Residuated Frames: the Join of Proof Theory and Algebra for Substructural Logics

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(partly based on ongoing work with N. Galatos)

短歌 (A Short Song)

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  われてもすえに　あはんとぞおもふ

崇徳院

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  The stream goes down a mountain rapidly
  Even if it runs into a rock
  And is forced to divide into two
  They will join together in the end.

  Sutoku-in (1119 – 1164)

Motivation

- Algebraization of logical concepts

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Logic	$\mapsto$	Variety
Provability	$\mapsto$	Validity

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Interpolation	$\mapsto$	Amalgamation
Disjunction property	$\mapsto$	...

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- Algebraization of proof theoretic methods?

Maehara's Methods	$\mapsto$	???
Cut-free Proof Analyses	$\mapsto$	???

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- Cut elimination admits algebraic proof (Okada, Belardinelli-Jipsen-Ono)

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- What about consequences of cut-elimination, such as Interpolation and Disjunction Property?

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  - algebraic uniform proof of IP and AP
  - algebraic uniform proof of DP and an algebraic counterpart of DJP

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$$\Sigma \mathrel{\tilde{\Rightarrow}} C$$

with $C = (\Gamma, \_, \Delta \Rightarrow \alpha)$.

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with $C = (\Gamma, \_, \Delta \Rightarrow \alpha)$.

- $\tilde{\Rightarrow}$ gives rise to a residuated structure:

$$\begin{aligned} \Sigma, \Pi \tilde{\Rightarrow} C &\equiv \Gamma, \Sigma, \Pi, \Delta \Rightarrow \alpha \\ &\equiv \Sigma \tilde{\Rightarrow} C[\_, \Pi] \\ &\equiv \Pi \tilde{\Rightarrow} C[\Pi, \_] \end{aligned}$$

Sequent calculus for FLe

- Inference rules of **FLe**

$$\frac{}{\alpha \tilde{\Rightarrow} \alpha} \qquad \frac{\Gamma \tilde{\Rightarrow} \alpha \quad \alpha \tilde{\Rightarrow} C}{\Gamma \tilde{\Rightarrow} C}$$

$$\frac{\alpha, \Gamma \tilde{\Rightarrow} \beta}{\Gamma \tilde{\Rightarrow} \alpha \to \beta} \qquad \frac{\Gamma \tilde{\Rightarrow} \alpha \quad \beta \tilde{\Rightarrow} C}{\Gamma, \alpha \to \beta \tilde{\Rightarrow} C}$$

$$\frac{\Gamma \tilde{\Rightarrow} \alpha \quad \Gamma \tilde{\Rightarrow} \beta}{\Gamma \tilde{\Rightarrow} \alpha \wedge \beta} \qquad \frac{\alpha_i \tilde{\Rightarrow} C}{\alpha_1 \wedge \alpha_2 \tilde{\Rightarrow} C}$$

$$\frac{\Gamma \tilde{\Rightarrow} \alpha_i}{\Gamma \tilde{\Rightarrow} \alpha_1 \vee \alpha_2} \qquad \frac{\alpha \tilde{\Rightarrow} C \quad \beta \tilde{\Rightarrow} C}{\alpha \vee \beta \tilde{\Rightarrow} C}$$

- Write $\Gamma \tilde{\Rightarrow}_{\mathbf{FLe}} C$ if $\Gamma \tilde{\Rightarrow} C$ is provable.

Interpolation Property

- Craig Interpolation: Let $\phi \in Fm(X)$ and $\psi \in Fm(Y)$. If $\phi \Rightarrow_{\mathbf{FLe}} \psi$, then there is $i \in Fm(X \cap Y)$ such that

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- Step 2: Prove Maehara's Lemma: Let $\Gamma \in (Fm(X) \cup Fm(Y))^*$ and $\alpha \in Fm(X) \cup Fm(Y)$ and Suppose $\Gamma \Rightarrow_{\mathbf{FLe}} \alpha$. Then,

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  for any $\Delta \in Fm^*(X)$ and $C \in Fm^{con}(Y)$ such that $(\Gamma \Rightarrow \alpha) \equiv (\Delta \tilde{\Rightarrow} C)$, there is $i \in Fm(X \cap Y)$ such that

$$\Delta \tilde{\Rightarrow}_{\mathbf{FLe}} i \quad \text{and} \quad i \tilde{\Rightarrow}_{\mathbf{FLe}} C.$$

  The same holds with X and Y exchanged.

Maehara's Lemma Rephrased

- From relation $\Rightarrow_{\mathbf{FLe}}$, define a new relation $\Rightarrow_M$ between $(Fm(X) \cup Fm(Y))^*$ and $(Fm(X) \cup Fm(Y))$:

  $\Gamma \Rightarrow_M \alpha$ (Γ maeharaly implies α) iff
  for any $\Delta \in Fm^*(X)$ and $C \in Fm^{con}(Y)$ such that $(\Gamma \Rightarrow \alpha) \equiv (\Delta \tilde{\Rightarrow} C)$, there is $i \in Fm(X \cap Y)$ such that

  $$\Delta \tilde{\Rightarrow}_{\mathbf{FLe}} i \quad \text{and} \quad i \tilde{\Rightarrow}_{\mathbf{FLe}} C.$$

  The same holds with X and Y exchanged.

- Maehara's Lemma: the new relation $\Rightarrow_M$ is preserved under the rules of **FLe** (including cut).

Maehara's Lemma Rephrased

- Proof of IP: If $\phi \Rightarrow_{\mathbf{FLe}} \psi$ with $\phi \in Fm(X)$ and $\psi \in Fm(Y)$, SFP implies that it has a derivation only using formulas in $Fm(X) \cup Fm(Y)$.

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- Subformula property is much easier to prove than cut-elimination when using algebraic methods.

Simple Residuated Frame

- $Q^{con} = Q^* \times Q$. $(\epsilon, a) \in Q^{con}$ is written as a.
- A simple residuated frame $\mathbf{W} = (Q, \sqsubset)$: $\sqsubset$ is a relation between Q^* and Q^{con} such that

$$xy \sqsubset (z, a) \iff y \sqsubset (xz, a)$$

for any $x, y \in Q^*$ and $(z, a) \in Q^{con}$.

- $\sqsubset$ can be generated from its restriction to $Q^* \times Q$ by residuation.
- $(Q^*, Q^{con}, \sqsubset)$ is a residuate frame in Nick's sense.

Simple Residuated Frame

- Example 1 (sequent calculus): $(Fm(X), \stackrel{\sim}{\Rightarrow}_{\mathbf{FLe}})$ is a simple frame.
- Example 2 (the dual frame): If $\mathbf{A} = (A, \wedge, \vee, \cdot, \rightarrow, 1)$ is a commutative residuated lattice, then $\mathbf{A}_+ = (A, \sqsubset)$ is a simple residuated frame, where $\sqsubset$ is generated by $a_1, \ldots, a_n \sqsubset b \Longleftrightarrow a_1 \cdots a_n \leq_{\mathbf{A}} b$.

From Frames to Algebras

- Let $\mathbf{W} = (Q, \sqsubset)$ be a simple residuated frame. For any $X \subseteq Q^*$ and $U \subseteq Q^{con}$,

$$\begin{aligned} X^{\rhd} &= \{u \in Q^{con} | \forall x \in X (x \sqsubset u)\} \\ U^{\lhd} &= \{x \in Q^* | \forall u \in U (x \sqsubset u)\} \end{aligned}$$

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- Proposition: If $\mathbf{W} = (Q, \sqsubset)$ is a simple frame, then $\mathbf{R}(\mathbf{W}) = (\mathbf{W}_\gamma, \cap, \cup_\gamma, \bullet_\gamma, \to, \{\epsilon\}_\gamma)$ is a commutative residuated lattice.

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- The dual algebra $\mathbf{W}^+$ of $\mathbf{W}$: the least subalgebra of $\mathbf{R}(\mathbf{W})$ containing $\{a^{\lhd} | a \in Q\}$.

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- Fundamental Theorem 1 (BOJ, Galatos-Jipsen): If $\mathbf{W}$ is a simple Gentzen frame, then $(\cdot)^{\lhd} : \mathbf{Q} \longrightarrow \mathbf{W}^{+}$ is a homomorphism. Moreover, if $\sqsubset$ is antisymmetric on $Q \times Q$, then $(\cdot)^{\lhd}$ is an embedding.

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- Corollay: If $\mathbf{A}$ is a commutative residuated lattice, then $\mathbf{A} \cong (\mathbf{A}_{+})^{+}$.

Gentzen Rules

$$\frac{}{a \sqsubset a} \qquad \frac{x \sqsubset a \quad a \sqsubset u}{x \sqsubset u}$$

$$\frac{ax \sqsubset b}{x \sqsubset a \to b} \qquad \frac{x \sqsubset a \quad b \sqsubset u}{x(a \to b) \sqsubset u}$$

$$\frac{x \sqsubset a \quad x \sqsubset b}{x \sqsubset a \wedge b} \qquad \frac{a_i \sqsubset u}{(a_1 \wedge a_2) \sqsubset u}$$

$$\frac{x \sqsubset a_i}{x \sqsubset a_1 \vee a_2} \qquad \frac{a \sqsubset u \quad b \sqsubset u}{a \vee b \sqsubset u}$$

Maehara's Lemma Revisited

- Instead of working on the concrete

  $$(Fm(X \cap Y), \Rightarrow_{\mathbf{FLe}}) \quad (Fm(X), \Rightarrow_{\mathbf{FLe}}) \quad (Fm(Y), \Rightarrow_{\mathbf{FLe}}),$$

  we consider in general

  $$\mathbf{W}_A = (A, \sqsubset_A) \quad \mathbf{W}_B = (B, \sqsubset_B) \quad \mathbf{W}_C = (C, \sqsubset_C)$$

  such that $A \subseteq B \cap C$ and

  $$x \sqsubset_A u \Longleftrightarrow x \sqsubset_B u \Longleftrightarrow x \sqsubset_C u$$

  for any $x \in A^*$ and $u \in A^{con}$.

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- Maehara's Lemma (Frame Version): If $\mathbf{W}_B$ and $\mathbf{W}_C$ are simple Gentzen frames, so is $\mathbf{W}_M$.
- Lemma: If $\sqsubset_B$ and $\sqsubset_C$ are antisymmetric, so is $\sqsubset_M$
- Corollary: $(\cdot)^{\lhd} : \mathbf{B} \cup \mathbf{C} \longrightarrow \mathbf{W}_M^+$ is a homomorphism. Moreover, if $\sqsubset_B$ and $\sqsubset_C$ are antisymmetric, $(\cdot)^{\lhd}$ is an embedding.

Algebraic Proof of Interpolation

- Take $\mathbf{W}_A = (Fm(X \cap Y), \Rightarrow_{\mathbf{FLe}})$ $\mathbf{W}_B = (Fm(X), \Rightarrow_{\mathbf{FLe}})$ and $\mathbf{W}_C = (Fm(Y), \Rightarrow_{\mathbf{FLe}})$.

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- $\mathbf{W}_M^+$ is a commutative residuated lattice and $(\cdot)^{\lhd} : Fm(X) \cup Fm(Y) \longrightarrow \mathbf{W}_M^+$ is a homomorphism.
- Let $\phi \in Fm(X)$ and $\psi \in Fm(Y)$. If $\phi \Rightarrow_{\mathbf{FLe}} \psi$, then

$$\mathbf{W}_M^+ \models \phi \Rightarrow \psi$$

$$\mathbf{W}_M^+, (\cdot)^{\lhd} \models \phi \Rightarrow \psi$$

$$\phi^{\lhd} \subset \psi^{\lhd}$$

$$\phi \in \psi^{\lhd}$$

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- There is $i \in Fm(X \cap Y)$ such that $\phi \Rightarrow_{\mathbf{FLe}} i$ and $i \Rightarrow_{\mathbf{FLe}} \psi$.

Direct Proof of Amalgamation

- Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be commutative residuated lattices such that $\mathbf{A}$ is a subalgebra of both $\mathbf{B}$ and $\mathbf{C}$.

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- Take $\mathbf{W}_A = \mathbf{A}_+$, $\mathbf{W}_B = \mathbf{B}_+$ and $\mathbf{W}_C = \mathbf{C}_+$. Then $\mathbf{W}_M^+$ is a commutative residuated lattice and $(\cdot)^{\lhd} : \mathbf{B} \cup \mathbf{C} \longrightarrow \mathbf{W}_M^+$ is an embedding (since $\mathbf{W}_B$ and $\mathbf{W}_C$ are antisymmetric).

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- We thus have two embeddings

$$(\cdot)^{\lhd}_B : \mathbf{B} \longrightarrow \mathbf{W}_M^+$$
$$(\cdot)^{\lhd}_C : \mathbf{C} \longrightarrow \mathbf{W}_M^+$$

and cleary $a^{\lhd}_B = a^{\lhd}_C$ for any $a \in A$.

Generalizing Amalgamation

- Square diagramatic properties: Amalgamation, Congruence Extension, Transferrable Injection ...
- One can uniformly prove them for the variety of commutative residuated lattices.
- Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be commutative residuated lattices and $f : \mathbf{A} \longrightarrow \mathbf{B}$ and $g : \mathbf{A} \longrightarrow \mathbf{C}$ homomorphisms. ($\mathbf{B} \cap \mathbf{C} = \emptyset$)

Generalizing Amalgamation

- From the simple Genzen frames $\mathbf{B}_+$ and $\mathbf{C}_+$, define $\mathbf{W}_M = (B \cup C, \sqsubset_M)$ by

  $x \sqsubset_M a$ iff

  for $y \in B^*$ and $u \in C^{con}$ such that $yu = (x, a)$, there is $i \in A$ such that

  $$y \sqsubset_{\mathbf{B}} f(i) \quad \text{and} \quad g(i) \sqsubset_{\mathbf{C}} u.$$

  The same holds with $\mathbf{B}$ and $\mathbf{C}$ exchanged.

- Generalized Maehara's Lemma: $\mathbf{W}_M$ is a simple Genzen frame.
- Corollary: $(\cdot)^{\triangleleft} : \mathbf{B} \cup \mathbf{C} \longrightarrow \mathbf{W}_M^+$ is a homomorphism.

Generalizing Amalgamation

- Hence we have homomorphisms

$$h : \mathbf{B} \longrightarrow \mathbf{W}_M^+$$
$$k : \mathbf{C} \longrightarrow \mathbf{W}_M^+$$

  Moreover, $f(a)^{\triangleleft} = g(a)^{\triangleleft}$ for any $a \in A$, i.e., $h \circ f = k \circ g$.

- Lemma:
  1. If f is injective, so is k.
  2. If f is surjective, so is k.

- A uniform proof of TI, CEP and AP for commutative residuated lattices.

- Proposition: In general, $g(ker\ f) = (ker\ k) \cap g(A)$.

Disjunction Property

- Disjunction Property: If $\Rightarrow_{\mathbf{FLe}} \phi \vee \psi$, then either $\Rightarrow_{\mathbf{FLe}} \phi$ or $\Rightarrow_{\mathbf{FLe}} \psi$.
- Proof:
  - Step 1: Prove cut-elimination
  - Step 2: Show Disjunction Lemma: If $\Rightarrow \alpha \vee \beta$ is cut-free provable, either $\Rightarrow \alpha$ or $\Rightarrow \beta$ is cut-free provable.

Disjunction Lemma Rephrased

- We need to consider meta-disjunction on r.h.s. So consider pairs of formulas $Fm \times Fm$. Write $\alpha \oplus \beta$ for $(\alpha, \beta) \in Fm \times Fm$.
- From relation $\Rightarrow_{\mathbf{FLe}}$, define a new relation $\Rightarrow_D$ between Fm^* and $(L \times L)$.

  $\Gamma \Rightarrow_D \alpha \oplus \beta$ iff the following hold:

  If $\Gamma = \emptyset$ either $\Rightarrow_{\mathbf{FLe}} \alpha$ or $\Rightarrow_{\mathbf{FLe}} \beta$.

  Otherwise $\Gamma \Rightarrow_{\mathbf{FLe}} \alpha \vee \beta$.
- $\Gamma \Rightarrow_{\mathbf{FLe}} \alpha \longleftrightarrow \Gamma \Rightarrow_D \alpha \oplus \alpha$
- $\Gamma \Rightarrow_D \alpha \oplus \beta \longrightarrow \Gamma \Rightarrow_{\mathbf{FLe}} \alpha \vee \beta$
- Disjunction Lemma: $\Rightarrow_D$ is preserved by the rules of **FLe** other than cut.

Disjunction Lemma Rephrased

- A less-simple frame $(Q, Q', \sqsubset)$: $\sqsubset$ is a relation between Q^* and $(Q, Q')^{con} = Q^* \times Q'$ such that

$$xy \sqsubset (z, a) \Longleftrightarrow y \sqsubset (xz, a).$$

- It is simple in case $Q = Q'$.
- Given a less-simple frame $\mathbf{W}$, the completion $\mathbf{R}(\mathbf{W})$ is defined as before. It is a commutative residuated lattice.
- Let $\mathbf{Q}$ be a partial algebra in the language of residuated lattices. A less-simple cut-free Gentzen frame over $\mathbf{Q}$ is $(Q, Q', \sqsubset)$ such that Q is the underlying set of $\mathbf{Q}$, Q can be embedded into Q' and $\sqsubset$ is preserved under the rules of $\mathbf{FLe}$ other than cut.

Fundamntal Theorem

- Fundamental Theorem 2 (BJO, GJ): If $\mathbf{W}$ is a simple Gentzen frame over $\mathbf{Q}$, then $(\cdot)^{\lhd} : \mathbf{Q} \longrightarrow \mathbf{R}(\mathbf{W})$ is a quasi homomorphism: for any $a, b \in Q$, $\star \in \{\wedge, \vee, \cdot, \rightarrow\}$, and any $a \in X \subset a^{\lhd}$, $b \in Y \subset b^{\lhd}$,

$$a \star b \in X \star_{R(W)} Y \subseteq (a \star b)^{\lhd}.$$

Moreover, if $\sqsubset$ is antisymmetric on $Q \times Q$, then $(\cdot)^{\lhd}$ is injective (quasi-embedding).

Disjunction Lemma Revisited

- Let W be a simple Genzen fram over Q. The quasi-dual algebra W^{++} of W is the subalgebra of $R(W)$ with the underlying set

$$\{X | X \text{ is closed and } a \in X \subseteq a^{\lhd} \text{ for some } a \in Q\}.$$

- If $\sqsubset$ is antisymmetric, the quasi-embedding $(\cdot)^{\lhd}$ can be reversed. Namely, there is a surjective homomorphism $f : \mathbf{W}^{++} \longrightarrow \mathbf{Q}$ given by $f(\mathbf{X}) = a$ if $a \in \mathbf{X} \subseteq a^{\lhd}$.

Disjunction Lemma Revisited

- Given a simple Gentzen frame $\mathbf{W} = (Q, \sqsubset)$ over a commutative residuated lattice $\mathbf{Q}$, build a less-simple frame $\mathbf{W}_D = (Q, Q \times Q, \sqsubset_D)$ where $\sqsubset_D$ is generated by

  $x \sqsubset_D a \oplus b \Longleftrightarrow$

  If $x = \epsilon$ either $\epsilon \sqsubset a$ or $\epsilon \sqsubset b$. Otherwise $x \sqsubset a \vee b$.

- Disjunction Lemma (Frame Version): W_D is a cut-free Genzen frame.
- Corollary: $(\cdot)^{\lhd} : \mathbf{Q} \longrightarrow \mathbf{W}_D^{++}$ is a quasi-homomorphism.
- Lemma: Let $X, Y \in W_D^{++}$ such that $a \in X \subseteq a^{\lhd}$ and $b \in Y \subseteq b^{\lhd}$. If $1 \leq X \vee_{W_D^{++}} Y$, then $\epsilon \sqsubset a$ or $\epsilon \sqsubset b$.
- Moreover, if $\sqsubset$ is antisymmetric, there is a surjective homomorphism $f : W_D^{++} \longrightarrow Q$. If $1 \leq X \vee_{W_D^{++}} Y$, then $\epsilon \sqsubset f(X)$ or $\epsilon \sqsubset f(Y)$.

Algebraic Proof of Disjunction Property

- From $(Fm, \Rightarrow_{\mathbf{FLe}})$ construct $\mathbf{W}_D$.
- If $\Rightarrow_{\mathbf{FLe}} \phi \vee \psi$, then $\mathbf{W}_D^{++}, v \models \phi \vee \psi$ with valuation v given by $v(p) = p^{\lhd}$.
- Since $\phi \in v(\phi) \subseteq \phi^{\lhd}$ and $\psi \in v(\phi) \subseteq \psi^{\lhd}$, we have $\Rightarrow_{\mathbf{FLe}} \phi$ or $\Rightarrow_{\mathbf{FLe}} \psi$.

Algebraic counterpart of DP

- Theorem: For any substructural logic $\mathbf{L}$, the following are equivalent.
  1. $\mathbf{L}$ satisfies the Disjunction Property
  2. For any $\mathbf{A} \in \mathcal{V}(\mathbf{L})$, there is $\mathbf{D} \in \mathcal{V}(\mathbf{L})$ and a surjective homomorphism $f : \mathbf{D} \longrightarrow \mathbf{A}$ such that $1 \leq_{\mathbf{D}} a \vee b$ implies $1 \leq_{\mathbf{A}} f(a)$ or $1 \leq_{\mathbf{A}} f(b)$.
- Proof of condition 2 for $\mathbf{FLe}$: Just take A_+ as W and construct W_D.

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    - General theory to translate proof theoretic arguments to arguments on frames.