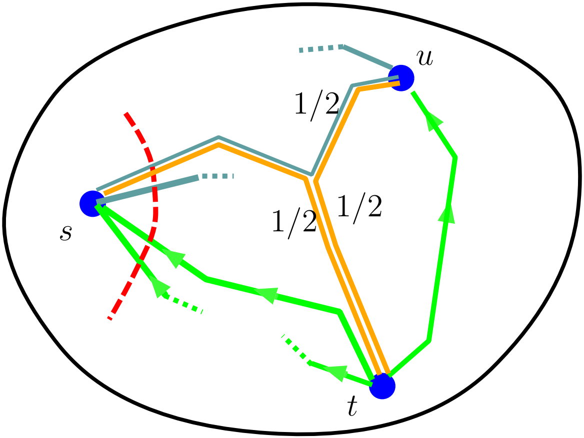
Cherkassky's algorithmic proof



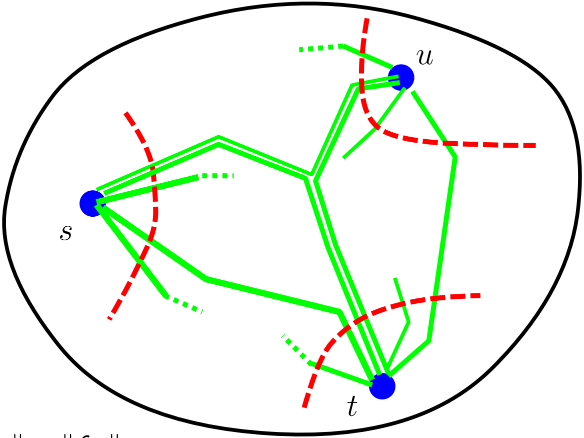
Cherkassky's algorithmic proof



Cherkassky's algorithmic proof



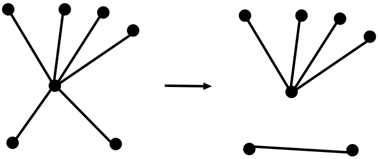
Cherkassky's algorithmic proof



$$\|f_{st}| + \|f_{tu}\| + \|f_{us}\| =$$
$$\frac{1}{2}\{(s,\{t,u\})\text{-mincut} + (t,\{s,u\})\text{-mincut} + (u,\{s,t\})\text{-mincut }\}$$

Splitting-off method

Splitting-off operation:



Def. $(G = (V, E), c)$: Eulerian $\overset{\text{def}}{\Longleftrightarrow}$ $c(\partial x) \in 2\mathbf{Z}$ $(\forall x \in V)$

Theorem (Rothschild-Winston 66, Lovász 76)

Suppose $(G = (V, E), c)$ is Eulerian.

$$\max_{\text{integral flow}} \|f_{st}\| + \|f_{s't'}\| = \min\{(ss', tt')\text{-mincut}, (st', ts')\text{-mincut}\}$$

$$\max_{\text{integral flow}} \sum_{st \in \binom{S}{2}} \|f_{st}\| = \frac{1}{2} \sum_{t \in S} \ (t, S \setminus t)\text{-mincut}$$

$\rightarrow$ blackboard

Fractionality Problem

Fractionality

$\mathrm{frac}(H) :=$ the least positive integer k with property:
$\exists\ 1/k$-integral maximum flow for $\forall(G, c; H)$.

Problem (Karzanov, ICM Kyoto 90)

Classify commodity graphs having finite fractionality.

$\mathrm{frac}(\,|\,) = 1$ (Ford-Fulkerson 56)

$\mathrm{frac}(\,|\,|\,) = 2$ (Hu 63)

$\mathrm{frac}(\underbrace{|\,|\,|\cdots|}_{k\geq 3}) = +\infty$

$\mathrm{frac}(\triangle) = \mathrm{frac}(\boxtimes) = \mathrm{frac}(K_n) = 2$ (Lovász 76, Cherkassky 77)

$\mathrm{frac}(|\,\triangle) = 2$ (Karzanov 98)

$\mathrm{frac}(|\,\boxtimes) = \mathrm{frac}(K_2 + K_n) = 4$ (Lomonosov 04)

$\mathrm{frac}(\triangle\,\triangle) = ?$

Part II: Multiflow-metric duality and beyond

1. Japanese Theorem (Onaga-Kakusho 71, Iri 71)
2. T-duality

Multiflow-Metric Duality (Onaga-Kakusho, Iri 71 $\sim$)

Multiflow is a *linear programming* $\to$ LP-dual by *metric*

LP-duality:
$$\begin{array}{lll} \max. & \mu^\top x \\ \text{s.t.} & Ax \leq c \\ & x \geq 0 \end{array} \quad = \quad \begin{array}{ll} \min. & y^\top c \\ \text{s.t.} & y^\top A \geq \mu \\ & y \geq 0 \end{array}$$

Metric: $d : V \times V \to \mathbf{R}_+$,
$$d(x,x) = 0 \quad (x \in V),$$
$$d(x,y) = d(y,x) \quad (x,y \in V),$$
$$d(x,y) + d(y,z) \geq d(x,z) \quad (x,y,z \in V).$$

l_1-metric: $\|x - y\|_1 = \sum_{i=1}^n |x_i - y_i| \quad (x,y \in \mathbf{R}^n)$

l_∞-metric: $\|x - y\|_\infty = \max_{i=1,2,\ldots,n} |x_i - y_i| \quad (x,y \in \mathbf{R}^n)$

graph-metric: $\mathrm{dist}_{G,l}(x,y) = \min\{\sum_{e \in P} l(e) \mid (x,y)\text{-path } P\}$

Multiflow-Metric Duality

- feasibility (Onaga-Kakusho, Iri 71)
  - cut condition v.s. cut decomposability (cf. Avis-Deza 91)
- concurrent flow (Shahrokhi-Matula 90)
  - approximate max-flow min-cut theorem (Leighton-Rao 88)
  - conductances in Markov chains (Sinclair 89)
  - low-distortional embedding v.s. approximation of sparsest cuts (Linial-London-Rabinovich 95, Aumann-Rabani 98, . . .)
- maximization (Karzanov-Lomonosov 70s $\sim$)

Our version

$(G = (V, E), c)$: undirected network with terminal set $S \subseteq V$
$\mu : \binom{S}{2} \to \mathbf{R}_+$: terminal weight

μ-weighted maximum multiflow problem

Maximize $\sum_{st \in \binom{S}{2}} \mu(st) \| f_{st} \|$ *over all multiflows* $f = \{f_{st}\}_{st \in \binom{S}{2}}$

Our version

$(G = (V, E), c)$: undirected network with terminal set $S \subseteq V$
$\mu : \binom{S}{2} \to \mathbf{R}_+$: terminal weight

μ-weighted maximum multiflow problem

Maximize $\sum_{st \in \binom{S}{2}} \mu(st) \| f_{st} \|$ *over all multiflows* $f = \{f_{st}\}_{st \in \binom{S}{2}}$

Theorem (Multiflow-Metric Duality)

$$\max \sum \mu(st) \| f_{st} \| \quad = \quad \text{Min.} \sum_{xy \in E} c(xy) d(x, y)$$

$$\text{s.t. } d\text{: metric on } V, d(s,t) \geq \mu(st) \ (s, t \in S)$$

$\to$ blackboard

T-duality

$\max \sum \mu(st)\|f_{st}\| = \mathrm{Min} \sum c(xy)d(x,y)$ s.t.d: metric on $V, \ldots$

Theorem (Karzanov 98, H. 09)

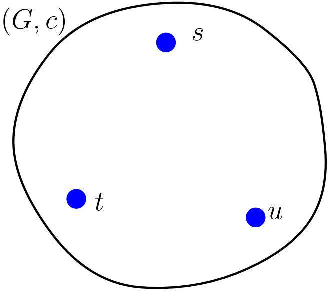
$$\max \sum \mu(st)\|f_{st}\| \quad = \quad \text{Min.} \sum_{xy \in E} c(xy)\|\rho(x) - \rho(y)\|_\infty$$
$$\text{s.t. } \rho : V \to T_\mu, \ \rho(s) \in T_{\mu,s} \ (s \in S).$$

Tight span (Isbell 64, Dress 84)

$T_\mu :=$ Minimal $\{p \in \mathbf{R}_+^S \mid p(s) + p(t) \geq \mu(s,t) \ s,t \in S\}$
$T_{\mu,s} := T_\mu \cap \{p(s) = 0\} \ (s \in S)$

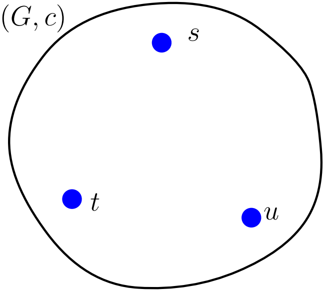
$\to$ blackboard

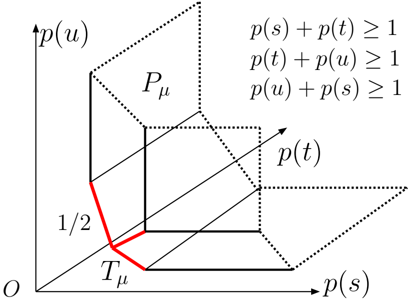
Example



$$\max \|f_{st}\| + \|f_{tu}\| + \|f_{su}\|$$

Example

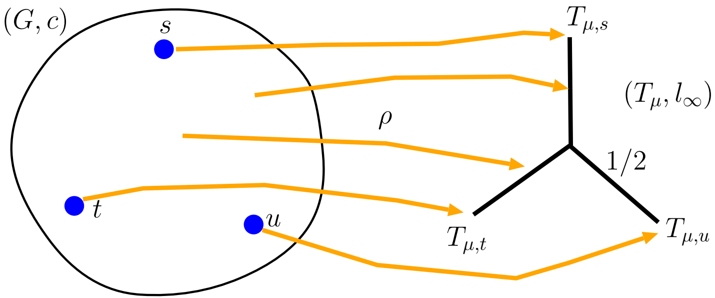




$$\max \|f_{st}\| + \|f_{tu}\| + \|f_{su}\| \quad = \quad \min \sum_{xy \in E} c(xy) \|\rho(x) - \rho(y)\|$$

$$\text{s.t. } \rho : V \to T_\mu, \ \rho(s) \in T_{\mu,s}$$

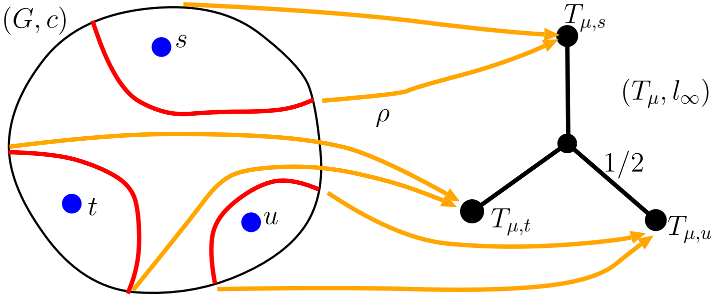
Example



$$\max \|f_{st}\| + \|f_{tu}\| + \|f_{su}\| \quad = \quad \min \sum_{xy \in E} c(xy)\|\rho(x) - \rho(y)\|$$

$$\text{s.t. } \rho : V \to T_\mu, \ \rho(s) \in T_{\mu,s}$$

Example



$$\max \|f_{st}\| + \|f_{tu}\| + \|f_{su}\|$$
$$= \frac{1}{2}\{(s,\{t,u\})\text{-mincut} + (t,\{s,u\})\text{-mincut} + (u,\{s,t\})\text{-mincut}\}.$$

A Solution of the Fractionality Problem

Fractionality

$\mathrm{frac}(\mu)$:= the least positive integer k with property:
$\exists$ $1/k$-integral max flow for $\forall$ μ-max multiflow problem

Theorem (H. 07-09, STOC2010)

- dim $T_\mu \leq 2 \rightarrow \mathrm{frac}(\mu) \leq 24$
- dim $T_\mu \geq 3 \rightarrow \mathrm{frac}(\mu) = +\infty$

Problems 51,52 in:

A. Schrijver, "Combinatorial Optimization", 2003.

Digression: what is tight span ?

Tight span (Isbell 64, Dress 84)

$$T_\mu := \mathrm{Minimal}\{p \in \mathbf{R}_+^S \mid p(s) + p(t) \geq \mu(s,t) \ s,t \in S\}$$

64 Isbell: category of metric spaces, injective hull

84 Dress: phylogenetic tree

94 Chrobak-Larmore: *k*-server problem

98 Karzanov, Chepoi: connection to multiflows

06 Hirai: nonmetric version

Part III: Multiflows as LP-relaxations of NP-hard problems

1. Approximate max-flow min-cut theorems (Leighton-Rao 88, ...)
2. Minimum 0-extensions

Of course, multiflow is an LP-relaxation of edge-disjoint paths, but...

Multicut

$(G = (V, E), c)$: undirected network
H: commodity graph of k edges
Def. multicut w.r.t. H
$\overset{\text{def}}{\Longleftrightarrow}$ edge subset $\mathcal{E}$ with $P \cap \mathcal{E} \neq \emptyset$ for every H-path.

Minimum multicut problem

Minimize $c(\mathcal{E})$ over multicuts $\mathcal{E}$.

Weak duality

$$\max \sum_{st \in H} \|f_{st}\| \leq \text{Min. multicut}$$

Multicut

$(G = (V, E), c)$: undirected network
H: commodity graph of k edges
Def. multicut w.r.t. H
$\overset{\text{def}}{\Longleftrightarrow}$ edge subset $\mathcal{E}$ with $P \cap \mathcal{E} \neq \emptyset$ for every H-path.

Minimum multicut problem

Minimize $c(\mathcal{E})$ over multicuts $\mathcal{E}$.

Theorem (Garg-Vazirani-Yannakakis 96)

$$\max \sum_{st \in H} \|f_{st}\| \leq \text{Min. multicut} \leq O(\log k) \max \sum_{st \in H} \|f_{st}\|$$

Maximum concurrent flow and Sparsest cut

$(G = (V, E), c)$: undirected network
H: commodity graph of k edges
$q : H \to \mathbf{Z}_+$: demand function

Maximum concurrent flow

Maximize π s.t. $\pi \geq 0$: $\exists f = \{f_{st}\}_{st \in H}$, $\|f_{st}\| = \pi q(st)$ $(\forall st \in H)$

Multiflow-metric duality (Shahrokhi-Matula 90)

$$\text{Max } \pi = \min \frac{c \cdot d}{q \cdot d} \quad \text{s.t.} \quad \text{metric } d \text{ on } V.$$

Sparsest cut problem

$$\text{Minimize } \frac{c(\partial X)}{q(\partial X)} \text{ over } \emptyset \neq X \subset V.$$

Conductance in Markov chain (Sinclair 89)

Sparsest cut and Low-distortional embedding

Theorem (Bourgain 85)

For any n-point metric d, there is an l_1-metric d^* such that $d^* \leq d \leq O(\log n) d^*$

$\rightarrow d^* = \sum \lambda_i \delta_{X_i}$, where δ_{X_i}: cut metric.

Theorem (Linial-London-Rabinovich 95, Aumann-Rabani 98)

$$\min_d \frac{c \cdot d}{q \cdot d} \leq \min_X \frac{c(\partial X)}{q(\partial X)} \leq O(\log k) \min_d \frac{c \cdot d}{q \cdot d}.$$

V. V. Vazirani, "Approximation Algorithms", 2001.
J. Matousek, "Lectures on Discrete Geometry", 2002.
($\exists$ *Japanese translations !*)

Minimum 0-extension problem

$(G = (V, E), c)$: undirected network
S: terminal set with $|S| = k$
μ: metric on S
Def: extension d of μ on $V \overset{\text{def}}{\Longleftrightarrow}$ metric d on V with $d|_S = \mu$.
Def: 0-extension d of μ on V
$\overset{\text{def}}{\Longleftrightarrow}$ extension d s.t. $\forall x \in V, \exists \in S$ with $d(s, x) = 0$.

Minimum 0-extension problem

Minimize $c \cdot d$ over 0-extensions d.

Minimum 0-extension problem

$(G = (V, E), c)$: undirected network
S: terminal set with $|S| = k$
μ: metric on S

Def: extension d of μ on V $\overset{\text{def}}{\Longleftrightarrow}$ metric d on V with $d|_S = \mu$.

Def: 0-extension d of μ on V
$\overset{\text{def}}{\Longleftrightarrow}$ extension d s.t. $\forall x \in V, \exists \in S$ with $d(s, x) = 0$.

Minimum 0-extension problem

Minimize $c \cdot d$ over 0-extensions d.

Minimum 0-extension problem (alternative form)

Min $\sum_{xy} c(xy)\mu(\rho(x), \rho(y))$ s.t. $\rho : V \to S$, $\rho|_S =$ identity.

Multiway cut, computer vision, . . .
A special class of metric labeling problem (Kleinberg-Tardos 98)

Metric relaxation

A. Karzanov: Minimum 0-extensions of graph metrics, Europ. J. combin. 1998

Metric relaxation (Karzanov 98)

Minimize $c \cdot d$ over extensions d

$= \text{Min } c \cdot d$ s.t. metric d on V with $d|_S = \mu$.
$= \text{Max} \sum \mu(st) \| f_{st} \|$ s.t. f: multiflow in $(G, c; S)$

Metric relaxation

A. Karzanov: Minimum 0-extensions of graph metrics, Europ. J. combin. 1998

Metric relaxation (Karzanov 98)

Minimize $c \cdot d$ over extensions d

$= \text{Min } c \cdot d$ s.t. metric d on V with $d|_S = \mu$.
$= \text{Max } \sum \mu(st) \| f_{st} \|$ s.t. f: multiflow in $(G, c; S)$

Theorem (Calinescu-Karloff-Rabani 04)

$\max \sum \mu(st) \| f_{st} \| \leq \text{Min 0-extension} \leq O(\log k) \max \sum \mu(st) \| f_{st} \|$

Metric relaxation

A. Karzanov: Minimum 0-extensions of graph metrics, Europ. J. combin. 1998

Metric relaxation (Karzanov 98)

Minimize $c \cdot d$ over extensions d

$= \text{Min } c \cdot d$ s.t. metric d on V with $d|_S = \mu$.
$= \text{Max } \sum \mu(st) \|f_{st}\|$ s.t. f: multiflow in $(G, c; S)$

Theorem (Calinescu-Karloff-Rabani 04)

$$\max \sum \mu(st)\|f_{st}\| \leq \text{Min 0-extension} \leq O(\log k) \max \sum \mu(st)\|f_{st}\|$$

Theorem (Karzanov 98)

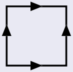
If μ is the graph metric of a frame,
then *metric relaxation exactly solves minimum 0-extension.*

Frames and 2-dimensional tight spans

Definition

frame $\overset{\text{def}}{\Longleftrightarrow}$ a bipartite graph with properties:

- no isometric cycle of length > 4
- orientable

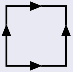


Frames and 2-dimensional tight spans

Definition

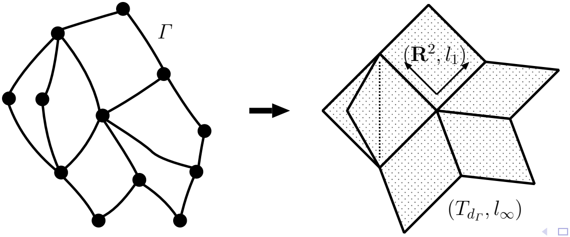
frame $\overset{\text{def}}{\Longleftrightarrow}$ a bipartite graph with properties:

- no isometric cycle of length > 4
- orientable



Theorem (Karzanov 98)

If μ is the graph metric of a frame, then T_μ is obtained by *filling a folder to each maximal $K_{2,m}$-subgraph.*



Concluding remarks

Multiflow theory is a frontier of combinatorial optimization !!

There are many important topics I did not mention here (sorry):

- Multiflows on planar graphs (Okamura-Seymour, ..)
- FPTAS for multiflows (Garg-Könemann, ...)
- Mader's A-path theorem and generalizations
  (nonzero A-paths (Chudnovsky et.al.), ...)
- Disjoint path problems (Robertson-Seymour, ...)
  $\rightarrow$ go to RAMP symposium 10/28-29 (Kobayashi's talk)
- ...