A Direct Proof of the Amalgamation Property for Commutative Residuated Lattices

Kazushige Terui
National Institute of Informatics

Amalgamation Property

- $\mathcal{V}$ has the amalgamation property if for any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{V}$ with embeddings

$$f_1 : \mathbf{A} \longrightarrow \mathbf{B} \qquad f_2 : \mathbf{A} \longrightarrow \mathbf{C},$$

  there are $\mathbf{D} \in \mathcal{V}$ and embeddings

$$g_1 : \mathbf{B} \longrightarrow \mathbf{D} \qquad g_2 : \mathbf{C} \longrightarrow \mathbf{D}$$

  such that $g_1 \circ f_1 = g_2 \circ f_2$.
- For BCK-algebras (Wroński 1984)
- For Commutative integral residuated lattices (Kowalski 2003)
- For Commutative (contractive) residuated lattices (Takamura 2004)

The long and winding road to AP

- To prove AP, one shows:

The long and winding road to AP

- To prove AP, one shows:
  1. Cut-Elimination Theorem

The long and winding road to AP

- To prove AP, one shows:
  1. Cut-Elimination Theorem
  2. Maehara's Lemma

The long and winding road to AP

- To prove AP, one shows:
  1. Cut-Elimination Theorem
  2. Maehara's Lemma
  3. Craig's Interpolation

The long and winding road to AP

- To prove AP, one shows:
  1. Cut-Elimination Theorem
  2. Maehara's Lemma
  3. Craig's Interpolation
  4. Local Deduction Theorem

The long and winding road to AP

- To prove AP, one shows:
  1. Cut-Elimination Theorem
  2. Maehara's Lemma
  3. Craig's Interpolation
  4. Local Deduction Theorem
  5. Deductive Interpolation

The long and winding road to AP

- To prove AP, one shows:
  1. Cut-Elimination Theorem
  2. Maehara's Lemma
  3. Craig's Interpolation
  4. Local Deduction Theorem
  5. Deductive Interpolation
  6. Equational Interpolation

The long and winding road to AP

- To prove AP, one shows:
  1. Cut-Elimination Theorem
  2. Maehara's Lemma
  3. Craig's Interpolation
  4. Local Deduction Theorem
  5. Deductive Interpolation
  6. Equational Interpolation
  7. Wroński's Construction

The long and winding road to AP

- To prove AP, one shows:
  1. Cut-Elimination Theorem
  2. Maehara's Lemma
  3. Craig's Interpolation
  4. Local Deduction Theorem
  5. Deductive Interpolation
  6. Equational Interpolation
  7. Wroński's Construction
  8. Amalgamation Property

The long and winding road to AP

- To prove AP, one shows:
  1. Cut-Elimination Theorem
  2. Maehara's Lemma
  3. Craig's Interpolation
  4. Local Deduction Theorem
  5. Deductive Interpolation
  6. Equational Interpolation
  7. Wroński's Construction
  8. Amalgamation Property
- Advantage: relationship between different concepts

The long and winding road to AP

- To prove AP, one shows:
  1. Cut-Elimination Theorem
  2. Maehara's Lemma
  3. Craig's Interpolation
  4. Local Deduction Theorem
  5. Deductive Interpolation
  6. Equational Interpolation
  7. Wroński's Construction
  8. Amalgamation Property
- Advantage: relationship between different concepts
- Disadvantages: Too long. Pure algebraists wouldn't like it. Does not work for noncommutative cases.

Our goal

- To give a direct proof of the AP for $\mathcal{CRL}$.

Our goal

- To give a direct proof of the AP for $\mathcal{CRL}$.
  - APs for $\mathcal{CIRL}$, $\mathcal{CCRL}$, etc. follow immediately.

Our goal

- To give a direct proof of the AP for $\mathcal{CRL}$.
  - APs for $\mathcal{CIRL}$, $\mathcal{CCRL}$, etc. follow immediately.
  - No Cut-Elimination! No Local Deduction! (No Proof Theory!)

Our goal

- To give a direct proof of the AP for $\mathcal{CRL}$.
  - APs for $\mathcal{CIRL}$, $\mathcal{CCRL}$, etc. follow immediately.
  - No Cut-Elimination! No Local Deduction! (No Proof Theory!)
  - Wroński's construction replaced by phase semantic completion.

Our goal

- To give a direct proof of the AP for $\mathcal{CRL}$.
  - APs for $\mathcal{CIRL}$, $\mathcal{CCRL}$, etc. follow immediately.
  - No Cut-Elimination! No Local Deduction! (No Proof Theory!)
  - Wroński's construction replaced by phase semantic completion.
  - Maehara's lemma turned into an algebraic argument.

Our goal

- To give a direct proof of the AP for $\mathcal{CRL}$.
  - APs for $\mathcal{CIRL}$, $\mathcal{CCRL}$, etc. follow immediately.
  - No Cut-Elimination! No Local Deduction! (No Proof Theory!)
  - Wroński's construction replaced by phase semantic completion.
  - Maehara's lemma turned into an algebraic argument.
- Maehara's Lemma: If $\Gamma \Rightarrow \phi$ is provable (without cut), then for any partition $\Gamma_1, \Gamma_2 \equiv \Gamma$, there is an interpolant i:

$$\Gamma_1 \Rightarrow i \qquad i, \Gamma_2 \Rightarrow \phi$$

The monoid and condition set $[\![\ ]\!]$

- We assume $\mathbf{A} = \mathbf{B} \cap \mathbf{C}$
- Consider $((B \cup C)^*, \circ, \epsilon)$, the free commutative monoid generated by $B \cup C$.
- Given $d \in B \cup C$, define the interpolating set $[\![d]\!] \subseteq (B \cup C)^*$ by: $t \in [\![d]\!]$ holds $\iff$

The monoid and condition set $[\![\ ]\!]$

- We assume $\mathbf{A} = \mathbf{B} \cap \mathbf{C}$
- Consider $((B \cup C)^*, \circ, \epsilon)$, the free commutative monoid generated by $B \cup C$.
- Given $d \in B \cup C$, define the interpolating set $[\![d]\!] \subseteq (B \cup C)^*$ by: $t \in [\![d]\!]$ holds $\iff$
  1. if $d \in B$, then for any partition $t_1 \circ t_2 = t$ with $t_1 \in C^*$ and $t_2 \in B^*$, there is $i \in A$ such that
  $$t_1 \leq_{\mathbf{C}} i \qquad i \cdot t_2 \leq_{\mathbf{B}} d.$$
  2. if $d \in C$, then for any partition $t_1 \circ t_2 = t$ with $t_1 \in B^*$ and $t_2 \in C^*$, there is $i \in A$ such that
  $$t_1 \leq_{\mathbf{B}} i \qquad i \cdot t_2 \leq_{\mathbf{C}} d.$$

Phase semantic completion

- $X \subseteq (B \cup C)^*$ is closed if $X = \bigcap_{i \in I}(\{t_i\} \to [\![d_i]\!])$.
- $Cl(X) =$ the least closed set containing X.
- $\mathbf{D} = \langle Closedsets, \cap, \cup_{Cl}, \bullet_{Cl}, \to, Cl(\{\epsilon\})\rangle$, where
  - $X \to Y = \{u \mid \{u\} \circ X \subseteq Y\}$
  - $X \cup_{Cl} Y = Cl(X \cup Y)$, $\quad X \bullet_{Cl} Y = Cl(X \bullet Y)$
- Lemma: $\mathbf{D}$ is a commutative residuated lattice.

$[\![\ ]\!]$ is an embedding

- We have

$$
\begin{aligned}
[\![\ ]\!] &: \mathbf{B} \longrightarrow \mathbf{D} \\
[\![\ ]\!] &: \mathbf{C} \longrightarrow \mathbf{D}
\end{aligned}
$$

- The $g_1 \circ f_1 = g_2 \circ f_2$ requirement trivially holds.

$[\![\ ]\!]$ is an embedding

- We have

$$[\![\ ]\!] : \mathbf{B} \longrightarrow \mathbf{D}$$
$$[\![\ ]\!] : \mathbf{C} \longrightarrow \mathbf{D}$$

- The $g_1 \circ f_1 = g_2 \circ f_2$ requirement trivially holds.
- Lemma: $[\![d]\!] = Cl(\{d\})$
- Corollary: $[\![\ ]\!]$ is injective.
- Lemma: If $d, e \in B$,
  1. $[\![d \cdot e]\!] = [\![d]\!] \bullet_{Cl} [\![e]\!]$.
  2. $[\![d \to e]\!] = [\![d]\!] \to [\![e]\!]$.
  3. $\cdots$

Proof Idea

- Just Maehara's Lemma!
- Proof of $[\![d]\!] \bullet_{Cl} [\![e]\!] \subseteq [\![d \cdot e]\!]$:
  - Let $t \in [\![d]\!]$ and $u \in [\![e]\!]$. We show:

$$\frac{t \in [\![d]\!] \quad u \in [\![e]\!]}{t \circ u \in [\![d \cdot e]\!]}$$

Proof Idea

- Just Maehara's Lemma!
- Proof of $[\![d]\!] \bullet_{Cl} [\![e]\!] \subseteq [\![d \cdot e]\!]$:
  - Let $t \in [\![d]\!]$ and $u \in [\![e]\!]$. We show:

$$\frac{t \in [\![d]\!] \quad u \in [\![e]\!]}{t \circ u \in [\![d \cdot e]\!]}$$

- Proof of $[\![d \cdot e]\!] = Cl(\{d \cdot e\}) \subseteq Cl([\![d]\!] \bullet [\![e]\!]) = [\![d]\!] \bullet_{Cl} [\![e]\!]$:
  - Let $[\![d]\!] \bullet [\![e]\!] \subseteq \{t\} \to [\![f]\!]$.
  - Since $d \in [\![d]\!]$ and $e \in [\![e]\!]$, we have $d \circ e \circ t \in [\![t]\!]$. We show:

$$\frac{d \circ e \circ t \in [\![t]\!]}{(d \cdot e) \circ t \in [\![f]\!]}$$

  - So $d \cdot e \in \{t\} \to [\![f]\!]$.

Integral and contractive cases

- Integrality and contractivity are preserved by the operation $(\bullet \cup \bullet)^*$ and phase semantic completion.
  - (Cf. Latter is a necessary and sufficient condition for cut-elimination for $\mathbf{FL}$+ simple structural rules; Terui 2006, Ciabattoni-T 2006)
- I.e., If $\mathbf{B}$ and $\mathbf{C}$ are integral and/or contractive, so is $\mathbf{D}$.
- Theorem: Any of $\mathcal{CRL}, \mathcal{CIRL}, \mathcal{CCRL}$ satisfies the (strong) AP.

Integral and contractive cases

- Integrality and contractivity are preserved by the operation $(\bullet \cup \bullet)^*$ and phase semantic completion.
  - (Cf. Latter is a necessary and sufficient condition for cut-elimination for $\mathbf{FL}$+ simple structural rules; Terui 2006, Ciabattoni-T 2006)
- I.e., If $\mathbf{B}$ and $\mathbf{C}$ are integral and/or contractive, so is $\mathbf{D}$.
- Theorem: Any of $\mathcal{CRL}, \mathcal{CIRL}, \mathcal{CCRL}$ satisfies the (strong) AP.

- Conclusion: Phase semantic construction provides a simple and general methodology. It could be used to show AP for other algebras, and to show other properties.