Coherence Spaces for Real Functions and Operators*

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Abstract

We discuss how to represent real numbers, real functions and operators based on coherence spaces and stable/linear maps. Specifically, we introduce a representation of the real line by a coherence space, which is admissible in the sense of the type-two theory of effectivity (TTE). This implies that a real function is realized by a stable map if and only if it is continuous, thus further leads to an admissible representation of the space of continuous real functions. In contrast, a real function is realized by a linear map if and only if it is uniformly continuous. Our presentation is concrete and self-contained, so that it can be read without any prerequisite in computable analysis and realizability.

1 Introduction

Coherence spaces, introduced by Girard [Gi87], are a drastic simplification of stable domain theory due to Berry [Be78]. Originally it was introduced as a denotational semantics for System F and used to interpret lambda terms by *stable* maps. As is well known, stable maps better capture the *sequential* computation in the sense of PCF than *continuous* maps in Scott domains. In particular, it is known that the *parallel-or* function is not expressible as stable map. Coherence spaces are equipped with another type of morphism, called *linear* maps, which gave birth to linear logic. Thus coherence spaces provide a simple denotational basis for both "stable" and "linear" functional computations.

While computation is usually carried out over finite objects such as integers, lists and trees, we are here interested in more abstract objects such as real numbers, real functions and operators. Since the latter cannot be directly manipulated by a computer, we first have to *represent* them in an appropriate way. The aim of this paper is to give a *self-contained* account on how to represent such abstract entities by coherence spaces and to see how the distinction between stable and linear maps shows up in this setting.

There have been a lot of attempts to concretely representing reals and real functions (as well as more abstract mathematical entities) since the very first

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work by Turing [Tu36]. It constitutes a large research field, collectively called *computable analysis*. Among various approaches, let us only mention two particularly successful ones: the *type-two theory of effectivity (TTE)* [KW85, We00, BHW08] and the theory of *domain representations* [Sco70, Bl97, ES99, SHT08].

In TTE, the real line $\mathbb{R}$, for instance, is represented by a partial surjective function $\rho :\subseteq \{0,1\}^\omega \longrightarrow \mathbb{R}$ from the Cantor space $\{0,1\}^\omega$ (or the Baire space $\mathbb{N}^\omega$) to $\mathbb{R}$. Once a representation has been given, the whole computation is carried out on the Cantor/Baire spaces, whose objects are concrete and directly manipulated by a computer. Among various representations, one can distinguish good ones from bad ones with respect to continuity and computability. One of the most important achievements of TTE is a suitable criterion for good representations, called *admissibility*. It works not only for the real line but also for various topological spaces.

On the other hand, a domain representation of $\mathbb{R}$ is given by a partial surjection $\rho : D \longrightarrow \mathbb{R}$ from a domain D. This approach, though similar to the previous one, leads to a *typed* account on representation: once $\mathbb{R}$ has been represented by D, it is natural to represent the function space $\mathcal{C}(\mathbb{R},\mathbb{R})$ by $D \Rightarrow D$, an exponential object in the category at issue.

In this paper, we develop a similar theory of representations based on coherence spaces. After some preliminaries in Section 2, we introduce *coherence representations* in Section 3. In our framework, the real line $\mathbb{R}$ is represented by a partial surjection $\rho_{\boldsymbol{R}} :\subseteq \boldsymbol{R} \longrightarrow \mathbb{R}$ from a suitable coherence space $\boldsymbol{R}$ (similarly to [DCC00]). Our theory is thus typed as the domain representations approach. We then import the concept of admissibility from TTE. It is, however, not as easy as it may seem at first, because coherence spaces are far more liberal than the Cantor and Baire spaces. We are thus led to restrict the class of representations to *spanned* ones which behave similarly to TTE representations. Admissibility can be naturally defined with this restriction and it is shown that our representation $\rho_{\boldsymbol{R}} :\subseteq \boldsymbol{R} \longrightarrow \mathbb{R}$ is indeed admissible in this sense. As a consequence, we obtain a natural result that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous if and only if it is realized by a stable map $F : \boldsymbol{R} \longrightarrow_{st} \boldsymbol{R}$ (Section 4). This result then induces an admissible representation of $\mathcal{C}(\mathbb{R},\mathbb{R})$ based on the exponential coherence space $\boldsymbol{R} \Rightarrow \boldsymbol{R}$ (Section 6).

All the above suggests that coherence spaces could be a reasonable denotational semantics for functional programming languages for real number computation (e.g., [Es96, ES14], just to mention a few). We must however admit that there is not much novelty, if we only consider stable maps as morphisms. An entirely new phenomenon arises when we consider linear maps as well. In Section 5, we show that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is *uniformly* continuous if and only if it is realized by a *linear* map $F : \boldsymbol{R} \longrightarrow_{lin} \boldsymbol{R}$. Thus linearity in coherence spaces corresponds to uniformity of real functions. Although this result rests on our specific way of representing $\mathbb{R}$, we believe that it is worth noting since it well illustrates a distinction between stable and linear maps in an analytic setting.

Related work. This paper is based on a presentation made at the *Twelfth International Conference on Computability and Complexity in Analysis (CCA'15)*. It is fair to say that some of our results may be indirectly obtained from the corresponding results in TTE, by establishing an equivalence between spanned coherence representations and TTE representations. In addition, some of our developments admit a more abstract account based on the modern theory of realizability [Lo94]. Indeed, it is well known that representations in TTE (and in our setting) are intimately related to modest sets in the sense of realizability [Bi99, Ba00, Ba02]. These aspects are to be discussed in a separate paper written by the first author [Ma16]. The latter also generalizes our results on $\mathbb{R}$ to a wider class of topological spaces and uniform spaces.

In contrast to [Ma16], we try to make our presentation as concrete, expository and self-contained as possible so that it can be read without any background on computable analysis. Although this risks reproving similar results and losing a global perspective, we hope that it will be useful to invite people working on denotational semantics of linear logic to the field of computable analysis. Apart from connection to computable analysis, this paper exhibits several new coherence spaces as well as their curious properties, which will be valuable as source of inspiration and deeper understanding of coherence spaces.

2 Coherence spaces

We here recall some basics of coherence spaces. See [Gi87, Me09] for further information.

Definition 2.1 (coherence spaces) *A* coherence space $\boldsymbol{X} = (X, \coh)$ *consists of a set X of* tokens *and a reflexive symmetric relation $\coh$ on X, called* coherence.

Throughout this paper, we assume that *every token set X is at most countable.* This assumption, which is needed for Theorem 4.5, is quite reasonable in practice, since we would like to think of tokens as computational objects (see [As90] for computability over coherence spaces).

A *clique* of $\boldsymbol{X}$ is a set of pairwise coherent tokens in X. By abuse of notation, we denote the set of cliques by $\boldsymbol{X}$. We also write $\boldsymbol{X}_{\mathsf{fin}}$ and $\boldsymbol{X}_{\mathsf{max}}$ to denote the sets of finite cliques and maximal cliques, respectively.

Given tokens $x, y \in X$, we write $x \frown y$ (*strict coherence*) if $x \coh y$ and $x \neq y$. Notice that coherence and strict coherence are mutually definable from each other. Given cliques $a, b \in \boldsymbol{X}$, we write $a \coh b$ if $a \cup b \in \boldsymbol{X}$. This means that any token in a is coherent with any token in b.

The set $\boldsymbol{X}$ is ordered by inclusion $\subseteq$, and endowed with the *Scott topology* generated by $\{\langle a\rangle : a \in \boldsymbol{X}_{\mathsf{fin}}\}$, where

$$\langle a\rangle := \{b \in \boldsymbol{X} : a \subseteq b\}.$$

Thus a coherence space can be seen as a poset, and it is in fact a *Scott domain* whose compact elements are exactly finite cliques. Note that it is a T_0-space, and is *countably based* (i.e., has a countable base) due to our assumption that the token set X is countable.

A typical coherence space is $\boldsymbol{PF} := (\mathbb{N} \times \mathbb{N}, \coh)$, where $\coh$ is defined by

$$(m_1, n_1) \coh (m_2, n_2) \iff m_1 = m_2 \text{ implies } n_1 = n_2.$$

An equivalent definition can be given in terms of $\frown$:

$$(m_1, n_1) \frown (m_2, n_2) \iff m_1 \neq m_2.$$

We have $f \in \boldsymbol{PF}$ iff f is (the graph of) a partial function $f :\subseteq \mathbb{N} \longrightarrow \mathbb{N}$, where we abuse the same notation for both a function and its graph. Maximal cliques correspond to total functions. A set $U \subseteq \boldsymbol{PF}$ is open iff there is a set B of finite partial functions such that $f \in U \iff g \subseteq f$ for some $g \in B$.

Definition 2.2 (stable and linear maps) *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be coherence spaces. A function $F : \boldsymbol{X} \to \boldsymbol{Y}$ is said to be* stable*, written $F : \boldsymbol{X} \longrightarrow_{st} \boldsymbol{Y}$, if it is continuous and for any cliques $a, b \in \boldsymbol{X}$,*

$$a \frown\!\!\!\!\smile b \quad \Longrightarrow \quad F(a \cap b) = F(a) \cap F(b).$$

A function $F : \boldsymbol{X} \to \boldsymbol{Y}$ is said to be linear*, written $F : \boldsymbol{X} \longrightarrow_{lin} \boldsymbol{Y}$, if it satisfies*

$$a = \sum_i a_i \implies F(a) = \sum_i F(a_i),$$

where $\sum$ means disjoint union of cliques.

It is easy to see that every linear map is stable.

There are alternative definitions. Given a function $F : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$, call $(a, y) \in \boldsymbol{X}_{\mathsf{fin}} \times Y$ a *minimal pair* of F if $F(a) \ni y$ and there is no proper subset $a' \subsetneq a$ such that $F(a') \ni y$. The set of minimal pairs of F is called the *trace* of F and denoted by $\mathsf{tr}(F)$.

Now, F is a stable map iff it is monotone w.r.t. $\subseteq$ and:

(st) if $F(a) \ni y$, there is a *unique* $a_0 \subseteq a$ such that $(a_0, y) \in \mathsf{tr}(F)$.

Indeed, suppose that F is stable and $F(a) \ni y$. Then continuity ensures the existence of a finite $a_0 \subseteq a$ such that $F(a_0) \ni y$, and stability ensures that a_0 is unique if it is chosen to be minimal.

If F is furthermore linear, preservation of disjoint union ensures that a_0 is a singleton: $a = \sum_{x \in a}\{x\} \Rightarrow F(a) = \sum_{x \in a} F(x)$. Thus, F is a linear map iff it is monotone and:

(lin) if $F(a) \ni y$, there is a *unique* $x \in a$ such that $(\{x\}, y) \in \mathsf{tr}(F)$.

By abuse of notation, we denote the set $\{(x, y) \mid (\{x\}, y) \in \mathsf{tr}(F)\}$ by $\mathsf{tr}(F)$ if F is supposed to be a linear map.

Below are some typical constructions of coherence spaces. Let $\boldsymbol{X}_i = (X_i, \frown\!\!\!\!\smile_i)$ be a coherence space for $i = 1, 2$. We denote by $X_1 \uplus X_2$ the disjoint sum $\{(1, x) : x \in X_1\} \cup \{(2, x') : x' \in X_2\}$. We define:

- $\top := (\varnothing, \varnothing)$.
- $\boldsymbol{X}_1 \times \boldsymbol{X}_2 := (X_1 \uplus X_2, \frown\!\!\!\!\smile)$, where $(i, x) \frown\!\!\!\!\smile (j, y)$ holds iff either $i \neq j$ or $i = j \wedge x \frown\!\!\!\!\smile_i y$.
- $\boldsymbol{X}_1 \Rightarrow \boldsymbol{X}_2 := ((\boldsymbol{X}_1)_{\mathsf{fin}} \times X_2, \frown\!\!\!\!\smile)$, where $(a, x) \frown (b, y)$ holds iff $a \frown\!\!\!\!\smile_1 b$ implies $x \frown_2 y$.
- $\boldsymbol{X}_1 \multimap \boldsymbol{X}_2 := (X_1 \times X_2, \frown\!\!\!\!\smile)$, where $(z, x) \frown (w, y)$ holds iff $z \frown\!\!\!\!\smile_1 w$ implies $x \frown_2 y$.

It is well known that the category **Coh** of coherence spaces and stable maps equipped with $(\top, \times, \Rightarrow)$ is cartesian closed. Likewise, the category **Lin** of coherence spaces and linear maps can be enhanced with the structure of a *Seely category*, a model of linear logic (see [Me09]).

We do not describe the categorical structure in detail, but let us just remark the following. Given $c \in \boldsymbol{X} \times \boldsymbol{Y}$, there uniquely exist cliques $a \in \boldsymbol{X}$ and $b \in \boldsymbol{Y}$ such that $c = a \uplus b$. Writing $c = \langle a, b \rangle$, it is easy to see that $\langle a, b \rangle \frown_{\boldsymbol{X} \times \boldsymbol{Y}} \langle a', b' \rangle$ holds iff $a \frown_{\boldsymbol{X}} a' \wedge b \frown_{\boldsymbol{Y}} b'$. Given $F : \boldsymbol{X} \longrightarrow_{st} \boldsymbol{Y}$, we have $\mathsf{tr}(F) \in \boldsymbol{X} \Rightarrow \boldsymbol{Y}$. Conversely, given $\alpha \in \boldsymbol{X} \Rightarrow \boldsymbol{Y}$, there is a stable map $\widehat{\alpha} : \boldsymbol{X} \longrightarrow_{st} \boldsymbol{Y}$ defined by

$$\widehat{\alpha}(a) := \{y \in Y : (a_0, y) \in \alpha \text{ for some } a_0 \subseteq a\}.$$

Similarly, every linear map $F : \boldsymbol{X} \longrightarrow_{lin} \boldsymbol{Y}$ leads to a clique $\mathsf{tr}(F) \in \boldsymbol{X} \multimap \boldsymbol{Y}$, and every clique $\alpha \in \boldsymbol{X} \multimap \boldsymbol{Y}$ leads to a linear map $\widehat{\alpha}(a) := \{y \in Y : (x, y) \in \alpha \text{ for some } x \in a\}$. These data constitute the bijective correspondences:

$$\mathbf{Coh}(\boldsymbol{X}, \boldsymbol{Y}) \cong \boldsymbol{X} \Rightarrow \boldsymbol{Y}, \quad \mathbf{Lin}(\boldsymbol{X}, \boldsymbol{Y}) \cong \boldsymbol{X} \multimap \boldsymbol{Y}.$$

3 Representations

3.1 Representations of sets

We are interested in computation over *abstract* mathematical spaces, which cannot be directly dealt with by computers. The basic idea of TTE is to represent an abstract space $\mathbb{X}$ by a surjective partial function $\rho :\subseteq \{0,1\}^\omega \longrightarrow \mathbb{X}$ from the *concrete* space $\{0,1\}^\omega$ (Cantor space). We basically follow the same idea, the only difference being that we think of coherence spaces as *concrete*. Let us begin with representations of plain sets without any topological structures.

Definition 3.1 (representation) *Let S be an arbitrary set. A tuple $(\boldsymbol{X}, \rho, S)$ is called a* representation *of S if $\boldsymbol{X}$ is a coherence space and $\rho :\subseteq \boldsymbol{X} \longrightarrow S$ is a partial surjective function. Below, $(\boldsymbol{X}, \rho, S)$ is denoted as $\boldsymbol{X} \xrightarrow{\rho} S$ or simply as ρ. If $\rho(a) = r$, we say that r is* realized *by a clique a (via representation ρ).*

Representations allow us to express abstract functions as stable maps.

Definition 3.2 (stable realizability) *Let $\boldsymbol{X} \xrightarrow{\rho_{\boldsymbol{X}}} S$ and $\boldsymbol{Y} \xrightarrow{\rho_{\boldsymbol{Y}}} T$ be representations. We say that a total function $f : S \longrightarrow T$ is* realized by *a stable map $F : \boldsymbol{X} \longrightarrow_{st} \boldsymbol{Y}$ via representations $\rho_{\boldsymbol{X}}, \rho_{\boldsymbol{Y}}$ if the following diagram commutes:*

$$\begin{tikzcd} \boldsymbol{X} \arrow[r, "F"] \arrow[d, "\rho_{\boldsymbol{X}}"'] & \boldsymbol{Y} \arrow[d, "\rho_{\boldsymbol{Y}}"] \\ S \arrow[r, "f"] & T \end{tikzcd} \tag{1}$$

Such a function f is also called stably realizable.

Many of the constructions on coherence spaces are inherited by representations and stably realizable maps. Typically, we have:

Definition 3.3 (product and exponential) *Let* $\boldsymbol{X} \xrightarrow{\rho_{\boldsymbol{X}}} S$ *and* $\boldsymbol{Y} \xrightarrow{\rho_{\boldsymbol{Y}}} T$ *be representations.*

- *The* product $\boldsymbol{X} \times \boldsymbol{Y} \xrightarrow{\langle \rho_{\boldsymbol{X}}, \rho_{\boldsymbol{Y}} \rangle} S \times T$ *is naturally defined, where* $\mathsf{dom}(\rho_{\boldsymbol{X}\times\boldsymbol{Y}}) := \{\langle a, b\rangle : a \in \mathsf{dom}(\rho_{\boldsymbol{X}}), b \in \mathsf{dom}(\rho_{\boldsymbol{Y}})\}$ *and* $\rho_{\boldsymbol{X}\times\boldsymbol{Y}}(\langle a, b\rangle) := \langle \rho_{\boldsymbol{X}}(a), \rho_{\boldsymbol{Y}}(b)\rangle$.

- *The* exponential $\boldsymbol{X} \Rightarrow \boldsymbol{Y} \xrightarrow{[\rho_{\boldsymbol{X}} \to \rho_{\boldsymbol{Y}}]} \mathcal{SR}(\rho_{\boldsymbol{X}}, \rho_{\boldsymbol{Y}})$ *is defined as follows. Define* $[\rho_{\boldsymbol{X}} \to \rho_{\boldsymbol{Y}}] :\subseteq \boldsymbol{X} \Rightarrow \boldsymbol{Y} \longrightarrow T^S$ *by*

  $$[\rho_{\boldsymbol{X}} \to \rho_{\boldsymbol{Y}}](\alpha) := f \quad \Longleftrightarrow \quad f : S \longrightarrow T \text{ is realized by } \widehat{\alpha} : \boldsymbol{X} \longrightarrow_{st} \boldsymbol{Y}.$$

  $\mathcal{SR}(\rho_{\boldsymbol{X}}, \rho_{\boldsymbol{Y}}) \subseteq T^S$ *is the range of* $[\rho_{\boldsymbol{X}} \to \rho_{\boldsymbol{Y}}]$*, which consists of stably realizable functions.*

As expected, the category $\mathbf{Rep}(\mathbf{Coh})$ of representations and stably realizable functions is cartesian closed. Furthermore it is regular and locally cartesian closed, since $\mathbf{Rep}(\mathbf{Coh})$ is equivalent to the category of modest sets over a universal coherence space. Relationship with the realizability theory will be discussed in [Ma16].

3.2 Representation of the real line

We now illustrate how to represent a space. Our principal example is the real line $\mathbb{R}$.

Let $\mathbb{D} := \mathbb{Z} \times \mathbb{N}$, where each $(m, n) \in \mathbb{D}$ is identified with a *dyadic rational number* $m/2^n$. Notice that we distinguish $(1, 0)$ and $(2, 1)$ as elements of $\mathbb{D}$, while $1/2^0$ and $2/2^1$ are identical as points of $\mathbb{R}$. We will explicitly write $x \in \mathbb{D}$ when we take the former standpoint.

We use the following notations: for $x = (m, n) \in \mathbb{D}$, $\mathsf{den}(x) := n$ (the exponent of the denominator), and

$$[x] \; := \; [(m-1)/2^n;\; (m+1)/2^n]$$

(the closed interval of $\mathbb{R}$ with center x and length 2^{-n+1}). We also write $\mathbb{D}_n := \{(m, n) \mid m \in \mathbb{Z}\}$ so that $x \in \mathbb{D}_n$ iff $\mathsf{den}(x) = n$.

A coherence space for the real line is given by $\boldsymbol{R} := (\mathbb{D}, \frown)$, where

$$x \frown y \quad \Longleftrightarrow \quad \mathsf{den}(x) \neq \mathsf{den}(y) \text{ and } [x] \cap [y] \neq \varnothing.$$

Let a be a clique in $\boldsymbol{R}$ and $n \in \mathbb{N}$. Since $\neg(x \frown x')$ for all $x, x' \in \mathbb{D}_n$, a contains at most one element x_n such that $x_n \in \mathbb{D}_n$. Moreover, it is not hard to see that a can be extended to a larger clique $a' := a \cup \{x_n\}$ if $a \cap \mathbb{D}_n = \varnothing$. Hence if a is a maximal clique, it can be identified with a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in \mathbb{D}_n$ for each $n \in \mathbb{N}$. On the other hand, notice that the second condition can be rephrased as follows:

$$[x] \cap [y] \neq \varnothing \quad \Longleftrightarrow \quad |x - y| \leqslant 2^{-\mathsf{den}(x)} + 2^{-\mathsf{den}(y)}. \tag{2}$$

Hence $a \in \boldsymbol{R}_{\max}$ expresses a (rapidly-converging) *Cauchy sequence*:

$$N < m, n \implies |x_m - x_n| \leqslant 2^{-m} + 2^{-n} \leqslant 2^{-N}.$$

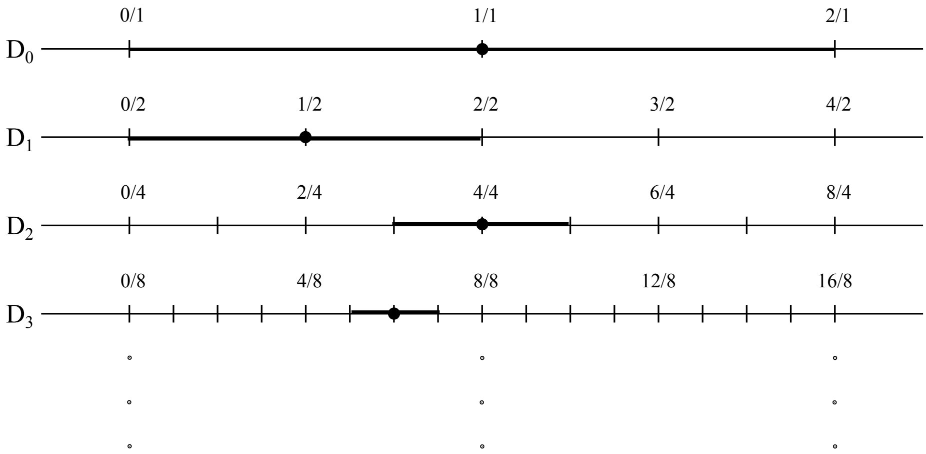

Figure 1: A clique $a = \{(1, 0), (1, 1), (4, 2), (6, 3), \dots\}$

This allows us to define a function $\rho_{\boldsymbol{R}}(a) := \lim_{n\to\infty} x_n$ with $\mathsf{dom}(\rho_{\boldsymbol{R}}) := \boldsymbol{R}_{\mathsf{max}}$. We often write a^* instead of $\rho_{\boldsymbol{R}}(a)$. Since any real number can be approximated by a sequence of dyadic rationals, $\rho_{\boldsymbol{R}} : \mathsf{dom}(\boldsymbol{R}) \longrightarrow \mathbb{R}$ is surjective. Hence we have obtained a representation $\boldsymbol{R} \xrightarrow{\rho_{\boldsymbol{R}}} \mathbb{R}$. Figure 1 illustrates a clique $a = \{(1, 0), (1, 1), (4, 2), (6, 3), \dots\}$, where each thick line indicates the interval $[x]$ associated to a token $x \in a$.

Our next goal is to show that $\rho_{\boldsymbol{R}} : \boldsymbol{R}_{\mathsf{max}} \longrightarrow \mathbb{R}$ is continuous, where $\boldsymbol{R}_{\mathsf{max}} \subseteq \boldsymbol{R}$ is endowed with the subspace topology.

Lemma 3.4 *For every $a \in \boldsymbol{R}_{\mathsf{max}}$, we have*

$$x \in a \quad \Longrightarrow \quad a^* \in [x].$$

Proof. Suppose that $a = (x_n)_{n\in\mathbb{N}}$ and $x = x_m$. From (2), we have $|x - x_n| \leqslant 2^{-m} + 2^{-n}$ for any $n \in \mathbb{N}$. Since x_n tends to a^* as $n \to \infty$. we obtain $|x - a^*| \leqslant 2^{-m}$, i.e., $a^* \in [x]$. ∎

Given a nonempty set $b \subseteq \mathbb{D}$, we write $[b] := \bigcap_{x\in b}[x]$. Note that $b \subseteq c$ implies $[c] \subseteq [b]$. The following is an easy consequence of the previous lemma.

Lemma 3.5 *For every $a \in \boldsymbol{R}_{\mathsf{max}}$, $[a] = \{a^*\}$.*

Proof. Let $a = (x_n)_{n\in\mathbb{N}} \in \boldsymbol{R}_{\mathsf{max}}$. We have $a^* \in [a]$ by Lemma 3.4. Given any $r \in [a]$, we have $|r - a^*| \leqslant |r - x_{n+1}| + |x_{n+1} - a^*| \leqslant 2^{-n}$ for all $n \in \mathbb{N}$ since $r \in [a] \subseteq [x_{n+1}]$, which leads to $r = a^*$.

∎

Lemma 3.6 *For every $a \in \boldsymbol{R}_{\mathsf{max}}$ and every open set $U \subseteq \mathbb{R}$ with $a^* \in U$, there exists $x \in a$ such that $[x] \subseteq U$.*

Proof. Let $a = (x_n)_{n\in\mathbb{N}} \in \boldsymbol{R}_{\mathsf{max}}$. One can take $\delta > 0$ such that $(a^* - \delta, a^* + \delta) \subseteq U$. Then for a sufficiently large n, we have $[x_n] \subseteq (a^* - \delta, a^* + \delta)$, since $[x_n]$ is a closed interval which contains a^* and whose length tends to 0 as $n \to \infty$.

∎

Lemma 3.7 *$\rho_{\boldsymbol{R}} : \boldsymbol{R}_{\mathsf{max}} \longrightarrow \mathbb{R}$ is continuous.*

Proof. Let $U \subseteq \mathbb{R}$ be an open set. We claim:

$$\rho_{\boldsymbol{R}}^{-1}[U] \; = \; \bigcup\{\langle b\rangle : b \in \boldsymbol{R}_{\mathsf{fin}} \text{ and } [b] \subseteq U\} \cap \mathsf{dom}(\rho_{\boldsymbol{R}}),$$

where the right hand side is a union of basic open sets in $\boldsymbol{R}_{\mathsf{max}}$, hence is open. Indeed, $a \in \rho_{\boldsymbol{R}}^{-1}[U]$ implies $a^* \in U$. By Lemma 3.6, we obtain $[x] \subseteq U$ for some $x \in a$. Since $a \in \langle\{x\}\rangle$, it belongs to the right hand side. Conversely, suppose that $a \in \langle b\rangle$ with $b \in \boldsymbol{R}_{\mathsf{fin}}$ and $[b] \subseteq U$. Then $a^* \in [a] \subseteq [b] \subseteq U$ by Lemma 3.5. That is, $a \in \rho_{\boldsymbol{R}}^{-1}[U]$. ■

We investigate further properties of the representation $\rho_{\boldsymbol{R}}$ which will be needed later.

Lemma 3.8 (hybridding) *Let $a = (x_m)_{m\in\mathbb{N}}$ be a maximal clique and $b = \{y_n : n \in I\}$ be a clique of $\boldsymbol{R}$ with $I \subseteq \mathbb{N}$ such that $a^* \in [b]$. Define $c = (z_k)_{k\in\mathbb{N}}$ by*

$$\begin{array}{rcll} z_k & := & y_k & (k \in I) \\ & := & x_k & (\textit{otherwise}) \end{array}$$

Then c is a maximal clique such that $b \subseteq c$ and $c^* = a^*$.

Proof. Notice that every $[z_k]$ contains a^* in common. c is a clique, since $\mathsf{den}(z_k) \neq \mathsf{den}(z_{k'})$ and $a^* \in [z_k] \cap [z_{k'}] \neq \emptyset$ for all $k, k' \in \mathbb{N}$ with $k \neq k'$. Maximality is obvious. $c^* = a^*$ follows by Lemma 3.5. ■

Lemma 3.9 *For any $b \in \boldsymbol{R}$ and $r \in \mathbb{R}$, we have:*

$$r \in [b] \quad \Longleftrightarrow \quad b \textit{ has an extension } c \in \boldsymbol{R}_{\mathsf{max}} \textit{ with } c^* = r.$$

Proof. The backward direction can be easily seen by noting that $b \subseteq c$ implies $c^* \in [c] \subseteq [b]$. For the forward direction, choose a maximal clique a with $a^* = r$. Applying Lemma 3.8 to the cliques a and b, we obtain a maximal clique c such that $b \subseteq c$ and $c^* = r$. ■

4 Admissible Representations

4.1 Spanned representations

We have seen how to represent the real line. A similar idea leads to representation of other metric spaces, and a more general class of topological spaces.

In general, a space may have many representations. Some are good, while others are terrible from a computational perspective. In TTE, a criterion for reasonable representations has been established, that is the concept of admissibility [We00, Sc02a]. Roughly speaking, a representation is admissible if it is continuous and "weakly final" among all continuous representations. This idea has been adapted to domain representations by Hamrin [Ha05].

We are now going to adapt it to coherence spaces. However, it turns out that a straightforward translation as in [Ha05, Da07] does not work, since it does not even make $\boldsymbol{R} \xrightarrow{\rho_{\boldsymbol{R}}} \mathbb{R}$ admissible. This is because we consider stable maps, which are far more restrictive than continuous ones. We are thus led to impose an additional requirement.

Recall that each member of the Cantor space $\{0,1\}^\omega$ (infinite sequence) can be approximated by its finite prefixes in $\{0,1\}^*$, and that the set $\{0,1\}^*$ with the prefix ordering $\sqsubseteq$ forms a tree. In short, $\{0,1\}^\omega$ is "spanned" by the tree $(\{0,1\}^*, \sqsubseteq)$. The point of the definition below is to impose a similar structure on the domain of a representation.

Definition 4.1 (spanned representation) *A representation $\boldsymbol{X} \xrightarrow{\rho} S$ is said to be* spanned *if there is a set $\mathcal{F} \subseteq \boldsymbol{X}_{\mathsf{fin}}$ (called a* spanning forest*) with the following properties:*

- *For any $a, b \in \mathcal{F}$ with $a \frown b$, either $a \subseteq b$ or $b \subseteq a$ holds.*
- *$a \in \mathsf{dom}(\rho)$ iff there is a maximal chain $\{a_i\}_{i \in I}$ in $\mathcal{F}$ such that $a = \bigcup a_i$.*

We denote by **SpnRep(Coh)** *the full subcategory of* **Rep(Coh)** *that consists of spanned representations.*

It would be perhaps more appropriate to call it a *spannable* representation since the spanning forest $\mathcal{F}$ is not part of data, though we stick to calling it a *spanned* representation. Observe that the first condition makes $(\mathcal{F}, \subseteq)$ a forest in the sense of graph theory: $a \subseteq c$ and $b \subseteq c$ with $a, b, c \in \mathcal{F}$ imply $a \subseteq b$ or $b \subseteq a$. On the other hand, the second condition states that every $a \in \mathsf{dom}(\rho)$ is approximated by a (unique) maximal chain in $\mathcal{F}$, and conversely any $a \in \mathcal{F}$ has an extension in $\mathsf{dom}(\rho)$, that is, $\langle a \rangle \cap \mathsf{dom}(\rho) \neq \varnothing$ (*density*).

For instance, the representation $\boldsymbol{R} \xrightarrow{\rho_{\boldsymbol{R}}} \mathbb{R}$ is spanned by the tree $\mathcal{F}$ of finite initial segments of Cauchy sequences: $a_0 \in \mathcal{F}$ iff $a_0 = \{x_0, \ldots, x_m\} \in \boldsymbol{R}_{\mathsf{fin}}$ for some $m \in \mathbb{N}$ and $x_i \in \mathbb{D}_i$.

A more direct example is the following.

Example 4.2 (coherence space for the Cantor space) *Let $\boldsymbol{C} := (\{0,1\}^*, \frown)$, where $w \frown u$ iff $w \sqsubseteq u$ or $u \sqsubseteq w$. Then any maximal clique in $\boldsymbol{C}$ can be identified with an infinite sequence in the Cantor space $\{0,1\}^\omega$, so that we obtain a function $\rho_{\boldsymbol{C}} : \boldsymbol{C}_{\mathsf{max}} \longrightarrow \{0,1\}^\omega$. It is not hard to see that $\rho_{\boldsymbol{C}}$ is continuous (in fact a homeomorphism).*

On the other hand, any $w \in \{0,1\}^\omega \cup \{0,1\}^*$ leads to a clique $w^\circ \in \boldsymbol{C}$ defined by:

$$w^\circ \ := \ \{u \in \{0,1\}^* : u \sqsubseteq w\}.$$

This allows us to define a spanning forest $\mathcal{F} := \{w^\circ : w \in \{0,1\}^*\} \subseteq \boldsymbol{C}_{\mathsf{fin}}$. Thus $\boldsymbol{C} \xrightarrow{\rho_{\boldsymbol{C}}} \{0,1\}^\omega$ is an object of **SpnRep(Coh)**.

SpnRep(Coh) is closed under finite products.

Lemma 4.3 *If $\boldsymbol{X} \xrightarrow{\rho_{\boldsymbol{X}}} S$ and $\boldsymbol{Y} \xrightarrow{\rho_{\boldsymbol{Y}}} T$ belong to* **SpnRep(Coh)**, *so is the product representation $\boldsymbol{X} \times \boldsymbol{Y} \xrightarrow{\rho_{\boldsymbol{X} \times \boldsymbol{Y}}} S \times T$.*

Proof. Let $\mathcal{F}_{\boldsymbol{X}}$ and $\mathcal{F}_{\boldsymbol{Y}}$ be spanning forests for $\rho_{\boldsymbol{X}}$ and $\rho_{\boldsymbol{Y}}$, respectively. Note that each element of $\mathsf{dom}(\rho_{\boldsymbol{X}})$ is either a leaf of $\mathcal{F}_{\boldsymbol{X}}$ or the limit of an infinite path in $\mathcal{F}_{\boldsymbol{X}}$. For each $a \in \mathcal{F}_{\boldsymbol{X}}$, we write $h(a) = n$ if there are exactly n elements below a in the forest $(\mathcal{F}_{\boldsymbol{X}}, \subseteq)$: $a_1 \subsetneq \cdots \subsetneq a_n \subsetneq a$ (strict inclusion). We use the same notation for elements of $\mathcal{F}_{\boldsymbol{Y}}$.

Now a forest $\mathcal{F}_{\boldsymbol{X}\times\boldsymbol{Y}} \subseteq (\boldsymbol{X} \times \boldsymbol{Y})_{\mathsf{fin}}$ can be defined as follows:

$$\langle a, b\rangle \in \mathcal{F}_{\boldsymbol{X}\times\boldsymbol{Y}} \quad \Longleftrightarrow \quad \begin{cases} h(a) = h(b), \text{ or} \\ a \text{ is a leaf and } h(a) < h(b), \text{ or} \\ b \text{ is a leaf and } h(a) > h(b), \end{cases}$$

for all $a \in \mathcal{F}_{\boldsymbol{X}}$ and $b \in \mathcal{F}_{\boldsymbol{Y}}$. It is clear that $\mathcal{F}_{\boldsymbol{X}\times\boldsymbol{Y}}$ satisfies the second condition for spanning forests. For the first condition, let $\langle a_i, b_i\rangle \in \mathcal{F}_{\boldsymbol{X}\times\boldsymbol{Y}}$ $(i = 1, 2)$ be cliques such that $\langle a_1, b_1\rangle \frown \langle a_2, b_2\rangle$. Notice that we have $a_1 \subseteq a_2 \vee a_2 \subseteq a_1$ and $b_1 \subseteq b_2 \vee b_2 \subseteq b_1$ by assumption. Our goal is to show that either $a_1 \subseteq a_2 \wedge b_1 \subseteq b_2$ or $a_2 \subseteq a_1 \wedge b_2 \subseteq b_1$ holds. Suppose for contradiction that $a_1 \subsetneq a_2$ and $b_2 \subsetneq b_1$ so that $h(a_1) < h(a_2)$ and $h(b_2) < h(b_1)$. We never have $h(a_1) < h(b_1)$ since a_1 is not a leaf. On the other hand, $h(b_1) \leqslant h(a_1)$ implies $h(b_2) < h(a_2)$, which is impossible since b_2 is not a leaf. Thus it is impossible to have both $a_1 \subsetneq a_2$ and $b_2 \subsetneq b_1$ together. ■

4.2 Admissibility

As will be discussed in [Ma16], **SpnRep(Coh)** is categorically equivalent to the category $\mathbf{Rep}(\mathbb{B})$ of TTE representations (although we do not use this fact in this paper). Hence we may naturally import the concept of admissibility from the latter.

Definition 4.4 (admissibility) *Let $\mathbb{Y}$ be a topological space. A representation $\boldsymbol{Y} \xrightarrow{\rho} \mathbb{Y}$ in* **SpnRep(Coh)** *is* admissible *if it is continuous as a partial function and for any subspace $\mathbb{Y}_0 \subseteq \mathbb{Y}$ and any continuous representation $\boldsymbol{X} \xrightarrow{\gamma} \mathbb{Y}_0$ in* **SpnRep(Coh)** *there exists a stable map $F : \boldsymbol{X} \longrightarrow_{st} \boldsymbol{Y}$ which realizes the inclusion map $i : \mathbb{Y}_0 \longrightarrow \mathbb{Y}$:*

$$\begin{array}{ccc} \boldsymbol{X} & \xrightarrow{\quad F \quad} & \boldsymbol{Y} \\ {\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle\rho} \\ \mathbb{Y}_0 & \xrightarrow{\quad i \quad} & \mathbb{Y} \end{array} \tag{3}$$

Given a topological space $\mathbb{X}$, its admissible representations are interchangeable in the following sense. Let $\boldsymbol{X}_0 \xrightarrow{\rho_0} \mathbb{X}$ and $\boldsymbol{X}_1 \xrightarrow{\rho_1} \mathbb{X}$ be admissible representations. Then the identity map $\mathsf{id} : \mathbb{X} \longrightarrow \mathbb{X}$ is realized by stable maps $F : \boldsymbol{X}_0 \longrightarrow_{st} \boldsymbol{X}_1$ and $G : \boldsymbol{X}_1 \longrightarrow_{st} \boldsymbol{X}_0$. Hence realizability of a function $f : \mathbb{X} \longrightarrow \mathbb{Y}$ does not depend on the choice of an admissible representation. However, notice that F and G need not be inverses of each other, since we do not require uniqueness.

Admissible representations enjoy a very pleasant property that stable realizability does coincide with *sequential continuity* (see below).

Theorem 4.5 *Let $\mathbb{X}$ and $\mathbb{Y}$ be topological spaces admissibly represented by $\boldsymbol{X} \xrightarrow{\rho_{\mathbb{X}}} \mathbb{X}$ and $\boldsymbol{Y} \xrightarrow{\rho_{\mathbb{Y}}} \mathbb{Y}$. A function $f : \mathbb{X} \longrightarrow \mathbb{Y}$ is stably realizable if and only if it is* sequentially continuous*, that is, it preserves the limit of any convergent sequence:*

$$x_n \to x \ (n \to \infty) \implies f(x_n) \to f(x) \ (n \to \infty).$$

Let us make some comments before proving the theorem. Several variants of this theorem are known in the literature. In TTE, Kreitz and Weihrauch [KW85] first proved the theorem for countably-based T_0-spaces. It was later extended to arbitrary topological spaces and beyond by Schröder [Sc01, Sc02a, Sc02b, Sc02c], modifying the definition of admissibility. Hamrin [Ha05] proved a similar result about net-continuity for domain representations.

In general, continuity of a function $f : \mathbb{X} \longrightarrow \mathbb{Y}$ implies sequential continuity. The other direction holds when the topology on $\mathbb{X}$ satisfies an additional property. A subset S of $\mathbb{X}$ is called *sequentially open* if any sequence $(x_n)_n \in \mathbb{X}^\omega$ converging to a point in S eventually lies in S. A *sequential space* is a topological space whose open sets and sequentially open sets coincide. It is this property that makes the two notions of continuity coincide. It is known that every countably based space is a sequential space. Hence the following topological spaces are all sequential: the real line $\mathbb{R}$, coherence spaces $\boldsymbol{X}$, and their subspaces (since every subspace of a countably based space is also countably based).

As a corollary of Theorem 4.5, we have:

Corollary 4.6 *Let $\mathbb{X}$ and $\mathbb{Y}$ be topological spaces which are admissibly represented by $\boldsymbol{X} \xrightarrow{\rho_{\mathbb{X}}} \mathbb{X}$ and $\boldsymbol{Y} \xrightarrow{\rho_{\mathbb{Y}}} \mathbb{Y}$. Suppose that $\mathbb{X}$ is a sequential space. Then a function $f : \mathbb{X} \longrightarrow \mathbb{Y}$ is stably realizable if and only if it is continuous.*

Although Theorem 4.5 follows from the corresponding result for TTE in [Sc02a] via the categorical equivalence $\mathbf{SpnRep}(\mathbf{Coh}) \simeq \mathbf{Rep}(\mathbb{B})$ [Ma16], we nevertheless give a self-contained proof here. Let us begin with a technical construction.

Let $\mathbb{X}$ be a topological space and $(x_n)_{n\in\mathbb{N}}$ a sequence in $\mathbb{X}^\omega$ which converges to $x_\infty \in \mathbb{X}$. We assume that $x_\infty \neq x_n$ for every $n \in \mathbb{N}$. Then the subspace $\mathbb{X}_0 := \{x_n : n \in \mathbb{N} \cup \{\infty\}\}$ can be represented as follows.

We use the coherence space $\boldsymbol{C}$ in Example 4.2 as the source of representation. Let

$$\begin{array}{rcl} a_n & := & (0^n 1^\omega)^\circ \quad (n \in \mathbb{N}) \\ a_\infty & := & (0^\omega)^\circ. \end{array}$$

We may then define a representation $\boldsymbol{C} \xrightarrow{\gamma} \mathbb{X}_0$ by $\gamma(a_n) := x_n$ for every $n \in \mathbb{N} \cup \{\infty\}$.

Lemma 4.7

1. *$(a_n)_{n\in\mathbb{N}}$ converges to a_∞ in $\boldsymbol{C}$.*

2. *γ is continuous.*

Proof. For claim 1, suppose that a_∞ belongs to a basic open set $\langle b \rangle$, where $b \in \boldsymbol{X}_{\mathsf{fin}}$. By definition, $a_\infty \in \langle b \rangle$ iff $b \subseteq (0^\omega)^\circ$. Such b must be of the form $\{0^{m_1}, \ldots, 0^{m_k}\}$, where $m_1, \ldots m_k \in \mathbb{N}$. Let $m := \max\{m_1, \ldots, m_k\}$. Then for any $n \in \mathbb{N}$, we have $a_n \in \langle b \rangle$ iff $n \geqslant m$. Hence all a_n but finitely many belong to $\langle b \rangle$.

For claim 2, observe that $a_m \in \langle (0^n 1)^\circ \rangle$ iff $(0^n 1)^\circ \subseteq (0^m 1)^\circ$ iff $m = n$. Hence $\langle (0^n 1)^\circ \rangle \cap \mathsf{dom}(\gamma)$ is a singleton $\{a_n\}$ for every $n \in \mathbb{N}$. Now let $U \subseteq \mathbb{X}_0$ be an open set. We prove that $\gamma^{-1}[U]$ is open by case distinction.

- If $x_\infty \notin U$, then by the above observation,

$$\gamma^{-1}[U] = \{a_n : x_n \in U\} = \bigcup_{n : x_n \in U} \langle (0^n 1)^\circ \rangle \cap \mathsf{dom}(\gamma) \ ,$$

  which is a union of basic open sets, hence is open.

- If $x_\infty \in U$, then there is $m \in \mathbb{N}$ such that all x_n with $n \geqslant m$ belong to U, since (x_n) converges to x_∞. In this case, we have:

$$\gamma^{-1}[U] = \{a_n : x_n \in U\} \cup \Big(\langle (0^m)^\circ \rangle \cap \mathsf{dom}(\gamma) \Big),$$

  which is open in the similar way.

Hence γ is continuous. ■

We are now ready to prove Theorem 4.5.

Proof. ($\Leftarrow$) Let $f : \mathbb{X} \longrightarrow \mathbb{Y}$ be a sequentially continuous function and $\mathbb{Y}_0 := f[\mathbb{X}]$ (the range of f). Then the composed function $f \circ \rho_{\boldsymbol{X}} : \mathsf{dom}(\rho_{\boldsymbol{X}}) \longrightarrow \mathbb{Y}_0$ is sequentially continuous. Recall that $\mathsf{dom}(\rho_{\boldsymbol{X}})$ is sequential, since it is a subspace of a sequential space $\boldsymbol{X}$. Hence $f \circ \rho_{\boldsymbol{X}}$ is in fact continuous. Since $\mathsf{dom}(f \circ \rho_{\boldsymbol{X}}) = \mathsf{dom}(\rho_{\boldsymbol{X}})$, it belongs to **SpnRep(Coh)**. We may now apply admissibility of $\rho_{\boldsymbol{Y}}$ to $\gamma := f \circ \rho_{\boldsymbol{X}}$ (see (3)) to obtain a stable map $F : \boldsymbol{X} \longrightarrow_{st} \boldsymbol{Y}$ that makes the diagram (1) commute.

($\Rightarrow$) Suppose that f is realized by a stable map F. Let $(x_n)_n \in \mathbb{X}^\omega$ be a sequence converging to $x_\infty \in \mathbb{X}$ and let $\mathbb{X}_0 := \{x_n : n \in \mathbb{N} \cup \{\infty\}\}$ as above. Our goal is to show that $f(x_n)$ converges to $f(x_\infty)$ in $\mathbb{Y}$ as $n \to \infty$. By the lemma above, there is a continuous representation $\boldsymbol{C} \xrightarrow{\gamma} \mathbb{X}_0$. Since $\mathbb{X}_0$ is a subspace of $\mathbb{X}$, admissibility of $\rho_{\boldsymbol{X}}$ implies the existence of a stable map $G : \boldsymbol{C} \longrightarrow_{st} \boldsymbol{X}$ that makes the left square below commute (the right square commutes by assumption):

$$\begin{tikzcd}
\boldsymbol{C} \arrow[r, "G"] \arrow[d, "\gamma"'] & \boldsymbol{X} \arrow[r, "F"] \arrow[d, "\rho_{\boldsymbol{X}}"'] & \boldsymbol{Y} \arrow[d, "\rho_{\boldsymbol{Y}}"] \\
\mathbb{X}_0 \arrow[r, "i"] & \mathbb{X} \arrow[r, "f"] & \mathbb{Y}
\end{tikzcd}$$

Since stable maps are continuous, the composed map $\rho_{\boldsymbol{Y}} \circ F \circ G$ is also continuous. We also know that the sequence $(a_n)_n$ converges to a_∞ in $\boldsymbol{C}$ by the previous lemma. Now consider the sequence $(f(x_n))_n$. Since we have

$$f(x_n) = f \circ i \circ \gamma(a_n) = \rho_Y \circ F \circ G(a_n)$$

for every $n \in \mathbb{N} \cup \{\infty\}$, we conclude that the sequence $(f(x_n))_n$ converges to $f(x_\infty)$. ■

The following is an easy consequence of Lemma 4.3.

Proposition 4.8 *If $\boldsymbol{X} \xrightarrow{\rho_{\boldsymbol{X}}} \mathbb{X}$ and $\boldsymbol{Y} \xrightarrow{\rho_{\boldsymbol{Y}}} \mathbb{Y}$ are admissible representations, so is the product representation $\boldsymbol{X} \times \boldsymbol{Y} \xrightarrow{\langle \rho_{\boldsymbol{X}}, \rho_{\boldsymbol{Y}} \rangle} \mathbb{X} \times \mathbb{Y}$.*

4.3 Admissibility of $\rho_{\boldsymbol{R}}$

Having established a general property of admissible representations, we now provide an instance. Let us begin with a generalization of Lemma 3.6.

Lemma 4.9 *Let $\boldsymbol{X} \xrightarrow{\gamma} \mathbb{X}$ be a continuous representation. For every open set $U \subseteq \mathbb{X}$ and $a \in \mathsf{dom}(\gamma)$ with $\gamma(a) \in U$, there exists a finite subclique $a_0 \subseteq a$ such that*

$$\gamma(a) \in \gamma\langle a_0 \rangle \subseteq U,$$

where $\gamma\langle a_0 \rangle = \{\gamma(b) : b \in \langle a_0 \rangle \cap \mathsf{dom}(\gamma)\}$.

Proof. By continuity of γ, $\gamma^{-1}[U]$ is an open set of the form $\bigcup_{i \in I} \langle a_i \rangle$, and a must belong to some $\langle a_i \rangle$ $(i \in I)$. ■

Recall that $\rho_{\boldsymbol{R}}$ is continuous (Lemma 3.7) and admits a spanning forest so that it belongs to **SpnRep**(**Coh**). We now prove:

Theorem 4.10 *The representation $\boldsymbol{R} \xrightarrow{\rho_{\boldsymbol{R}}} \mathbb{R}$ is admissible.*

Proof. Let $\mathbb{R}_0$ be a subspace of $\mathbb{R}$ and $\boldsymbol{X} \xrightarrow{\gamma} \mathbb{R}_0$ a continuous representation in **SpnRep**(**Coh**). Our goal is to find a stable map F which makes the following diagram commute:

$$\begin{tikzcd} \boldsymbol{X} \arrow[r, "F"] \arrow[d, "\gamma"'] & \boldsymbol{R} \arrow[d, "\rho_{\boldsymbol{R}}"] \\ \mathbb{R}_0 \arrow[r, "i"] & \mathbb{R} \end{tikzcd} .$$

A naive attempt would be to define:

$$F(a) \;=\; \{y \in \mathbb{D} : \gamma\langle a_0 \rangle \subseteq [y] \text{ for some finite } a_0 \subseteq a\}.$$

Although F is intuitively correct, it is not stable. First of all, $F(a)$ should not contain two distinct elements y, z with $y, z \in \mathbb{D}_n$, since they are not coherent while $F(a)$ must be a clique. Thus we have to "choose" one element $y \in \mathbb{D}_n$ for each $n \in \mathbb{N}$ if there are several candidates.

This is not enough, however. The condition (st) (see Section 2) forces that $F(a) \ni y$ implies the *unique* existence of a minimal $a_0 \subseteq a$ such that $F(a_0) \ni y$. It is here that a spanning forest $\mathcal{F}$ for γ plays a role. The idea is to look for such a minimal a_0 in $\mathcal{F}$. Since $\mathcal{F}$ is a forest, minimality ensures uniqueness. Moreover, restricting to the elements of $\mathcal{F}$ is not harmful, since any element of $\mathsf{dom}(\gamma)$ is approximated by a maximal chain in $\mathcal{F}$.

Let us now proceed to the formal definition. Let $\Psi : \mathcal{F} \times \mathbb{N} \rightrightarrows \mathbb{D}$ be a multifunction defined by

$$\Psi(a, n) := \{y \in \mathbb{D}_n : \gamma\langle a\rangle \subseteq [y]\}$$

for each $a \in \mathcal{F}$ and $n \in \mathbb{N}$, and let $\psi :\subseteq \mathcal{F} \times \mathbb{N} \longrightarrow \mathbb{D}$ be a partial function such that

- $\psi(a, n)$ is defined and $\psi(a, n) \in \Psi(a, n)$ just in case a is a minimal element in the set $\{b \in \mathcal{F} : \Psi(b, n) \neq \varnothing\}$.

Thus ψ is a choice function. Notice that $\psi(a, n) = y$ implies $\gamma\langle a\rangle \subseteq [y]$ and $\mathsf{den}(y) = n$. Finally we define

$$F(a) := \{y \in \mathbb{D} : \exists a_0 \subseteq a,\ \exists n \in \mathbb{N},\ \psi(a_0, n) = y\}.$$

This implements the ideas explained above. Let us now verify that F is the desired map in 4 steps.

(i) $F(a)$ is a clique of $\boldsymbol{R}$ for every $a \in \boldsymbol{X}$.
Let $y, z \in F(a)$ with $y \neq z$. This means that there are $a_0, a_1 \in \mathcal{F}$ and $n, m \in \mathbb{N}$ such that $\psi(a_0, n) = y$, $\psi(a_1, m) = z$ and $a_0, a_1 \subseteq a$. Since $a_0 \frown a_1$, either $a_0 \subseteq a_1$ or $a_1 \subseteq a_0$ hold. If $n = m$, then $a_0 = a_1$ by the minimality condition and thus $y = \psi(a_0, n) = \psi(a_1, m) = z$, contradicting the assumption $y \neq z$. Hence $\mathsf{den}(y) = n \neq m = \mathsf{den}(z)$.

We also have $[y] \cap [z] \neq \varnothing$. Indeed, there is $c \in \langle a_0\rangle \cap \langle a_1\rangle \cap \mathsf{dom}(\gamma)$ by the density condition (see Definition 4.1; note also $a_0 \subseteq a_1$ or $a_1 \subseteq a_0$), thus $\gamma(c) \in \gamma\langle a_0\rangle \cap \gamma\langle a_1\rangle \subseteq [y] \cap [z]$. Therefore, $y \frown z$.

(ii) F is a stable map.
It is sufficient to verify the condition (st): $F(a) \ni y$ implies the unique existence of a minimal $a_0 \subseteq a$ such that $F(a_0) \ni y$. But it is clear from the definition of F.

(iii) $F(a) \in \boldsymbol{R}_{\mathsf{max}}$ for every $a \in \mathsf{dom}(\gamma)$.
First recall that there is a chain $\{a_k\}_{k\in I}$ in $\mathcal{F}$ such that $a = \bigcup_{k\in I} a_k$. For maximality of $F(a)$, it suffices to show that for every $n \in \mathbb{N}$ there is $y \in F(a)$ with $\mathsf{den}(y) = n$.

Since $\gamma(a) \in \mathbb{R}$, there is $z \in \mathbb{D}_n$ such that $\gamma(a)$ belongs to the interior U of $[z]$. By Lemma 4.9, there is a finite subclique a' of a such that $\gamma\langle a'\rangle \subseteq U \subseteq [z]$. Without loss of generality, we may assume that $a' = a_k$ for some $k \in I$. This shows that $\Psi(a_k, n) \neq \varnothing$, which implies that $\psi(a_m, n) = y$ is defined for some $a_m \subseteq a_k$. Hence we obtain $y \in F(a)$ with $\mathsf{den}(y) = n$ as required.

(iv) $F(a)^* = \gamma(a)$ for every $a \in \mathsf{dom}(\gamma)$.

Suppose that $y \in F(a)$ with $\mathsf{den}(y) = n$, namely there is $a_0 \subseteq a$ such that $\psi(a_0, n) = y$. That is, $\gamma(a) \in \gamma\langle a_0\rangle \subseteq [y]$. Since it holds for every y in $F(a)$, we have $\gamma(a) \in [F(a)]$. Since $F(a) \in \boldsymbol{R}_{\mathsf{max}}$ by (iii), we conclude $F(a)^* = \gamma(a)$ by Lemma 3.5. ∎

As a consequence of Corollary 4.6, Proposition 4.8 and Theorem 4.10:

Corollary 4.11 *A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous if and only if it is realized by a stable map $F : \boldsymbol{R}^n \longrightarrow \boldsymbol{R}$.*

Hence we can express any continuous real function by a stable map. On the other hand, expressing real *operators* needs some extra work. We will address this issue in Section 6.

4.4 Traces of realizable functions

Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is realized by a stable map $F : \boldsymbol{R} \longrightarrow \boldsymbol{R}$. Then the trace $\mathsf{tr}(F) \in \boldsymbol{R} \Rightarrow \boldsymbol{R}$ must contain enough data to recover f. Hence it is interesting to look into its structure. We here prove two basic properties of the traces of realizers. Recall that each clique $\alpha \in \boldsymbol{R} \Rightarrow \boldsymbol{R}$ is a subset of $\boldsymbol{R}_{\mathsf{fin}} \times \mathbb{D}$. Given any $\alpha \in \boldsymbol{R} \Rightarrow \boldsymbol{R}$ and any $n \in \mathbb{N}$, we denote the subclique $\{(a, y) \in \alpha : y \in \mathbb{D}_n\}$ by $\alpha^{(n)}$.

Proposition 4.12 *If $(a_0, y) \in \mathsf{tr}(F)$, then $f[a_0] \subseteq [y]$.*

Proof. Suppose that a real number r belongs to the interval $[a_0]$. By Lemma 3.9, a_0 has an extension $a \in \boldsymbol{R}_{\mathsf{max}}$ such that $a^* = r$. We have $F(a) \ni y$, so $F(a)^* \in [y]$ by Lemma 3.4. Hence $f(r) = f(a^*) = F(a)^* \in [y]$. ∎

Proposition 4.13 *For every $a \in \boldsymbol{R}_{\mathsf{max}}$ and $n \in \mathbb{N}$, there is a unique $(a_0, y) \in \mathsf{tr}(F)^{(n)}$ such that $a_0 \subseteq a$.*

Proof. Since F is a realizer of f, we must have $F(a) \in \boldsymbol{R}_{\mathsf{max}}$. So $F(a) \ni y$ for some $y \in \mathbb{D}_n$, and we find $(a_0, y) \in \mathsf{tr}(F)^{(n)}$ such that $a_0 \subseteq a$. There is no other $(a_1, z) \in \mathsf{tr}(F)^{(n)}$ with the same property; if there were, we would have $a_0 \frown\!\!\!\!\smile a_1$ but not $y \frown z$, contradicting $\mathsf{tr}(F)$ being a clique. ∎

Figure 2 illustrates how the trace $\mathsf{tr}(F)$ approximates a function f: $(a_0, y) \in \mathsf{tr}(F)$ means that there is an approximating rectangle $[a_0] \times [y]$ such that $f[a_0] \subseteq [y]$. Such a rectangle is uniquely determined as soon as $a \in \boldsymbol{R}_{\mathsf{max}}$ and $n \in \mathbb{N}$ (corresponding to the height of the rectangle) are specified. Each $\mathsf{tr}(F)^{(n)}$ is then a collection of rectangles of height 2^{-n+1} which cover the graph of f.

5 Linear Maps

5.1 Linearity and uniform continuity

We now turn our attention to another aspect of coherence spaces: linearity. We have seen in the previous section that stable realizability coincides with

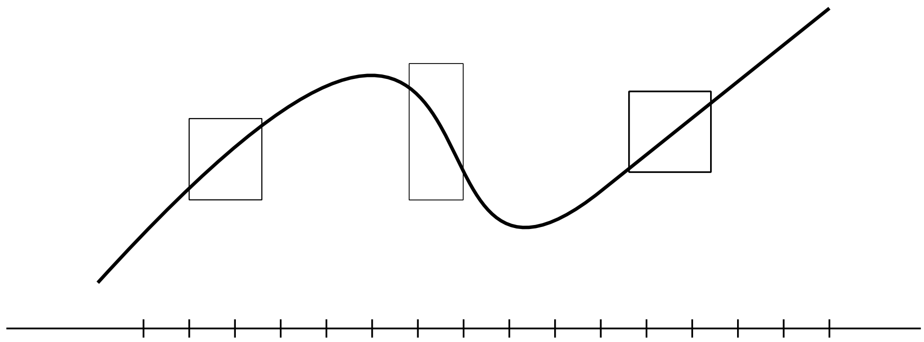

Figure 2: The trace of a stably realizable function

sequential continuity for any topological spaces with admissible representations. Although we do not yet have such a general result concerning linearity, we do have a curious observation if we restrict to real functions:

$$\text{linear realizability} \quad \Longleftrightarrow \quad \text{uniform continuity.}$$

This section is devoted to a proof of this fact. Let us begin with a formal definition.

Definition 5.1 (linear realizability) *Let $\boldsymbol{X} \xrightarrow{\rho_{\boldsymbol{X}}} \mathbb{X}$ and $\boldsymbol{Y} \xrightarrow{\rho_{\boldsymbol{Y}}} \mathbb{Y}$ be representations. A total function $f : \mathbb{X} \longrightarrow \mathbb{Y}$ is* linearly realizable *via $\rho_{\boldsymbol{X}}, \rho_{\boldsymbol{Y}}$ if it is realized by a linear map $F : \boldsymbol{X} \longrightarrow_{lin} \boldsymbol{Y}$ (see Definition 3.2).*

Recall that a function $g : \mathbb{R} \longrightarrow \mathbb{R}$ is uniformly continuous if and only if it has a *modulus of continuity*, that is a function $\mu : \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$|x - y| \leqslant 2^{-\mu(n)} \implies |g(x) - g(y)| \leqslant 2^{-n}$$

holds for any $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$.

Now let us fix a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is realized by a linear map F. Our first goal is to discover a modulus of continuity for f by looking into $\mathsf{tr}(F)$.

Recall that each clique $\alpha \in \boldsymbol{R} \multimap \boldsymbol{R}$ is a subset of $\mathbb{D} \times \mathbb{D}$. Given $\alpha \in \boldsymbol{R} \multimap \boldsymbol{R}$ and $n \in \mathbb{N}$, we denote the subclique $\{(x, y) \in \alpha : y \in \mathbb{D}_n\}$ by $\alpha^{(n)}$. As before, the trace $\mathsf{tr}(F)$ satisfies the following basic properties (see Propositions 4.12, 4.13).

Proposition 5.2 *If $(x, y) \in \mathsf{tr}(F)$, then $f[x] \subseteq [y]$.*

Proposition 5.3 *For every $a \in \boldsymbol{R}_{\mathsf{max}}$ and $n \in \mathbb{N}$, there is a unique $(x, y) \in \mathsf{tr}(F)^{(n)}$ such that $x \in a$.*

Proposition 5.3 can be understood interactively. Think of a as a Cauchy sequence and n as a degree of precision. Given a and n as inputs, F is expected to return an approximate value of $f(a^*)$ with precision 2^{-n}. As F is linear, it "sees" exactly one element x in the Cauchy sequence a and returns a dyadic number y. That is $(x, y) \in \mathsf{tr}(F)^{(n)}$.

Since F is not allowed to "see" any other element in a, the way F picks up an appropriate element x from a must be independent of the internal structure of a. That is, the number $\mathsf{den}(x)$ must be determined only by n. This is the main reason why linearity leads to uniformity. The next lemma ensures that this intuition is correct.

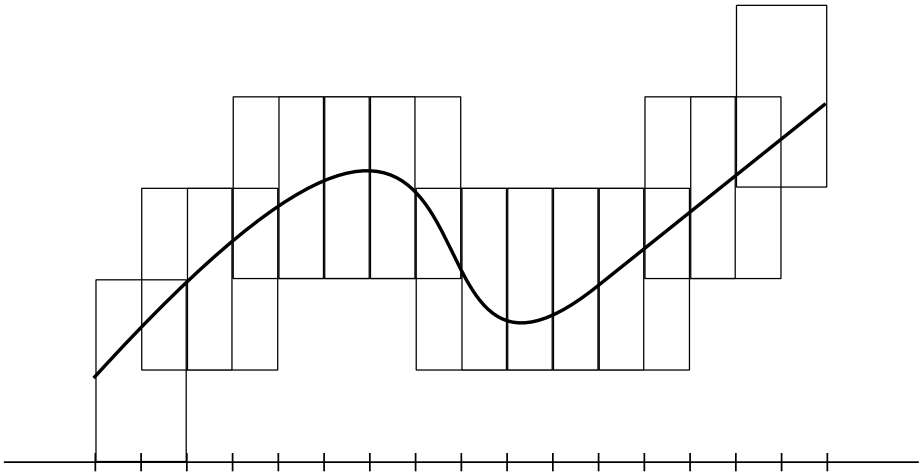

Figure 3: The trace of a linearly realizable function

Lemma 5.4 *There is a function $\mu : \mathbb{N} \longrightarrow \mathbb{N}$ such that for every $x \in \mathbb{D}_{\mu(n)}$ there is $y \in \mathbb{D}_n$ such that $(x, y) \in \mathsf{tr}(F)$.*

Proof. Let $n \in \mathbb{N}$. Fix an arbitrary $(x_\star, y_\star) \in \mathsf{tr}(F)^{(n)}$ which exists by Proposition 5.3. Let $m := \mathsf{den}(x_\star)$. We claim that

- for $x := x_\star + 2^{-m} \in \mathbb{D}_m$, there is $y \in \mathbb{D}_n$ such that $(x, y) \in \mathsf{tr}(F)^{(n)}$.

To see this, choose a maximal clique $a = \{z_i\}_{i\in\mathbb{N}}$ with $z_m = x_\star$ and $a^* = x_\star$. By Proposition 5.3, $(x_\star, y_\star) \in \mathsf{tr}(F)^{(n)}$ is the unique element such that $x_\star \in a$. Since $a^* \in [x]$, we may apply Lemma 3.8 to the cliques a and $b = \{x\}$, to obtain another clique c in which $z_m = x_\star$ is replaced by $\check{z}_m = x$. That is, we have:

$$\begin{array}{rcl} a & = & \{z_0, z_1, \ldots, z_m, z_{m+1}, \ldots\}, \\ c & = & \{z_0, z_1, \ldots, \check{z}_m, z_{m+1}, \ldots\}. \end{array}$$

By Proposition 5.3, there is $(z, y) \in \mathsf{tr}(F)^{(n)}$ such that $z \in c$. We have $z = \check{z}_m$, since otherwise $z \in a$ contradicting uniqueness of $(x_\star, y_\star) \in \mathsf{tr}(F)^{(n)}$. This proves the claim.

The same reasoning works for $x := x - 2^{-m}$ too. Hence by repetition we obtain $(x, y) \in \mathsf{tr}(F)^{(n)}$ for every $x \in \mathbb{D}_m$. Now the lemma follows by letting $\mu(n) := m$. ■

Figure 3 illustrates $\mathsf{tr}(F)^{(n)}$. Observe that there is a rectangle $[x] \times [y]$ for each $x \in \mathbb{D}_{\mu(n)}$. The graph of f is then covered by rectangles of "uniform size".

We are now ready to prove the main theorem of this section.

Theorem 5.5 *A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is linearly realizable if and only if it is uniformly continuous.*

Proof. ($\Rightarrow$) Suppose that f is realized by a linear map F as above. Then we claim that $\mu'(n) := \mu(n + 1)$ is a modulus of f. To see this, take $r, s \in \mathbb{R}$ such that $|r - s| \leqslant 2^{-\mu(n+1)}$. Then there is $x \in \mathbb{D}_{\mu(n+1)}$ such that $r, s \in [x]$. By the previous lemma, there is $y \in \mathbb{D}_{n+1}$ such that $(x, y) \in \mathsf{tr}(F)$. By Proposition 5.2 we obtain $f(r), f(s) \in [y]$. Therefore,

$$|f(r) - f(s)| \leqslant |f(r) - y| + |y - f(s)| \leqslant 2^{-(n+1)} + 2^{-(n+1)} = 2^{-n}.$$

($\Leftarrow$) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous. Then there is a function $\mu : \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$|r - s| \leqslant 2^{-\mu(n)+1} \quad \Longrightarrow \quad |f(r) - f(s)| \leqslant 2^{-n}.$$

We define a linear map $F : \boldsymbol{R} \longrightarrow_{lin} \boldsymbol{R}$ as follows. Let $\Psi : \mathbb{D} \times \mathbb{N} \rightrightarrows \mathbb{D}$ be a multifunction defined by

$$\Psi(x, n) := \{y \in \mathbb{D}_n : f[x] \subseteq [y]\}$$

for each $n \in \mathbb{N}$ and $x \in \mathbb{D}_{\mu(n)}$ (otherwise $\Psi(x, n) := \varnothing$), and let $\psi :\subseteq \mathbb{D} \times \mathbb{N} \longrightarrow \mathbb{D}$ be a partial function such that

- $\psi(x, n)$ is defined and $\psi(x, n) \in \Psi(x, n)$ just in case $\Psi(x, n) \neq \varnothing$.

Notice that $\psi(x, n) = y$ implies $f[x] \subseteq [y]$, $\mathsf{den}(x) = \mu(n)$ and $\mathsf{den}(y) = n$. Finally we define

$$F(a) := \{y \in \mathbb{D} : \exists x \in a,\ \exists n \in \mathbb{N},\ \psi(x, n) = y\}.$$

As in the proof of Theorem 4.10, the verification consists of the following 4 steps.

(i) $F(a)$ is a clique of $\boldsymbol{R}$ for every $a \in \boldsymbol{R}$.
Let $y, z \in F(a)$ with $y \neq z$. This means that there are $n, m \in \mathbb{N}$, $x \in \mathbb{D}_{\mu(n)}$ and $w \in \mathbb{D}_{\mu(m)}$ such that $\psi(x, n) = y$, $\psi(w, m) = z$ and $x, w \in a$ (so $x \frown w$). If $n = m$, then $\mathsf{den}(x) = \mathsf{den}(w) (= \mu(n))$ so that $x = w$ and $y = \psi(x, n) = \psi(w, m) = z$, contradicting $y \neq z$. Hence $\mathsf{den}(y) = n \neq m = \mathsf{den}(z)$.

We also have $[y] \cap [z] \neq \varnothing$, since $[x] \cap [w] \neq \varnothing$, $f[x] \subseteq [y]$ and $f[w] \subseteq [z]$. Therefore, $y \frown z$.

(ii) F is a linear map.
It is sufficient to verify the condition (lin) (see Section 2): $F(a) \ni y$ implies the unique existence of $x \in \mathbb{D}$ such that $F(\{x\}) \ni y$. Notice that $y \in \mathbb{D}$ determines the number $n := \mathsf{den}(y)$ uniquely. Hence there is a unique x such that $\psi(x, n) = y$ and $x \in a$.

(iii) $F(a) \in \boldsymbol{R}_{\mathsf{max}}$ for every $a = \{x_n\}_{n \in \mathbb{N}} \in \boldsymbol{R}_{\mathsf{max}}$.
It suffices to show that for every $n \in \mathbb{N}$ there is $y \in F(a)$ with $\mathsf{den}(y) = n$. By the definition of μ, $f[x_{\mu(n)}]$ is contained in an interval of length 2^{-n}, so there is $z \in \mathbb{D}_n$ such that $f[x_{\mu(n)}] \subseteq [z]$. Hence $\Psi(x_{\mu(n)}, n)$ is nonempty so that $y := \psi(x_{\mu(n)}, n)$ is defined. This y satisfies $y \in F(a)$ and $\mathsf{den}(y) = n$.

(iv) $F(a)^* = f(a^*)$ for every $a \in \boldsymbol{R}_{\mathsf{max}}$.
Suppose that $y \in F(a)$ with $\mathsf{den}(y) = n$, namely there is $x \in a$ such that $\psi(x, n) = y$. By Lemma 3.4, we have $f(a^*) \in f[x] \subseteq [y]$. Since it holds for every y in $F(a)$, we have $f(a^*) \in [F(a)]$. Since $F(a) \in \boldsymbol{R}_{\mathsf{max}}$ by (iii), we conclude

$F(a)^* = f(a^*)$ by Lemma 3.5. ■

Observe that the above proof rests on our specific way of representing the real line. In other words, linear realizability of a function does depend on the choice of an admissible representation. This motivates us to develop a theory of *linear admissible representations*, which is a subject of our companion paper [Ma16].

6 Admissible Representation of Function Spaces

6.1 Representing $\mathcal{C}(\mathbb{I}, \mathbb{R})$

Admissible representations studied in Section 4 can be extended to function spaces. Our ultimate goal is to show that $\rho_{\boldsymbol{RR}} := [\rho_{\boldsymbol{R}} \to \rho_{\boldsymbol{R}}]$ (see Section 3) is an admissible representation of the space $\mathcal{C}(\mathbb{R}, \mathbb{R})$ of continuous functions on the real line equipped with the compact-open topology. However, it is delicate since the domain $\mathbb{R}$ is not compact. Hence we begin with a more tractable case: $\mathcal{C}(\mathbb{I}, \mathbb{R})$, where $\mathbb{I}$ is a compact interval of $\mathbb{R}$.

Let $\mathbb{ID} := \mathbb{D} \cap \mathbb{I}$ and $\mathbb{ID}_n := \mathbb{D}_n \cap \mathbb{I}$. The latter is a finite set due to compactness. We define a coherence space $\boldsymbol{I} := (\mathbb{ID}, \frown)$ by restricting the token set $\mathbb{D}$ of $\boldsymbol{R} = (\mathbb{D}, \frown)$ to $\mathbb{ID}$. Function $\rho_{\boldsymbol{I}} : \boldsymbol{I}_{\mathsf{max}} \longrightarrow \mathbb{I}$ is defined just as $\rho_{\boldsymbol{R}}$. It is clear that $\boldsymbol{I} \xrightarrow{\rho_{\boldsymbol{I}}} \mathbb{I}$ is an admissible representation.

Lemma 6.1 $\boldsymbol{I}_{\mathsf{max}}$ *is a compact subset of* $\boldsymbol{I}$.

Proof. Let A be a set of finite cliques in $\boldsymbol{I}$ such that $\{\langle a \rangle : a \in A\}$ is an open cover of $\boldsymbol{I}_{\mathsf{max}}$. We may assume that each $a \in A$ is an initial segment of a Cauchy sequence (see §4.1): if $a \cap \mathbb{ID}_n = \varnothing$, then a can be replaced by finitely many cliques $a \cup \{x_1\}, \dots, a \cup \{x_k\}$, where $x_1, \dots, x_k \in \mathbb{ID}_n$, so that $\langle a \rangle = \bigcup \langle a \cup \{x_i\} \rangle$.

Let A_0 be the set of minimal elements of A (with respect to inclusion $\subseteq$). Then $\{\langle a \rangle : a \in A_0\}$ is still an open cover of $\boldsymbol{I}_{\mathsf{max}}$, since $a \subseteq b$ implies $\langle b \rangle \subseteq \langle a \rangle$ for all $a, b \in \boldsymbol{I}$. Our goal is to show that A_0 is finite.

Suppose that A_0 is infinite. Define the set $\mathcal{F} \subseteq \boldsymbol{I}_{\mathsf{fin}}$ by:

$$b \in \mathcal{F} \iff b \text{ is an initial segment of some } a \in A_0$$

Then $(\mathcal{F}, \subseteq)$ is an infinite tree with root $\varnothing$ and leaves in A_0. Moreover, it is finitely branching at the root and each internal node, since each $\mathbb{ID}_n$ is finite. Hence it contains an infinite branch $(b_i)_{i \in \mathbb{N}}$ by König's lemma. Clearly $b = \bigcup b_i$ belongs to $\boldsymbol{I}_{\mathsf{max}}$, hence to some member $\langle a \rangle$ $(a \in A_0)$ of the open covering.

However, this means that $a \subsetneq b_i$ for a large enough i, and $b_i \subseteq a'$ for some $a' \in A_0$ by the definition of $\mathcal{F}$. That is, $a \subsetneq a'$, contradicting minimality of a' in A_0. ■

We are ready to introduce a representation of $\mathcal{C}(\mathbb{I}, \mathbb{R})$. Recall that the category **Rep(Coh)** is cartesian closed, so that we have an exponential representation $\boldsymbol{I} \Rightarrow \boldsymbol{R} \xrightarrow{\rho_{\boldsymbol{IR}}} \mathcal{SR}(\rho_{\boldsymbol{I}}, \rho_{\boldsymbol{R}})$, where $\rho_{\boldsymbol{IR}} := [\rho_{\boldsymbol{I}} \to \rho_{\boldsymbol{R}}]$ and $\mathcal{SR}(\rho_{\boldsymbol{I}}, \rho_{\boldsymbol{R}})$ consists of stably realizable functions from $\mathbb{I}$ to $\mathbb{R}$ with respect to $\rho_{\boldsymbol{I}}$ and $\rho_{\boldsymbol{R}}$. Since the latter are both admissible, a function is realizable if and only if it is continuous (cf. Corollary 4.11). Hence $\mathcal{SR}(\rho_{\boldsymbol{I}}, \rho_{\boldsymbol{R}}) = \mathcal{C}(\mathbb{I}, \mathbb{R})$, and we have thus obtained a representation

$$\boldsymbol{I} \Rightarrow \boldsymbol{R} \xrightarrow{\rho_{\boldsymbol{IR}}} \mathcal{C}(\mathbb{I}, \mathbb{R}).$$

We assume that $\mathcal{C}(\mathbb{I}, \mathbb{R})$ is equipped with the *uniform topology* induced by the uniform norm $\|f\|_\infty := \max_{x \in \mathbb{I}} |f(x)|$ ($f \in \mathcal{C}(\mathbb{I}, \mathbb{R})$). We will now verify that $\rho_{\boldsymbol{IR}}$ is continuous with respect to this topology.

Lemma 6.2 *Suppose that $\alpha \in \mathsf{dom}(\rho_{\boldsymbol{IR}})$. Then $\alpha^{(n)}$ is a finite clique for every $n \in \mathbb{N}$.*

Proof. Suppose that $\alpha^{(n)} = \{(a_j, y_j) : j \in J\}$. Notice that $\alpha = \mathsf{tr}(F)$ for some realizer F of a function in $\mathcal{C}(\mathbb{I}, \mathbb{R})$. Hence by Proposition 4.13, $\{\langle a_j \rangle : j \in J\}$ is a *disjoint cover* of $\mathsf{dom}(\rho_{\boldsymbol{I}})$. Namely, for every $a \in \mathsf{dom}(\rho_{\boldsymbol{I}})$ there exists a unique $j \in J$ with $a \in \langle a_j \rangle$.

On the other hand, $\mathsf{dom}(\rho_{\boldsymbol{I}})$ is compact in $\boldsymbol{I}$ by Lemma 6.1. Since $\{\langle a_j \rangle : j \in J\}$ is a disjoint open cover of a compact set, J has to be finite by itself. ■

Lemma 6.3 $\rho_{\boldsymbol{IR}} :\subseteq \boldsymbol{I} \Rightarrow \boldsymbol{R} \longrightarrow \mathcal{C}(\mathbb{I}, \mathbb{R})$ *is continuous.*

Proof. Let $\alpha \in \mathsf{dom}(\rho_{\boldsymbol{IR}})$ and $n \in \mathbb{N}$. It is sufficient to give an open set U in $\boldsymbol{I} \Rightarrow \boldsymbol{R}$ such that

$$\beta \in U \cap \mathsf{dom}(\rho_{\boldsymbol{IR}}) \quad \Longrightarrow \quad \|\rho_{\boldsymbol{IR}}(\alpha) - \rho_{\boldsymbol{IR}}(\beta)\|_\infty \leqslant 2^{-n}.$$

Our choice is $U := \langle \alpha^{(n+1)} \rangle$. It is legitimate since $\alpha^{(n+1)} \in (\boldsymbol{I} \Rightarrow \boldsymbol{R})_{\mathsf{fin}}$ by the previous lemma.

Suppose that $\beta \in U \cap \mathsf{dom}(\rho_{\boldsymbol{IR}})$ and let $f := \rho_{\boldsymbol{IR}}(\alpha)$, $g := \rho_{\boldsymbol{IR}}(\beta)$. Given $r \in \mathbb{I}$, there is $a \in \mathsf{dom}(\rho_{\boldsymbol{I}})$ such that $a^* = r$. By Proposition 4.13, there is $(a_0, y) \in \alpha^{(n+1)}$ such that $a_0 \subseteq a$. We also have $(a_0, y) \in \beta$ since $\alpha^{(n+1)} \subseteq \beta$. Hence $y \in \widehat{\alpha}(a) \cap \widehat{\beta}(a)$. By Lemma 3.4, we obtain $f(r) = (\widehat{\alpha}(a))^* \in [y]$ and $g(r) = (\widehat{\beta}(a))^* \in [y]$. Since $[y]$ is of length 2^{-n}, $|f(r) - g(r)| \leqslant 2^{-n}$. Since this holds for every $r \in \mathbb{I}$, we conclude that $\|f - g\|_\infty \leqslant 2^{-n}$. ■

6.2 Admissibility of $\rho_{\boldsymbol{IR}}$

Our next goal is to show that $\boldsymbol{I} \Rightarrow \boldsymbol{R} \xrightarrow{\rho_{\boldsymbol{IR}}} \mathcal{C}(\mathbb{I}, \mathbb{R})$ belongs to **SpnRep**(**Coh**).

Lemma 6.4 *Let $\alpha, \beta \in \mathsf{dom}(\rho_{\boldsymbol{IR}})$ and $n \in \mathbb{N}$. If $\alpha^{(n)} \frown \beta^{(n)}$, then $\alpha^{(n)} = \beta^{(n)}$.*

Proof. Suppose that $\alpha^{(n)} \frown \beta^{(n)}$ and $(b_0, z) \in \beta^{(n)}$. We claim that $(b_0, z) \in \alpha^{(n)}$ so that we have $\beta^{(n)} \subseteq \alpha^{(n)}$ and $\alpha^{(n)} \subseteq \beta^{(n)}$ by a symmetric reasoning.

Choose a clique $b \in \langle b_0 \rangle \cap \mathsf{dom}(\rho_{\boldsymbol{I}})$. Then by Proposition 4.13, there is a unique $(a_0, y) \in \alpha^{(n)}$ such that $a_0 \subseteq b$. Since $(a_0, y) \frown (b_0, z)$, $a_0 \frown b_0$ and $\neg(z \smile y)$, we conclude $(b_0, z) = (a_0, y) \in \alpha^{(n)}$. ■

This allows us to define

$$\mathcal{F} := \{\alpha^{\leqslant n} : \alpha \in \mathsf{dom}(\rho_{\boldsymbol{IR}}) \text{ and } n \in \mathbb{N}\},$$

where $\alpha^{\leqslant n} := \bigcup_{i \leqslant n} \alpha^{(i)}$. $\mathcal{F}$ is indeed a spanning forest for $\mathsf{dom}(\rho_{\boldsymbol{IR}})$. By the previous lemma, if $\alpha^{\leqslant m}, \beta^{\leqslant n} \in \mathcal{F}$ and $\alpha^{\leqslant m} \frown \beta^{\leqslant n}$, then $\alpha^{(i)} = \beta^{(i)}$ for all

$i \leqslant \min(m, n)$, so either $\alpha^{\leqslant m} \subseteq \beta^{\leqslant n}$ or $\beta^{\leqslant n} \subseteq \alpha^{\leqslant m}$ depending on whether $m \leqslant n$ or $n \leqslant m$. Hence the representation $\rho_{\boldsymbol{IR}}$ belongs to **SpnRep**(**Coh**).

Before we conclude the admissibility of $\rho_{\boldsymbol{IR}}$ let us recall a fact from general topology that the evaluation map $\mathsf{ev} : \mathcal{C}(\mathbb{I}, \mathbb{R}) \times \mathbb{I} \longrightarrow \mathbb{R}$ is continuous because of the compactness of $\mathbb{I}$.

Theorem 6.5 *The representation* $\boldsymbol{I} \Rightarrow \boldsymbol{R} \xrightarrow{\rho_{\boldsymbol{IR}}} \mathcal{C}(\mathbb{I}, \mathbb{R})$ *is admissible.*

Proof. Let $\mathbb{Z}$ be a subspace of $\mathcal{C}(\mathbb{I}, \mathbb{R})$ and $\boldsymbol{Z} \xrightarrow{\gamma} \mathbb{Z}$ be a continuous representation in **SpnRep**(**Coh**). Our goal is to find a stable map $F : \boldsymbol{Z} \longrightarrow_{st} \boldsymbol{I} \Rightarrow \boldsymbol{R}$ which realizes the inclusion map $i : \mathbb{Z} \longrightarrow \mathcal{C}(\mathbb{I}, \mathbb{R})$:

$$\begin{tikzcd}
\boldsymbol{Z} \arrow[r, dotted, "F"] \arrow[d, "\gamma"'] & \boldsymbol{I} \Rightarrow \boldsymbol{R} \arrow[d, "\rho_{\boldsymbol{IR}}"] \\
\mathbb{Z} \arrow[r, "i"] & \mathcal{C}(\mathbb{I}, \mathbb{R})
\end{tikzcd}$$

Let $\mathbb{R}_0 := \{f(r) : f \in \mathbb{Z}, r \in \mathbb{I}\}$ and define $\delta :\subseteq \boldsymbol{Z} \times \boldsymbol{I} \longrightarrow \mathbb{R}_0$ by

$$\delta(\alpha, a) := \mathsf{ev}(\gamma(\alpha), \rho_{\boldsymbol{I}}(a))$$

for every $\alpha \in \mathsf{dom}(\gamma)$ and $a \in \mathsf{dom}(\rho_{\boldsymbol{I}})$. This map is surjective, and moreover continuous because ev is. Furthermore, a spanning forest for δ can be given as in the proof of Lemma 4.3. Hence $\boldsymbol{Z} \times \boldsymbol{I} \xrightarrow{\delta} \mathbb{R}_0$ belongs to **SpnRep**(**Coh**).

Now by admissibility of $\rho_{\boldsymbol{R}}$ we obtain a stable map G such that:

$$\begin{tikzcd}
\boldsymbol{Z} \times \boldsymbol{I} \arrow[r, "G"] \arrow[d, "\delta"'] & \boldsymbol{R} \arrow[d, "\rho_{\boldsymbol{R}}"] \\
\mathbb{R}_0 \arrow[r, "i"] & \mathbb{R}
\end{tikzcd}$$

By Currying G, we obtain a desired map $F : \boldsymbol{Z} \longrightarrow_{st} \boldsymbol{I} \Rightarrow \boldsymbol{R}$. ■

6.3 Representing $\mathcal{C}(\mathbb{R}, \mathbb{R})$

Let us now proceed to the full function space $\mathcal{C}(\mathbb{R}, \mathbb{R})$. As before, the exponential $\rho_{\boldsymbol{RR}} = [\rho_{\boldsymbol{R}} \to \rho_{\boldsymbol{R}}]$ forms a representation $\boldsymbol{R} \Rightarrow \boldsymbol{R} \xrightarrow{\rho_{\boldsymbol{RR}}} \mathcal{C}(\mathbb{R}, \mathbb{R})$.

Since $\mathbb{R}$ is not compact, the uniform topology is not suitable. Instead, we equip $\mathcal{C}(\mathbb{R}, \mathbb{R})$ with the *compact-open topology*, which is generated by the collection of function sets $T(K, U)$ as subbasis, where

$$T(K, U) := \{f \in C(\mathbb{R}, \mathbb{R}) : f[K] \subseteq U\},$$

for each compact $K \subseteq \mathbb{R}$ and open $U \subseteq \mathbb{R}$ (where $f[K]$ is the image of K under f). We now prove that $\rho_{\boldsymbol{RR}}$ is continuous with respect to the compact-open topology.

Lemma 6.6 *Suppose that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is realized by a stable map $F : \boldsymbol{R} \longrightarrow_{st} \boldsymbol{R}$. Then for any compact $K \subseteq \mathbb{R}$ and open $U \subseteq \mathbb{R}$, the function f belongs to $T(K, U)$ if and only if there are $(a_1, y_1), \dots, (a_k, y_k) \in \mathsf{tr}(F)$ such that*

$$K \subseteq [a_1] \cup \cdots \cup [a_k] \quad \textit{and} \quad [y_1] \cup \cdots \cup [y_k] \subseteq U. \tag{4}$$

Proof. ($\Leftarrow$) Our goal is to show $f(r) \in U$ for any $r \in K$. By the first condition, $r \in [a_i]$ for some $i \leqslant k$. Combining Proposition 4.12 with the second condition, we get

$$f(r) \in f[a_i] \subseteq [y_i] \subseteq U.$$

($\Rightarrow$) We claim that for each $r \in K$, there is $(a_r, y_r) \in \mathsf{tr}(F)$ such that r belongs to the interior of $[a_r]$ and $[y_r] \subseteq U$. Since the interiors of $\{[a_r]\}_{r \in K}$ cover K, compactness of K gives us finitely many $(a_{r_1}, y_{r_1}), \dots, (a_{r_k}, y_{r_k})$ satisfying the requirement (4).

Let us prove the claim. It is easy to see that for any $r \in K$, there is $a \in \rho_{\boldsymbol{R}}^{-1}(r)$ such that r belongs to the interior of $[x]$ for any member $x \in a$ (see Figure 1). Let $b := F(a) \in \boldsymbol{R}_{\mathsf{max}}$. Since f belongs to $T(K, U)$, we have $b^* = F(a)^* = f(r) \in U$. By Lemma 3.6, there exists $y_r \in b$ such that $[y_r] \subseteq U$. Since $F(a) \ni y_r$, there is a finite subclique $a_r \subseteq a$ such that $(a_r, y_r) \in \mathsf{tr}(F)$. Clearly r belongs to the interior of $[a_r] = \bigcap_{x \in a_r} [x]$. ■

Lemma 6.7 *The representation $\boldsymbol{R} \Rightarrow \boldsymbol{R} \overset{\rho_{\boldsymbol{RR}}}{\longrightarrow} \mathcal{C}(\mathbb{R}, \mathbb{R})$ is continuous.*

Proof. Let $K \subseteq \mathbb{R}$ be compact and $U \subseteq \mathbb{R}$ open. Then for every $f : \mathbb{R} \longrightarrow \mathbb{R}$ realized by a stable map F, $f \in T(K, U)$ if and only if there is a finite subclique $\alpha = \{(a_1, y_1), \dots, (a_k, y_k)\}$ of $\mathsf{tr}(F)$ satisfying the property (4). We denote the set of all such finite subcliques of $\mathsf{tr}(F)$ by $M(K, U)$. We then have:

$$\begin{array}{rcl} F \text{ realizes some } f \in T(K,U) & \iff & \mathsf{tr}(F) \text{ includes some } \alpha \in M(K,U) \\ & \iff & \mathsf{tr}(F) \in \displaystyle\bigcup_{\alpha \in M(K,U)} \langle \alpha \rangle. \end{array}$$

Recall that F realizes f iff $\mathsf{tr}(F) \in \rho_{\boldsymbol{RR}}^{-1}(f)$. Hence we obtain $\rho_{\boldsymbol{RR}}^{-1}[T(K, U)] = \bigcup_{\alpha \in M(K,U)} \langle \alpha \rangle \cap \mathsf{dom}(\rho_{\boldsymbol{RR}})$. This proves the continuity of $\rho_{\boldsymbol{RR}}$. ■

6.4 Admissibility of $\rho_{\boldsymbol{RR}}$

We have seen that the representation $\boldsymbol{R} \Rightarrow \boldsymbol{R} \overset{\rho_{\boldsymbol{RR}}}{\longrightarrow} \mathcal{C}(\mathbb{R}, \mathbb{R})$ is continuous. We now show that it belongs to **SpnRep(Coh)**, by exhibiting a spanning forest for $\rho_{\boldsymbol{RR}}$.

We consider a monotone sequence of compact intervals

$$\mathbb{I}_0 \subset \mathbb{I}_1 \subset \mathbb{I}_2 \subset \cdots$$

whose union is $\mathbb{R}$. The corresponding coherence spaces are denoted by $\boldsymbol{I}_0$, $\boldsymbol{I}_1$, $\boldsymbol{I}_2$, ... Notice that $\boldsymbol{I}_0 \subset \boldsymbol{I}_1 \subset \boldsymbol{I}_2 \subset \cdots \subset \boldsymbol{R}$ as the sets of cliques. Given a clique $\alpha \in \mathsf{dom}(\rho_{\boldsymbol{RR}})$, we decompose it into

$$\alpha^{(m,n)} := \{(a, y) \in \alpha : a \in (\boldsymbol{I}_m)_{\mathsf{fin}},\ y \in \mathbb{D}_n\},$$

so that $\alpha = \bigcup_{m,n \in \mathbb{N}} \alpha^{(m,n)}$. Then each $\alpha^{(m,n)}$ is finite by Lemma 6.2 and subject to Lemma 6.4. Let $\mathcal{F}$ be the set of cliques of the form $\bigcup_{i \leqslant m} \alpha^{(m,i)}$ for any $\alpha \in \mathsf{dom}(\rho_{\boldsymbol{RR}})$ and $m \in \mathbb{N}$. If $m \leqslant m'$, then $\alpha^{(m,n)} \frown \beta^{(m',n)}$ implies $\alpha^{(m,n)} \subseteq \beta^{(m',n)}$, since $\beta^{(m,n)} \subseteq \beta^{(m',n)}$ and $\alpha^{(m,n)} \frown \beta^{(m,n)}$ implies $\alpha^{(m,n)} = \beta^{(m,n)}$. Then it is not hard to see that $\mathcal{F}$ is a spanning forest for $\rho_{\boldsymbol{RR}}$.

Finally, the representation $\rho_{\boldsymbol{RR}}$ is admissible. The proof is exactly the same as that of Theorem 6.5, by noting that the evaluation map $\mathsf{ev} : \mathcal{C}(\mathbb{R}, \mathbb{R}) \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous when $\mathcal{C}(\mathbb{R}, \mathbb{R})$ is endowed with the compact-open topology (see, e.g., [Br93]). We therefore conclude:

Theorem 6.8 *The representation $\boldsymbol{R} \Rightarrow \boldsymbol{R} \xrightarrow{\rho_{\boldsymbol{RR}}} \mathcal{C}(\mathbb{R}, \mathbb{R})$ is admissible.*

As a consequence:

Corollary 6.9 *A real operator $\Phi : \mathcal{C}(\mathbb{R}, \mathbb{R}) \longrightarrow \mathcal{C}(\mathbb{R}, \mathbb{R})$ is sequentially continuous if and only if it is stably realizable.*

This allows us to represent various real operators as stable maps. An example is the integral operator

$$f \ \mapsto \ \lambda x. \int_0^x f(y) dy.$$

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