MacNeille completion and Buchholz' Omega rule for parameter-free second order logics

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Abstract

Buchholz' Ω-*rule* is a way to give a syntactic, possibly ordinal-free, proof of cut elimination for various impredicative systems of arithmetic. Our goal is to understand it from an algebraic point of view. Among many proofs of cut elimination for higher order logics, Maehara and Okada's algebraic proofs are of particular interest, since the essence of their arguments can be algebraically described as the (*Dedekind-*)*MacNeille* completion together with Girard's reducibility candidates. Interestingly, it turns out that the Ω-rule, formulated as a rule of logical inference, finds its algebraic foundation in the MacNeille completion. This observation naturally leads to an algebraic form of the Ω-rule that we call the Ω-*interpretation*, that partly appears in (Altenkirch-Coquand 2001). In this paper, we introduce sequent calculi for the parameter-free fragments of second order intuitionistic logic, and explain how use of reducibility candidates in (Maehara 1991) and (Okada 1996) can be avoided by means of the Ω-interpretation. It results in an algebraic proof of cut elimination formalizable in theories of finitely iterated inductive definitions, that can be compared with a result by (Aehlig 2005).

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1 Introduction

This paper concerns with cut elimination for subsystems of second order logics. It is of course very well known that the full second order classical/intuitionistic logics admit cut elimination. Then why are we interested in their subsystems? A primary reason is that proving cut elimination for a subsystem is often very hard if one is sensitive to the metatheory within which (s)he works. It is witnessed by the vast literature in the traditional proof theory. In fact, proof theorists are not just interested in proving cut elimination itself, but in identifying a *characteristic principle* P (e.g. ordinals, ordinal diagrams, combinatorial principles and inductive definitions) for each system of arithmetic and set theory, by proving cut elimination within a fairy weak metatheory (e.g. $\mathbf{PRA}$, $\mathbf{I\Sigma}_1$ and $\mathbf{RCA}_0$) extended by P. Our motivation is to understand those hard proofs and results from an algebraic perspective.

One can distinguish several types of cut elimination proofs for higher order logics/arithmetic: (i) syntactic proofs by ordinal assignment (e.g. Gentzen's consistency proof for $\mathbf{PA}$), (ii) syntactic but ordinal-free proofs, (iii) semantic proofs based on Schütte's semivaluation and its variants (e.g. [30]), (iv) algebraic proofs based on completions (the list is not intended to be exhaustive). Historically (i) and (iii) precede (ii) and (iv), but understanding (i) takes years just to catch up with the expanding universe of ordinal notations, while (iii) is slightly unsatisfactory for the truly constructive logician since it usually involves *reductio ad absurdum* and weak König's lemma. Hence we address (ii) and (iv) in this paper.

For (ii), a very useful and versatile technique is Buchholz' Ω-*rule*. Introduced in the context of ordinal analysis of $\mathbf{ID}$-theories [11] and further developed in, e.g., [14], it has later yielded an ordinal-free proof of cut elimination for fragments/extensions of Π^1_1-$\mathbf{CA}_0$ [12, 4, 3]. However, the Ω-rule is notoriously complicated, and is hard to grasp its meaning at a glance. Even the semantic soundness is not clear at all. While Buchholz gives an account based on the BHK interpretation [11], we will try to give an algebraic account in this paper.

For (iv), there is a very conspicuous algebraic proof of cut elimination for higher order logics which may be primarily ascribed to Maehara [24] and Okada [26, 28]. In contrast to (iii), these algebraic proofs are fully constructive; no use of reductio ad absurdum or any nondeterministic principle. More importantly, it extends to proofs of normalization for proof nets and typed lambda calculi [27]. While their arguments can be described in various dialects (e.g. phase semantics in linear logic), apparently most neutral and most widely accepted would be to speak in terms of algebraic completions: the essence of their arguments can be described as the (*Dedekind-*)*MacNeille completion* together with Girard's reducibility candidates, as we will explain in Section 6.

Having a syntactic technique at one hand and an algebraic methodology at the other, it is natural to ask the relationship between them. To make things concrete, we consider, in addition to the standard sequent calculus **LI2** for second order intuitionistic logic, a family of subcalculi $\mathbf{LIP} = \bigcup_{n\geq -1} \mathbf{LIP}_n$ for the *parameter-free* fragments. **LIP** is the intuitionistic counterpart of the classical sequent calculus studied in [32]. Although we primarily work on intuitionistic logic, all results in this paper (except Proposition 11) carry over to classical logic too. The parameter-free calculi provide a common ground for comparison in which one can talk about the algebraic MacNeille completion and the syntactic Ω-rule together.

Interestingly, it turns out that Buchholz' Ω-rule finds its algebraic foundation in the MacNeille completion, in the sense that the Ω-rule is not sound in Heyting-valued semantics in general, but is sound when the underlying algebra is the MacNeille completion of the Lindenbaum algebra. This observation leads to a curious way of interpreting formulas that we call the Ω-*interpretation*. The basic idea already appears in Altenkirch and Coquand [6], but ours is better founded, and accommodates the existential quantifier too.

The Ω-rule and Ω-interpretation are two sides of the same coin. Combining them together, we obtain an algebraic proof of cut elimination for **LIP**, that is comparable with Aehlig's result [1] for the parameter-free, negative fragments of second order Heyting arithmetic. As with [1], our proof does not rely on reducibility candidates, and is formalizable in theories of finitely iterated inductive definitions.

Organization. In Section 2 we recall some basics of the MacNeille completion. In Section 3 we give some background on iterated inductive definitions and then introduce sequent calculus **LIP** and its subcalculi. In Section 4 we transform the arithmetical Ω-rule into a logical one, and then review the cut elimination procedure based on the Ω-rule, taking **LIP** as example. In Section 5, we turn to the algebraic side of the Ω-rule, establish a connection with the MacNeille completion, and propose an algebraic counterpart of Ω. In Section 6, we review an algebraic proof of cut elimination for **LI2**, and then gives an algebraic proof for **LIP**. Appendix A fully describes the sequent calculi studied in this paper, and Appendix B consists of some proofs omitted in the main text.

2 MacNeille completion

Let $\mathbf{A} = \langle A, \wedge, \vee \rangle$ be a lattice. A *completion* of $\mathbf{A}$ is an embedding $e : \mathbf{A} \longrightarrow \mathbf{B}$ into a complete lattice $\mathbf{B} = \langle B, \wedge, \vee \rangle$. We often assume that e is an inclusion map so that $\mathbf{A} \subseteq \mathbf{B}$.

For example, let $[0,1]_{\mathbb{Q}} := [0,1] \cap \mathbb{Q}$ be the chain of rational numbers in the unit interval (seen as a lattice). Then it admits an obvious completion $[0,1]_{\mathbb{Q}} \subseteq [0,1]$. For another example, let $\mathbf{A}$ be a Boolean algebra. Then it also admits a completion $e : \mathbf{A} \longrightarrow \mathbf{A}^\sigma$, where $\mathbf{A}^\sigma := \langle \wp(\mathsf{uf}(\mathbf{A})), \cap, \cup, -, A, \emptyset \rangle$, the powerset algebra on the set of ultrafilters of $\mathbf{A}$, and $e(a) := \{u \in \mathsf{uf}(\mathbf{A}) : a \in u\}$.

A completion $\mathbf{A} \subseteq \mathbf{B}$ is $\bigvee$-*dense* if $x = \bigvee\{a \in A : a \leq x\}$ holds for every $x \in B$. It is $\bigwedge$-*dense* if $x = \bigwedge\{a \in A : x \leq a\}$. A $\bigvee$-dense and $\bigwedge$-dense completion is called a *MacNeille completion*.

▶ **Theorem 1.** *Every lattice $\mathbf{A}$ has a MacNeille completion unique up to isomorphism [8, 29]. MacNeille completion is* regular*, i.e., preserves all joins and meets that already exist in $\mathbf{A}$.*

Coming back to the previous examples:

- $[0,1]_\mathbb{Q} \subseteq [0,1]$ is MacNeille, since $x = \inf\{a \in \mathbb{Q} : x \leq a\} = \sup\{a \in \mathbb{Q} : a \leq x\}$ for any $x \in [0,1]$. It is regular since if $q = \lim_{n\to\infty} q_n$ holds in $\mathbb{Q}$, then it holds in $\mathbb{R}$ too.
- $e : \mathbf{A} \longrightarrow \mathbf{A}^\sigma$ is not regular when $\mathbf{A}$ is an infinite Boolean algebra. In fact, the Stone space $\mathsf{uf}(\mathbf{A})$ is compact, so collapses any infinite union of open sets into a finite one. It is actually a *canonical extension*, that has been extensively studied in ordered algebra and modal logic [23, 21, 20].

MacNeille completions behave better than canonical extensions in preservation of existing limits, but the price to pay is loss of generality. Let $\mathcal{DL}$ ($\mathcal{HA}$, $\mathcal{BA}$, resp.) be the variety of distributive lattices (Heyting algebras, Boolean algebras, resp.).

▶ **Theorem 2.**
- *$\mathcal{DL}$ is not closed under MacNeille completions [18].*
- *$\mathcal{HA}$ and $\mathcal{BA}$ are closed under MacNeille completions.*
- *$\mathcal{HA}$ and $\mathcal{BA}$ are the only nontrivial subvarieties of $\mathcal{HA}$ closed under MacNeille [9].*

As is well known, completion is a standard algebraic way to prove conservativity of extending first order logics to higher order ones. The above result indicates that MacNeille completions work for classical and intuitionistic logics, but not for proper intermediate logics. See [33] for more on MacNeille completions.

Now an easy but crucial observation follows.

▶ **Proposition 3.** *A completion $\mathbf{A} \subseteq \mathbf{B}$ is MacNeille iff the rules below are valid:*

$$\frac{\{a \leq y\}_{a \leq x}}{x \leq y} \qquad \frac{\{x \leq a\}_{y \leq a}}{x \leq y}$$

where x, y range over B and a over A.

The left rule has infinitely many premises indexed by the set $\{a \in A : a \leq x\}$. It states that if $a \leq x$ implies $a \leq y$ for every $a \in A$, then $x \leq y$. This is valid just in case $x = \bigvee\{a \in A : a \leq x\}$. Likewise, the right rule states that if $y \leq a$ implies $x \leq a$ for every $a \in A$, then $x \leq y$. This is valid just in case $y = \bigwedge\{a \in A : y \leq a\}$.

As we will see, the above looks very similar to the Ω-rule. This provides a link between lattice theory and proof theory.

3 Parameter-free second order intuitionistic logic

3.1 Arithmetic

We here recall theories of inductive definitions. Let $\mathbf{I\Sigma}_1$, $\mathbf{PA}$ and $\mathbf{PA2}$ be the first order arithmetic with Σ^0_1 induction, that with full induction, and the second order arithmetic with full induction and comprehension, respectively. Given a theory T of arithmetic, $T[X]$ denotes the extension of T with a single set variable X and atomic formulas of the form $X(t)$.

Great many subsystems of **PA2** are considered in the literature. For instance, the system Π^1_1-$\mathbf{CA}_0$ is obtained by restricting the induction and comprehension axiom schemata to Π^1_1 formulas. Even weaker are theories of iterated inductive definitions $\mathbf{ID}_n$ with $n < \omega$, that are obtained as follows.

$\mathbf{ID}_0$ is just **PA**. To obtain $\mathbf{ID}_{n+1}$, consider a formula $\varphi(X, x)$ in $\mathbf{ID}_n[X]$ which contains no first order free variables other than x and no negative occurrences of X. It can be seen as a monotone map $\varphi^{\mathbb{N}} : \wp(\mathbb{N}) \longrightarrow \wp(\mathbb{N})$ sending a set $X \subseteq \mathbb{N}$ to $\{n \in \mathbb{N} : \mathbb{N} \models \varphi(X, n)\}$, so has the least fixed point. Based on this intuition, one adds a unary predicate symbol I_φ for each such φ to the language of $\mathbf{ID}_n$ and axioms

$$\varphi(I_\varphi) \subseteq I_\varphi, \qquad \varphi(\tau) \subseteq \tau \to I_\varphi \subseteq \tau$$

for every abstract $\tau = \lambda x.\xi(x)$ in the new language. Here $\varphi(I_\varphi)$ is a shorthand for the abstract $\lambda x.\varphi(I_\varphi, x)$ and $\tau_1 \subseteq \tau_2$ is for $\forall x.\tau_1(x) \to \tau_2(x)$. The induction schema is extended to the new language. This defines the system $\mathbf{ID}_{n+1}$. Notice that $\mathbf{ID}_{n+1}$ does not involve any set variable. Finally, let $\mathbf{ID}_{<\omega}$ be the union of all $\mathbf{ID}_n$ with $n < \omega$.

Clearly $\mathbf{ID}_{<\omega}$ can be seen as a subsystem of Π^1_1-$\mathbf{CA}_0$. In fact, any fixed point atom $I_\varphi(t)$ can be replaced by second order formula

$$\boldsymbol{I}_\varphi(t) \ := \ \forall X.\forall x(\varphi(X, x) \to X(x)) \to X(t).$$

Given a formula ψ of $\mathbf{ID}_{<\omega}$, we write $\psi^{\boldsymbol{I}}$ for the formula of **PA2** obtained by repeating the above replacement. This makes the axioms of $\mathbf{ID}_{<\omega}$ all provable in Π^1_1-$\mathbf{CA}_0$.

The converse is not strictly true, but it is known that $\mathbf{ID}_{<\omega}$ has the same proof theoretic strength and the same arithmetical consequences with Π^1_1-$\mathbf{CA}_0$.

Let us point out that a typical use of inductive definition is to define a provability predicate. Let T be a sequent calculus system, and suppose that we are given a formula $\varphi(X, x)$ saying that there is a rule in T with conclusion sequent x (coded by a natural number) and premises $Y \subseteq X$. Then I_φ gives the set of all provable sequents in T. Notice that the premise set Y can be infinite. It is for this reason that **ID**-theories are suitable metatheories for infinitary proof systems. See [13] for more on inductive definitions.

3.2 Second order intuitionistic logic

In this subsection, we formally introduce sequent calculus **LI2** for the second order intuitionistic logic with full comprehension, that is an intuitionistic counterpart of Takeuti's classical calculus $\mathbf{G}^1\mathbf{LC}$ [31].

Consider a language L that consists of (first order) function symbols and predicate symbols. A typical example is the language $L_{\mathbf{PA}}$ of Peano arithmetic, which contains a predicate symbol for equality and function symbols for all primitive recursive functions. Let

- Var: a countable set of term variables $x, y, z, \dots$,
- $\mathsf{Tm}(L)$: the set of first order terms $t, u, v, \dots$ over L,
- VAR: the set of set variables $X, Y, Z, \dots$.

The set $\mathsf{FM}(L)$ of second order formulas is defined by:

$$\varphi, \psi \ ::= \ p(\vec{t}) \mid X(t) \mid \bot \mid \varphi \star \psi \mid Qx.\varphi \mid QX.\varphi,$$

where $p \in L$, $\star \in \{\wedge, \vee, \rightarrow\}$ and $Q \in \{\forall, \exists\}$. We define $\top := \bot \rightarrow \bot$. When the language L is irrelevant, we write $\mathsf{Tm} := \mathsf{Tm}(L)$ and $\mathsf{FM} := \mathsf{FM}(L)$. Given φ, let $\mathsf{FV}(\varphi)$ and $\mathsf{Fv}(\varphi)$ be the set of free set variables and that of free term variables in φ, respectively.

Typical formulas in $\mathsf{FM}(L_{\mathbf{PA}})$ are

$$\begin{array}{lcl} \boldsymbol{N}(t) & := & \forall X.[\forall x(X(x) \rightarrow X(x+1)) \wedge X(0) \rightarrow X(t)], \\ \boldsymbol{E}(t) & := & \forall X.\forall x.[t = x \wedge X(x) \rightarrow X(t)]. \end{array}$$

We assume the standard variable convention that α-equivalent formulas are syntactically identical, so that substitutions can be applied without variable clash. A *term substitution* is a function $\circ : \mathsf{Var} \longrightarrow \mathsf{Tm}$. Given $\varphi \in \mathsf{FM}$, the substitution instance φ° is defined as usual. Likewise, a *set substitution* is a function $\bullet : \mathsf{VAR} \longrightarrow \mathsf{ABS}$, where $\mathsf{ABS} := \{\lambda x.\xi : \xi \in \mathsf{FM}\}$ is the set of *abstracts*. Instance $\varphi^\bullet$ is obtained by replacing each atomic formula $X(t)$ with $X^\bullet(t)$ and applying β-reduction.

Let $\mathsf{SEQ} := \{\Gamma \Rightarrow \Pi : \Gamma, \Pi \subseteq_{\mathsf{fin}} \mathsf{FM}, |\Pi| \leq 1\}$ be the set of *sequents of* **LI2**. We write Γ, Δ to denote $\Gamma \cup \Delta$. Rules of **LI2** include:

$$\frac{}{\Gamma, \varphi \Rightarrow \varphi}\ (\mathsf{id}) \qquad \frac{\varphi(\tau), \Gamma \Rightarrow \Pi}{\forall X.\varphi(X), \Gamma \Rightarrow \Pi}\ (\forall X\ \mathsf{left}) \qquad \frac{\Gamma \Rightarrow \varphi(Y)}{\Gamma \Rightarrow \forall X.\varphi(X)}\ (\forall X\ \mathsf{right})$$

$$\frac{\Gamma \Rightarrow \varphi \quad \varphi, \Gamma \Rightarrow \Pi}{\Gamma \Rightarrow \Pi}\ (\mathsf{cut}) \qquad \frac{\varphi(Y), \Gamma \Rightarrow \Pi}{\exists X.\varphi(X), \Gamma \Rightarrow \Pi}\ (\exists X\ \mathsf{left}) \qquad \frac{\Gamma \Rightarrow \varphi(\tau)}{\Gamma \Rightarrow \exists X.\varphi(X)}\ (\exists X\ \mathsf{right})$$

where $\tau \in \mathsf{ABS}$ and rules $(\forall X$ right) and $(\exists X$ left) are subject to the eigenvariable condition $Y \notin \mathsf{FV}(\Gamma, \Pi)$. The inference rules for other connectives can be found in Appendix A. The indicated occurrence of $\forall X.\varphi(X)$ is the *main formula* and $\varphi(\tau)$ is the *minor formula* of rule $(\forall X$ left). The same terminology applies to other inference rules too.

An obvious observation essentially due to [31] is that if a Π^0_2 sentence φ is provable in **PA2**, then $\forall y.\boldsymbol{E}(y), \Gamma^{\boldsymbol{N}} \Rightarrow \varphi^{\boldsymbol{N}}$ is provable in **LI2**, where Γ is a finite set of true Π^0_1 sentences (equality axioms, basic axioms of Peano arithmetic and defining axioms of primitive recursive functions), and $\varphi^{\boldsymbol{N}}$ is obtained from φ by relativizing each first order quantifier Qx to $Qx \in \boldsymbol{N}$. In particular if φ is Σ^0_1, we obtain $\forall y.\boldsymbol{E}(y), \Gamma \Rightarrow \varphi$, and the assumption $\forall y.\boldsymbol{E}(y)$ can be eliminated by another relativization with respect to $\boldsymbol{E}$, so that we eventually obtain $\Gamma \Rightarrow \varphi$ in **LI2**. A consequence is that

$$\mathbf{I\Sigma}_1 \vdash \mathsf{CE}(\mathbf{LI2}) \rightarrow \mathsf{1CON}(\mathbf{PA2}),$$

where $\mathsf{CE}(\mathbf{LI2})$ is a Π^0_2 sentence stating that **LI2** admits cut elimination, and $\mathsf{1CON}(\mathbf{PA2})$ is that **PA2** is *1-consistent*, that is, all provable Σ^0_1 sentences are true.

Thus 1-consistency of **PA2** is reduced to cut elimination for **LI2**. We also have the converse, also provably in $\mathbf{I\Sigma}_1$. The reason is that cut elimination for **LI2** is "locally" provable in **PA2**, that is, whenever $\mathbf{LI2} \vdash \Gamma \Rightarrow \Pi$, **PA2** proves a Σ^0_1 statement "$\mathbf{LI2} \vdash^{cf} \Gamma \Rightarrow \Pi$" (that is, "$\Gamma \Rightarrow \Pi$ is cut-free provable in **LI2**"), and moreover, a derivation of the latter statement (in **PA2**) can be primitive recursively obtained from any derivation of the former (in **LI2**). Hence 1-consistency of **PA2** implies cut elimination for **LI2** (in $\mathbf{I\Sigma}_1$). See [7] for a concise explanation.

The equivalence holds because **PA2** and **LI2** have a "matching" proof theoretic strength. We are going to introduce subsystems of **LI2** that match $\mathbf{ID}_{<\omega} = \bigcup_{n\in\omega} \mathbf{ID}_n$ in this sense.

3.3 Parameter-free fragments

Now let us introduce parameter-free subsystems of **LI2**. We first define the set $\mathsf{FMP}_n \subseteq \mathsf{FM}$ of *parameter-free formulas at level* n for every $n \geq -1$.

FMP_{-1} is just the set of formulas in FM without second order quantifiers. It is also denoted by Fm. For $n \geq 0$, FMP_n is defined by:

$$\varphi, \psi \ ::= \ p(\vec{t}) \mid t \in X \mid \perp \mid \varphi \star \psi \mid Qx.\varphi \mid QX.\xi,$$

where $\star \in \{\wedge, \vee, \to\}$, $Q \in \{\forall, \exists\}$ and ξ is any formula in FMP_{n-1} such that $\mathsf{FV}(\xi) \subseteq \{X\}$. Thus $QX.\xi$ is free of set parameters, though may contain first order free variables. Finally, FMP is the union of all FMP_n.

For instance, both $\boldsymbol{N}(t)$ and $\boldsymbol{E}(t)$ belong to FMP_0 so that relativizations $\varphi^{\boldsymbol{N}}$, $\varphi^{\boldsymbol{E}}$ belong to FMP_0 too, whenever φ is an arithmetical formula. Furthermore, each fixed point atom I_φ with φ arithmetical translates to

$$\boldsymbol{I}_\varphi^{\boldsymbol{N}}(t) \ := \ \forall X.\forall x \in \boldsymbol{N}(\varphi^{\boldsymbol{N}}(X, x) \to X(x)) \to X(t),$$

that belongs to FMP_1. We write $\varphi^{\boldsymbol{IN}}$ to denote the translation of $\mathbf{ID}_1$-formula φ in FMP_1. Likewise, any formula φ of $\mathbf{ID}_n$ translates to a formula $\varphi^{\boldsymbol{IN}}$ in FMP_n. On the other hand, the second order definitions of positive connectives $\{\exists, \vee\}$:

$$\exists X.\varphi(X) \ := \ \forall Y.\forall X(\varphi(X) \to Y(*)) \to Y(*), \quad \varphi \vee \psi \ := \ \forall Y.(\varphi \to Y(*)) \wedge (\psi \to Y(*)) \to Y(*)$$

with $Y \notin \mathsf{FV}(\varphi)$ and $*$ a constant, are no more available. They do not belong to FMP, so restricting to the negative fragment $\{\forall, \wedge, \to\}$ causes a serious loss of expressivity in the parameter-free setting.

Sequent calculus $\mathbf{LIP}$ (resp. $\mathbf{LIP}_n$) is obtained from $\mathbf{LI2}$ by restricting the formulas to FMP (resp. FMP_n). Most importantly, when one applies rules ($\forall X$ left) and ($\exists X$ right) to introduce $QX.\varphi$ in $\mathbf{LIP}$ (resp. $\mathbf{LIP}_n$), the minor formula $\varphi(\tau)$ must belong to FMP (resp. FMP_n).

$\mathbf{LIP}$ is an intuitionistic counterpart of the classical calculus studied in [32], and $\mathbf{LIP}_{-1}$ is just the ordinary sequent calculus for first order intuitionistic logic, that is also denoted by $\mathbf{LI}$.

The following is well known. For every Π^0_2 sentence φ of $\mathbf{ID}_n$, $\mathbf{ID}_n \vdash \varphi$ implies $\mathbf{LIP}_n \vdash \forall y.\boldsymbol{E}(y), \Gamma^{\boldsymbol{N}} \Rightarrow \varphi^{\boldsymbol{IN}}$, where Γ is a finite set of true Π^0_1 sentences. In particular, if φ is a Σ^0_1 sentence of $\mathbf{PA}$, we obtain $\Gamma \Rightarrow \varphi$. As a consequence,

$$\mathbf{I\Sigma}_1 \vdash \mathsf{CE}(\mathbf{LIP}_n) \to \mathsf{1CON}(\mathbf{ID}_n), \qquad \mathbf{I\Sigma}_1 \vdash \mathsf{CE}(\mathbf{LIP}) \to \mathsf{1CON}(\mathbf{ID}_{<\omega}).$$

The converse is obtained by proving cut elimination for $\mathbf{LIP}_n$ locally in $\mathbf{ID}_n$.

4 Ω-rule

4.1 Introduction to Ω-rule

Cut elimination in higher order is tricky, since a principal reduction step

$$\dfrac{\dfrac{\Gamma \Rightarrow \varphi(Y)}{\Gamma \Rightarrow \forall X.\varphi(X)} \quad \dfrac{\varphi(\tau) \Rightarrow \Pi}{\forall X.\varphi(X) \Rightarrow \Pi}}{\Gamma \Rightarrow \Pi}\ (\mathsf{cut}) \qquad \Longrightarrow \qquad \dfrac{\Gamma \Rightarrow \varphi(\tau) \quad \varphi(\tau) \Rightarrow \Pi}{\Gamma \Rightarrow \Pi}\ (\mathsf{cut})$$

may yield a bigger cut formula so that one cannot argue by induction on the complexity of the cut formula. The Ω-rule, introduced by [11], is a way to circumvent this by enforcing that any ancestor of a cut formula is a subformula of that. It is used to give an ordinal-free proof of (partial) cut elimination for a parameter-free subsystem $\mathbf{BI}_1^-$ of analysis [12]. It is later extended to complete cut elimination for the same system [4], and to complete cut elimination for Π^1_1-$\mathbf{CA}_0 + \mathbf{BI}$ (bar induction) [3]. The Ω-rule further finds applications in modal fixed point logics [22, 25]. It is used to show strong normalization for parameter-free fragments of System F, provably in $\mathbf{ID}$-theories [5].

As a starter, let us consider the most direct translation of the arithmetical Ω-rule [12] into our setting[1]. We extend $\mathbf{LI}$ by enlarging the formulas to FMP_0 and adding rules ($\forall X$ right) and

$$\frac{\{\ \Delta, \Gamma \Rightarrow \Pi\ \}_{\Delta \in |\forall X.\varphi|^\flat}}{\forall X.\varphi, \Gamma \Rightarrow \Pi}\ (\Omega^\flat)$$

where $|\forall X.\varphi|^\flat$ consists of $\Delta \subseteq_{\mathsf{fin}} \mathsf{Fm}$ such that $\mathbf{LI} \vdash^{cf} \Delta \Rightarrow \varphi(Y)$ for some $Y \notin \mathsf{FV}(\Delta)$ (recall that "cf" indicates cut-free provability).

Rule $(\Omega^\flat)$ has infinitely many premises indexed by $|\forall X.\varphi|^\flat$. It is intended to be an alternative of ($\forall X$ left). Indeed, we can prove $\forall X.\varphi \Rightarrow \varphi(\tau)$ for an arbitrary abstract τ in the language as follows. Let $\Delta \in |\forall X.\varphi|^\flat$, that is, $\mathbf{LI} \vdash^{cf} \Delta \Rightarrow \varphi(Y)$ for some $Y \notin \mathsf{FV}(\Delta)$. We then have $\Delta \Rightarrow \varphi(\tau)$ in the extended system by substituting τ for Y. Hence rule $(\Omega^\flat)$ yields $\forall X.\varphi \Rightarrow \varphi(\tau)$.

Unfortunately, rule $(\Omega^\flat)$ cannot be combined with the standard rules for first order quantifiers.

▶ **Proposition 4.** *System* $\mathbf{LI} + (\forall X\ \mathsf{right}) + (\Omega^\flat)$ *is inconsistent.*

Proof. Consider formula $\varphi := X(c) \to X(x)$ with c a constant. We claim that $\forall X.\varphi \Rightarrow \bot$ is provable. Let $\Delta \in |\forall X.\varphi|^\flat$, that is, $\mathbf{LI} \vdash \Delta \Rightarrow Y(c) \to Y(x)$ for some $Y \notin \mathsf{FV}(\Delta)$. Since the sequent is first order and $Y(c) \to Y(x)$ is not provable, Craig's interpolation theorem yields $\Delta \Rightarrow \bot$. Hence $\forall X.\varphi \Rightarrow \bot$ follows by $(\Omega^\flat)$. Since both $\exists x.\forall X.\varphi \Rightarrow \bot$ and $\Rightarrow \exists x.\forall X.\varphi$ are provable, we obtain $\bot$. ◀

The problem is that $(\Omega^\flat)$ is not closed under term substitutions. In fact, $\forall X.\varphi[c/x] \Rightarrow \bot$ does not directly follow from $(\Omega^\flat)$. This causes an undesired effect on first order quantifiers, thus leading to inconsistency. To obtain a consistent system, we will replace rules ($\forall x$ right) and ($\exists x$ left) with Schütte's *ω-rules*, as is standard in proof theory for arithmetic.

4.2 Cut elimination by Ω-rule

We now introduce an infinitary sequent calculus $\mathbf{LI}\Omega_n$ for each $n \geq -1$ and use it for complete cut elimination for $\mathbf{LIP}$. The proof idea is entirely due to [3].

We first prepare an isomorphic copy of FMP, denoted by $\overline{\mathsf{FMP}}$. Corresponding to FMP_n is the subset $\overline{\mathsf{FMP}}_n \subseteq \overline{\mathsf{FMP}}$. In $\overline{\mathsf{FMP}}$, all second order quantifiers are overlined as $\overline{\forall} X.\vartheta$ and $\overline{\exists} X.\vartheta$. Given a formula $\varphi \in \mathsf{FMP}$, $\overline{\varphi} \in \overline{\mathsf{FMP}}$ is obtained by overlining all the second order quantifiers in it.

Formulas in $\overline{\mathsf{FMP}}$ are intended to be potential cut formulas, i.e., ancestors of cut formulas in a derivation (that are called *implicit* in [32]). The *level* of each such formula $\vartheta \in \overline{\mathsf{FMP}}$ is defined by $\mathsf{level}(\vartheta) := \min\{k : \vartheta \in \overline{\mathsf{FMP}}_k\}$.

[1] Actually the original rule has assumptions indexed by *derivations* of $\Delta \Rightarrow \varphi(Y)$, not by Δ's themselves. As an advantage, one obtains a concrete operator for cut elimination and reduces the complexity of inductive definition: the original semiformal system can be defined by inductive definition on a bounded formula, while ours requires a Π^0_1 formula. However, this point is irrelevant for the subsequent argument.

We are going to introduce a hybrid calculus $\mathbf{LI}\Omega_n$ for each $n \geq -1$ that consists of formulas $\mathsf{FMP} \cup \overline{\mathsf{FMP}}_n$. Those in FMP are treated as in $\mathbf{LIP}$, while those in $\overline{\mathsf{FMP}}_n$ (potential cut formulas) are as in $\mathbf{LIP}_n$ with ($\forall X$ left) and ($\exists X$ right) replaced by Ω-rules.

Calculus $\mathbf{LI}\Omega_{-1}$ is just $\mathbf{LIP}$ where sequents consist of formulas in FMP and cut formulas are restricted to $\overline{\mathsf{FMP}}_{-1} = \mathsf{FMP}_{-1} = \mathsf{Fm}$.

Suppose that $\mathbf{LI}\Omega_{k-1}$ has been defined for every $0 \leq k \leq n$. For each $\overline{\forall} X.\vartheta$ and $\overline{\exists} X.\vartheta$ of level k, let

$$\begin{array}{lcl} |\overline{\forall} X.\vartheta(X)| & := & \{\Delta : \mathbf{LI}\Omega_{k-1} \vdash^{cf} \Delta \Rightarrow \vartheta(Y) \text{ for some } Y \notin \mathsf{FV}(\Delta)\} \\ |\overline{\exists} X.\vartheta(X)| & := & \{(\Delta \Rightarrow \Lambda) : \mathbf{LI}\Omega_{k-1} \vdash^{cf} \vartheta(Y), \Delta \Rightarrow \Lambda \text{ for some } Y \notin \mathsf{FV}(\Delta, \Lambda)\}. \end{array}$$

Note that $\Delta \cup \Lambda \subseteq \mathsf{FMP} \cup \overline{\mathsf{FMP}}_{k-1}$. Calculus $\mathbf{LI}\Omega_n$ is defined as follows:

- Sequents consist of formulas in $\mathsf{FMP} \cup \overline{\mathsf{FMP}}_n$.
- Cut formulas are restricted to $\overline{\mathsf{FMP}}_n$.
- Rules ($\forall x$ right) and ($\exists x$ left) are replaced by:

$$\frac{\{\, \Gamma \Rightarrow \varphi(t)\}_{t \in \mathsf{Tm}}}{\Gamma \Rightarrow \forall x.\varphi(x)}\ (\omega\ \mathsf{right}) \qquad \frac{\{\, \varphi(t), \Gamma \Rightarrow \Pi\}_{t \in \mathsf{Tm}}}{\exists x.\varphi(x), \Gamma \Rightarrow \Pi}\ (\omega\ \mathsf{left})$$

- Other rules except for overlined quantifiers are the same as $\mathbf{LIP}$ (thus we have all of ($\forall X$ left), ($\forall X$ right), ($\exists X$ left) and ($\exists X$ right) for non-overlined quantifiers).
- Overlined quantifiers are treated by the following rules ($k = 0, \dots, n$):

$$\frac{\vartheta(Y), \Gamma \Rightarrow \Pi}{\overline{\exists} X.\vartheta(X), \Gamma \Rightarrow \Pi}\ (\overline{\exists} X\ \mathsf{left}) \qquad \frac{\Gamma \Rightarrow \vartheta(Y)}{\Gamma \Rightarrow \overline{\forall} X.\vartheta(X)}\ (\overline{\forall} X\ \mathsf{right})$$

$$\frac{\{\, \Delta, \Gamma \Rightarrow \Pi \,\}_{\Delta \in |\overline{\forall} X.\vartheta|}}{\overline{\forall} X.\vartheta, \Gamma \Rightarrow \Pi}\ (\Omega_k\ \mathsf{left}) \qquad \frac{\Gamma \Rightarrow \vartheta(Y) \quad \{\, \Delta, \Gamma \Rightarrow \Pi \,\}_{\Delta \in |\overline{\forall} X.\vartheta|}}{\Gamma \Rightarrow \Pi}\ (\tilde{\Omega}_k\ \mathsf{left})$$

$$\frac{\{\, \Gamma, \Delta \Rightarrow \Lambda \,\}_{(\Delta \Rightarrow \Lambda) \in |\overline{\exists} X.\vartheta|}}{\Gamma \Rightarrow \overline{\exists} X.\vartheta}\ (\Omega_k\ \mathsf{right}) \qquad \frac{\{\, \Gamma, \Delta \Rightarrow \Lambda \,\}_{(\Delta \Rightarrow \Lambda) \in |\overline{\exists} X.\vartheta|} \quad \vartheta(Y), \Gamma \Rightarrow \Pi}{\Gamma \Rightarrow \Pi}\ (\tilde{\Omega}_k\ \mathsf{right})$$

where k is the level of $\overline{\forall} X.\vartheta$, $\overline{\exists} X.\vartheta$ and rules ($\overline{\exists} X$ left), ($\overline{\forall} X$ right), ($\tilde{\Omega}_k$ left) and ($\tilde{\Omega}_k$ right) are subject to the eigenvariable condition ($Y \notin \mathsf{FV}(\Gamma, \Pi)$). See Appendix A for a complete list of inference rules.

First of all, rule ($\tilde{\Omega}_k$ left) is derivable by combining ($\overline{\forall} X$ right), (Ω_k left) and (cut). It is nevertheless included for a technical reason. The same applies to rule ($\tilde{\Omega}_k$ right).

On the other hand, rules (Ω_k left) and (Ω_k right) are our real concern. The former should be read as follows: whenever $\mathbf{LI}\Omega_{k-1} \vdash^{cf} \Delta \Rightarrow \vartheta(Y)$ implies $\mathbf{LI}\Omega_n \vdash \Delta, \Gamma \Rightarrow \Pi$ for every Δ with $Y \notin \mathsf{FV}(\Delta)$, one can conclude $\mathbf{LI}\Omega_n \vdash \overline{\forall} X.\vartheta, \Gamma \Rightarrow \Pi$.

It is admittedly too complicated. To get an intuition, observe a similarity with the characteristic rules of MacNeille completion. Indeed, Δ, $\overline{\forall} X.\vartheta$ and ($\Gamma \Rightarrow \Pi$) in (Ω_k left) correspond to a, x and y in the left rule of Proposition 3. In the next section, we will provide a further link between them.

Now let us list some key lemmas for cut elimination. The proofs are found in Appendix B.2 and B.3.

▶ **Lemma 5** (Embedding). $\mathbf{LIP}_n \vdash \Gamma \Rightarrow \Pi$ *implies* $\mathbf{LI}\Omega_n \vdash \Gamma \Rightarrow \Pi$.

▶ **Lemma 6.** $\mathbf{LI}\Omega_n \vdash \Gamma \Rightarrow \Pi$ *implies* $\mathbf{LI}\Omega_n \vdash^{cf} \Gamma \Rightarrow \Pi$.

▶ **Lemma 7** (Collapsing). $\mathbf{LI\Omega}_n \vdash^{cf} \Gamma \Rightarrow \Pi$ *implies* $\mathbf{LI\Omega}_{n-1} \vdash^{cf} \Gamma \Rightarrow \Pi$*, provided that* $\Gamma \cup \Pi \subseteq \mathsf{FMP} \cup \overline{\mathsf{FMP}}_{n-1}$.

Proof. By induction on the cut-free derivation of $\Gamma \Rightarrow \Pi$ in $\mathbf{LI\Omega}_n$. If it ends with ($\tilde{\Omega}_n$ left) (see above), we have $\mathbf{LI\Omega}_{n-1} \vdash^{cf} \Gamma \Rightarrow \vartheta(Y)$ by the induction hypothesis, noting that $\vartheta(Y) \in \overline{\mathsf{FMP}}_{n-1}$. Hence $\Gamma \in |\overline{\forall} X.\vartheta|$, so $\Gamma, \Gamma \Rightarrow \Pi$ is among the premises. Therefore $\mathbf{LI\Omega}_{n-1} \vdash^{cf} \Gamma \Rightarrow \Pi$ by the induction hypothesis again.

Rule ($\tilde{\Omega}_n$ left) is treated similarly. When $n = 0$, one has to replace (ω right) and (ω left) by ($\forall$ right) and ($\exists$ left) respectively, that is easy. ◀

▶ **Theorem 8** (Cut elimination). $\mathbf{LIP} \vdash \Gamma \Rightarrow \Pi$ *implies* $\mathbf{LIP} \vdash^{cf} \Gamma \Rightarrow \Pi$.

Proof. The sequent is provable in $\mathbf{LIP}_n$ for some $n < \omega$, so in $\mathbf{LI\Omega}_n$ by Lemma 5. Noting that $\Gamma \cup \Pi \subseteq \mathsf{FMP}$, we obtain a cut-free derivation in $\mathbf{LI\Omega}_{-1}$ by Lemmas 6 and 7, that is also a cut-free derivation in $\mathbf{LIP}$. ◀

Of course the above argument can be restricted to a proof of cut elimination for $\mathbf{LIP}_n$. From a metatheoretical point of view, the most significant part is to define provability predicates $\mathbf{LI\Omega}_{-1}, \dots, \mathbf{LI\Omega}_n$. $\mathbf{LI\Omega}_{-1}$ is finitary, so is definable in $\mathbf{PA} = \mathbf{ID}_0$. $\mathbf{LI\Omega}_0$ is obtained by an inductive definition relying on $\mathbf{LI\Omega}_{-1}$, so is definable in $\mathbf{ID}_1$. By repetition, we observe that $\mathbf{LI\Omega}_n$ is definable in $\mathbf{ID}_{n+1}$. Moreover, $\mathbf{LI\Omega}$ is definable with a uniform inductive definition in $\mathbf{ID}_\omega$. Once a suitable provability predicate has been defined, the rest of argument can be smoothly formalized. Hence we conclude:

$$\mathbf{ID}_{n+1} \vdash \mathsf{CE}(\mathbf{LIP}_n), \qquad \mathbf{ID}_\omega \vdash \mathsf{CE}(\mathbf{LIP}).$$

This is very well understood in the traditional proof thoery.

5 Ω-rule and MacNeille completion

We will now have a look at the Ω-rule from an algebraic point of view.

Let L be a language. A (complete) *Heyting-valued prestructure* for L is $\mathcal{M} = \langle \mathbf{A}, M, \mathcal{D}, L^{\mathcal{M}} \rangle$ where $\mathbf{A} = \langle A, \wedge, \vee, \to, \top, \bot \rangle$ is a complete Heyting algebra, M is a nonempty set (*term domain*), $\emptyset \neq \mathcal{D} \subseteq A^M$ (*abstract domain*) and $L^{\mathcal{M}}$ contains a function $f^{\mathcal{M}} : M^n \longrightarrow M$ for each n-ary function symbol $f \in L$ and $p^{\mathcal{M}} : M^n \longrightarrow A$ for each n-ary predicate symbol $p \in L$. Thus $p^{\mathcal{M}}$ is an $\mathbf{A}$-valued subset of M^n.

It is not our purpose to systematically develop a model theory for intuitionistic logic. We will use prestructures only for proving conservative extension and cut elimination. Hence we assume $M = \mathsf{Tm}$ and $f^{\mathcal{M}}(\vec{t}) = f(\vec{t})$ below, that simplifies the interpretation of formulas a lot.

A *valuation* on $\mathcal{M}$ is a function $\mathcal{V} : \mathsf{VAR} \longrightarrow \mathcal{D}$. $\mathcal{V}$ can be extended to an *interpretation* $\mathcal{V} : \mathsf{FM} \longrightarrow \mathbf{A}$ as follows:

$$\begin{array}{lcllcl}
\mathcal{V}(p(\vec{t})) & := & p^{\mathcal{M}}(\vec{t}) & \mathcal{V}(X(t)) & := & \mathcal{V}(X)(t) \\
\mathcal{V}(\bot) & := & \bot & \mathcal{V}(\varphi \star \psi) & := & \mathcal{V}(\varphi) \star \mathcal{V}(\psi) \\
\mathcal{V}(\forall x.\varphi(x)) & := & \bigwedge_{t \in \mathsf{Tm}} \mathcal{V}(\varphi(t)) & \mathcal{V}(\exists x.\varphi(x)) & := & \bigvee_{t \in \mathsf{Tm}} \mathcal{V}(\varphi(t)) \\
\mathcal{V}(\forall X.\varphi) & := & \bigwedge_{F \in \mathcal{D}} \mathcal{V}[F/X](\varphi) & \mathcal{V}(\exists X.\varphi) & := & \bigvee_{F \in \mathcal{D}} \mathcal{V}[F/X](\varphi)
\end{array}$$

where $\star \in \{\wedge, \vee, \to\}$ and $\mathcal{V}[F/X]$ is an update of $\mathcal{V}$ that maps X to F. $\mathcal{V}$ can also be extended to a function $\mathcal{V} : \mathsf{ABS} \longrightarrow \mathbf{A}^{\mathsf{Tm}}$ by $\mathcal{V}(\lambda x.\varphi)(t) := \mathcal{V}(\varphi[t/x])$. $\mathcal{M}$ is called a Heyting-valued *structure* if $\mathcal{V}(\tau) \in \mathcal{D}$ holds for every valuation $\mathcal{V}$ and every $\tau \in \mathsf{ABS}$. Clearly $\mathcal{M}$ is a Heyting-valued structure if $\mathcal{D} = \mathbf{A}^{\mathsf{Tm}}$. Such a structure is called *full*.

Given a sequent $\Gamma \Rightarrow \Pi$, let $\mathcal{V}(\Gamma) := \bigwedge\{\mathcal{V}(\varphi) : \varphi \in \Gamma\}$ ($:= \top$ if Γ is empty). $\mathcal{V}(\Pi) := \mathcal{V}(\psi)$ if $\Pi = \{\psi\}$, and $\mathcal{V}(\Pi) := \bot$ if Π is empty. It is routine to verify:

▶ **Lemma 9** (Soundness). *If* $\mathbf{LI2} \vdash \Gamma \Rightarrow \Pi$*, then* $\Gamma \Rightarrow \Pi$ *is* valid*, that is,* $\mathcal{V}(\Gamma^\circ) \leq \mathcal{V}(\Pi^\circ)$ *holds for every valuation* $\mathcal{V}$ *on every Heyting structure* $\mathcal{M}$ *and every term substitution* $\circ$*.*

This is not true for $\mathbf{LI} + (\Omega^\flat)$, because rule $(\Omega^\flat)$ is not closed under $\circ$. Then what if we forget term substitutions and consider $(\Omega^\flat)$ in isolation? Does the rule always preserve truth in every Heyting structure? This is to be discussed below.

A good starting point is an algebraic proof of conservative extension for **LI2** over **LI**, that proceeds as follows.

Let $\mathbf{L}$ be the Lindenbaum algebra for **LI**, that is, $\mathbf{L} := \langle \mathsf{Fm}/\!\sim, \wedge, \vee, \rightarrow, \top, \bot\rangle$ where $\varphi \sim \psi$ iff $\mathbf{LI} \vdash \varphi \leftrightarrow \psi$. The equivalence class of φ with respect to $\sim$ is denoted by $[\varphi]$. $\mathbf{L}$ is a Heyting algebra in which

$$(*) \qquad [\forall x.\varphi(x)] = \bigwedge_{t \in \mathsf{Tm}} [\varphi(t)], \qquad [\exists x.\varphi(x)] = \bigvee_{t \in \mathsf{Tm}} [\varphi(t)]$$

hold. Given a sequent $\Gamma \Rightarrow \Pi$, elements $[\Gamma]$ and $[\Pi]$ in $\mathbf{L}$ are naturally defined.

Let $\mathbf{L} \subseteq \mathbf{G}$ be a *regular* completion. Then $\mathcal{M}(\mathbf{G}) := \langle \mathbf{G}, \mathsf{Tm}, \mathbf{G}^{\mathsf{Tm}}, L^{\mathcal{M}(\mathbf{G})}\rangle$ is a full Heyting structure, where $p^{\mathcal{M}(\mathbf{G})}(\vec{t}) = [p(\vec{t})]$. Define a valuation $\mathcal{I}$ by $\mathcal{I}(X)(t) := [X(t)]$. We then have $\mathcal{I}(\varphi) = [\varphi]$ for every $\varphi \in \mathsf{Fm}$ by regularity (be careful here: $(*)$ may fail in $\mathbf{G}$ if it is not regular).

Now, suppose that **LI2** proves $\Gamma \Rightarrow \Pi$ with $\Gamma \cup \Pi \subseteq \mathsf{Fm}$. Then we have $\mathcal{I}(\Gamma) \leq \mathcal{I}(\Pi)$ by Lemma 9, so $[\Gamma] \leq [\Pi]$, that is, $\mathbf{LI} \vdash \Gamma \Rightarrow \Pi$. This proves that **LI2** is a conservative extension of **LI**.

Although this argument cannot be fully formalized in **PA2** because of Gödel's second incompleteness, it does admit a local formalization in it. Unfortunately, it does not scale down to $\mathbf{LIP}_n$ and $\mathbf{ID}_n$ simply because the latter does not have second order quantifiers, which are needed to write down $\mathcal{V}(\forall X.\varphi)$ and $\mathcal{V}(\exists X.\varphi)$. To circumvent this difficulty, a crucial observation is the following.

▶ **Theorem 10.** *Let* $\mathbf{L}$ *be the Lindenbaum algebra for* **LI** *and* $\mathbf{L} \subseteq \mathbf{G}$ *a regular completion.* $\mathcal{M}(\mathbf{G})$ *and* $\mathcal{I}$ *are defined as above. For every* sentence $\forall X.\varphi$ *in* FMP_0*, the following are equivalent.*

1. $\mathcal{I}(\forall X.\varphi) = \bigvee\{a \in \mathbf{L} : a \leq \mathcal{I}(\forall X.\varphi)\}$.
2. $\mathcal{I}(\forall X.\varphi) = \bigvee\{[\Delta] \in \mathbf{L} : \Delta \in |\forall X.\varphi|^\flat\}$.
3. *The inference below is sound for every* $y \in \mathbf{G}$*:*

$$\frac{\{\ \mathcal{I}(\Delta) \leq y\ \}_{\Delta \in |\forall X.\varphi|^\flat}}{\mathcal{I}(\forall X.\varphi) \leq y}$$

If $\mathbf{G}$ *is the MacNeille completion of* $\mathbf{F}$*, all the above hold.*

Proof. (1. $\Leftrightarrow$ 2.) Suppose that $a = [\Delta] \leq \mathcal{I}(\forall X.\varphi(X))$. Choose $Y \notin \mathsf{FV}(\Delta)$ and let $F_Y(t) := [Y(t)]$. We then have $[\Delta] \leq \mathcal{I}[F_Y/X](\varphi(X)) = [\varphi(Y)]$, that leads to $\mathbf{LI} \vdash^{cf} \Delta \Rightarrow \varphi(Y)$. Hence $\Delta \in |\forall X.\varphi(X)|^\flat$. Conversely, suppose that $\Delta \in |\forall X.\varphi(X)|^\flat$, i.e., $\mathbf{LI} \vdash^{cf} \Delta \Rightarrow \varphi(Y)$ with $Y \notin \mathsf{FV}(\Delta)$. It implies $[\Delta] = \mathcal{I}(\Delta) \leq \mathcal{I}[F/Y](\varphi(Y))$ for every $F \in \mathbf{G}^{\mathsf{Tm}}$ by Lemma 9 (restricted to **LI**). Hence $[\Delta] \leq \mathcal{I}(\forall X.\varphi(X))$.
(2. $\Rightarrow$ 3.) Just apply $[\Delta] = \mathcal{I}(\Delta)$.
(3. $\Rightarrow$ 2.) Let y be the right hand side of equation 2. so that $\mathcal{I}(\Delta) = [\Delta] \leq y$ holds for every $\Delta \in |\forall X.\varphi|^\flat$. Hence $\mathcal{I}(\forall X.\varphi) \leq y$. We also have $[\Delta] \leq \mathcal{I}(\forall X.\varphi(X))$ for every $\Delta \in |\forall X.\varphi(X)|^\flat$ as proved above. Hence $y \leq \mathcal{I}(\forall X.\varphi(X))$. ◀

The equivalence in Theorem 10 is suggestive, since 3. is an algebraic interpretation of rule $(\Omega^\flat)$, while 1. is a characteristic of the MacNeille completion (Proposition 3). Equation 2. suggests a way of interpreting second order formulas without using second order quantifiers at the meta-level. All these are true if the completion is MacNeille. It should be mentioned that essentially the same as 2. has been already observed by Altenkirch and Coquand [6] in the context of lambda calculus (without making any connection to the Ω-rule and the MacNeille completion). Indeed, they consider a logic which roughly amounts to the negative fragment of our $\mathbf{LIP}_0$ and employ equation 2. to give a "finitary" proof of (partial) normalization theorem for a parameter-free fragment of System F (see also [2, 5] for extensions). However, their argument is technically based on a downset completion, that is not MacNeille. As is well known, such a naive completion does not work well for the positive connectives $\{\exists, \vee\}$. In contrast, when $\mathbf{G}$ is the MacNeille completion of $\mathbf{L}$, we also have

$$\mathcal{I}(\exists X.\varphi) = \bigwedge\{[\Delta] \to [\Lambda] \in \mathbf{L} : (\Delta \Rightarrow \Lambda) \in |\exists X.\varphi|^\flat\},$$

where $(\Delta \Rightarrow \Lambda) \in |\exists X.\varphi(X)|^\flat$ iff $\mathbf{LI} \vdash^{cf} \varphi(Y), \Delta \Rightarrow \Lambda$ for some $Y \notin \mathsf{FV}(\Delta, \Lambda)$. We thus claim that the insight by Altenkirch and Coquand is augmented and better understood in terms of the MacNeille completion.

It is interesting to see that (second order) $\forall$ is interpreted by (first order) $\bigvee$ while $\exists$ is by $\bigwedge$. We call this style of interpretation the Ω-*interpretation*, that is the algebraic side of the Ω-rule, and that will play a key role in the next section. We conclude our discussion by reporting a counterexample for general soundness.

▶ **Proposition 11.** *There is a Heyting-valued structure in which* $(\Omega^\flat)$ *is not sound (even without term substitutions, see Lemma 9).*

Proof. Let A be the three-element chain $\{0 < 0.5 < 1\}$ and $\mathbf{A} := \langle A, \min, \max, \to, 1, 0\rangle$ be a Heyting algebra where $a \to b := 1$ if $a \leq b$ and $a \to b := b$ otherwise.

Consider the language that only consists of a term constant $*$. Then a full Heyting-valued structure $\mathcal{A} := \langle \mathbf{A}, \mathsf{Tm}, \mathbf{A}^{\mathsf{Tm}}, L^{\mathcal{A}}\rangle$ is naturally obtained. Let $\varphi := (X(*) \to \bot) \vee X(*)$. It is easy to see that $\mathcal{V}(\forall X.\varphi) = 0.5$ for every valuation $\mathcal{V}$.

Now consider the following instance:

$$\frac{\{\ \Delta \Rightarrow \bot\ \}_{\Delta \in |\forall X.\varphi|^\flat}}{\forall X.\varphi \Rightarrow \bot}\ (\Omega^\flat)$$

We claim that it is not sound for a valuation $\mathcal{V}$ such that $\mathcal{V}(X(t)) = 0$ for every $X \in \mathsf{VAR}$ and $t \in \mathsf{Tm}$. Suppose that $\Delta \in |\forall X.\varphi|$, i.e., $\mathbf{LI} \vdash^{cf} \Delta \Rightarrow \varphi(Y)$ with $Y \notin \mathsf{FV}(\Delta)$. Then $\mathcal{V}(\Delta) \leq \bigwedge_{F \in \mathbf{A}^{\mathsf{Tm}}}(\varphi) = 0.5$ by Lemma 9 (restricted to $\mathbf{LI}$). But Δ is first-order, so only takes value 0 or 1 under our assumption on $\mathcal{V}$. Hence $\mathcal{V}(\Delta) = 0$, that is, all premises are satisfied. However, $\mathcal{V}(\forall X.\varphi) = 0.5 > 0$, that is, the conclusion is not satisfied. ◀

This invokes a natural question. Is it possible to find a *Boolean-valued* counterexample? In other words, is the Ω-rule classically sound? This question is left open.

6 Algebraic cut elimination

6.1 Polarities and Heyting frames

This section is devoted to algebraic proofs of cut elimination. We begin with a very old concept due to Birkhoff [10], that provides a uniform framework for both MacNeille completion and cut elimination.

A *polarity* $\mathbf{W} = \langle W, W', R \rangle$ consists of two sets W, W' and a binary relation $R \subseteq W \times W'$. Given $X \subseteq W$ and $Z \subseteq W'$, let

$$X^{\rhd} := \{z \in W' : x \; R \; z \text{ for every } x \in X\}, \qquad Z^{\lhd} := \{x \in W : x \; R \; z \text{ for every } z \in Z\}.$$

For example, let $\mathbf{Q} := \langle \mathbb{Q}, \mathbb{Q}, \leq \rangle$. Then $X^{\rhd}$ is the set of upper bounds of X and $Z^{\lhd}$ is the set of lower bounds of Z. Hence $(X^{\rhd\lhd}, X^{\rhd})$ is a Dedekind cut for every $X \subseteq \mathbb{Q}$ bounded above.

The pair $(\rhd, \lhd)$ forms a *Galois connection*:

$$X \subseteq Z^{\lhd} \iff X^{\rhd} \supseteq Z$$

so induces a closure operator $\gamma(X) := X^{\rhd\lhd}$ on $\wp(W)$, that is, $X \subseteq \gamma(Y)$ iff $\gamma(X) \subseteq \gamma(Y)$ for any $X, Y \subseteq W$. Note that $X \subseteq W$ is closed iff there is $Z \subseteq W'$ such that $X = Z^{\lhd}$.

In the following, we write $\gamma(x) := \gamma(\{x\})$, $x^{\rhd} := \{x\}^{\rhd}$ and $z^{\lhd} := \{z\}^{\lhd}$. Let

$$\mathcal{G}(\mathbf{W}) := \{X \subseteq W : X = \gamma(X)\},$$

$X \wedge Y := X \cap Y$, $X \vee Y := \gamma(X \cup Y)$, $\top := W$ and $\bot := \gamma(\emptyset)$.

▶ **Lemma 12.** *If* $\mathbf{W}$ *is a polarity, then* $\mathbf{W}^+ := \langle \mathcal{G}(\mathbf{W}), \wedge, \vee \rangle$ *is a complete lattice.*

The lattice $\mathbf{W}^+$ is not always distributive because of the use of γ in the definition of $\vee$. To ensure distributivity, we have to impose a further structure on $\mathbf{W}$.

A *Heyting frame* is $\mathbf{W} = \langle W, W', R, \circ, \varepsilon, \setminus\!\!\setminus \rangle$, where

- $\langle W, W', R \rangle$ is a polarity,
- $\langle W, \circ, \varepsilon \rangle$ is a monoid,
- $\setminus\!\!\setminus : W \times W' \longrightarrow W'$ satisfies $x \circ y \; R \; z \iff y \; R \; x \setminus\!\!\setminus z$ for every $x, y \in W$ and $z \in W'$,
- the following inferences are valid:

$$\frac{x \circ y \; R \; z}{y \circ x \; R \; z}\,(e) \qquad \frac{\varepsilon \; R \; z}{x \; R \; z}\,(w) \qquad \frac{x \circ x \; R \; z}{x \; R \; z}\,(c)$$

Clearly $x \; R \; z$ is an analogue of a sequent and $(e), (w)$ and (c) correspond to exchange, weakening and contraction rules. By removing some/all of them, one obtains *residuated frames* that work for substructural logics as well [19, 16].

▶ **Lemma 13.** *If* $\mathbf{W}$ *is a Heyting frame,* $\mathbf{W}^+ := \langle \mathcal{G}(\mathbf{W}), \wedge, \vee, \rightarrow, \top, \bot \rangle$ *is a complete Heyting algebra, where* $X \rightarrow Y := \{y \in W : x \circ y \in Y \text{ for every } x \in X\}$.

See Appendix B.4 for a proof. Polarities and Heyting frames are handy devices to obtain MacNeille completions. Let $\mathbf{A} = \langle A, \wedge, \vee, \rightarrow, \top, \bot \rangle$ be a Heyting algebra. Then $\mathbf{W_A} := \langle A, A, \leq, \wedge, \top, \rightarrow \rangle$ is a Heyting frame. Notice that the third condition above amounts to $x \wedge y \leq z$ iff $y \leq x \rightarrow z$.

▶ **Theorem 14.** *If* $\mathbf{A}$ *is a Heyting algebra, then* $\gamma : \mathbf{A} \longrightarrow \mathbf{W}^+_{\mathbf{A}}$ *is a MacNeille completion.*

6.2 Algebraic cut elimination for full second order logic

We here outline an algebraic proof of cut elimination for the full second order calculus **LI2** that we attribute to Maehara [24] and Okada [26, 28]. This will be useful for a comparison with the parameter-free case $\mathbf{LIP}_{n+1}$, that is to be discussed in the next subsection.

Let $\wp_{\mathsf{fin}}(\mathsf{FM})$ be the set of finite sets of formulas, so that $\langle \wp_{\mathsf{fin}}(\mathsf{FM}), \cup, \emptyset \rangle$ is a commutative idempotent monoid. Recall that SEQ denotes the set of sequents of **LI2**. There is a natural map $\setminus\!\!\setminus : \wp_{\mathsf{fin}}(\mathsf{FM}) \times \mathsf{SEQ} \longrightarrow \mathsf{SEQ}$ defined by $\Gamma \setminus\!\!\setminus (\Sigma \Rightarrow \Pi) := (\Gamma, \Sigma \Rightarrow \Pi)$. So

$$\mathbf{CF} := \langle \wp_{\mathsf{fin}}(\mathsf{FM}), \mathsf{SEQ}, \Rightarrow^{cf}_{\mathbf{LI2}}, \cup, \emptyset, \setminus\!\!\setminus \rangle$$

is a Heyting frame, where $\Gamma \Rightarrow^{cf}_{\mathbf{LI2}} (\Sigma \Rightarrow \Pi)$ iff $\mathbf{LI2} \vdash^{cf} \Gamma, \Sigma \Rightarrow \Pi$. In the following, we simply write φ for sequent $(\emptyset \Rightarrow \varphi) \in \mathsf{SEQ}$.

Hence $\mathbf{CF}$ is a frame in which $\Gamma \in \varphi^{\lhd}$ holds iff $\Gamma \Rightarrow \varphi$ is cut-free provable in $\mathbf{LI2}$. In particular, $\varphi \in \varphi^{\lhd}$ always holds, so $\gamma(\varphi) \subseteq \varphi^{\lhd}$. It should also be noted that each $X \in \mathcal{G}(\mathbf{CF})$ is closed under weakening: if $\Delta \in X$ and $\Delta \subseteq \Sigma$, then $\Sigma \in X$.

Define a Heyting prestructure $\mathcal{CF} := \langle \mathbf{CF}^+, \mathsf{Tm}, \mathcal{D}, L^{\mathcal{CF}} \rangle$ by $p^{\mathcal{CF}}(\vec{t}) := \gamma(p(\vec{t}))$ for each predicate symbol p and

$$\mathcal{D} := \{F \in \mathcal{G}(\mathbf{CF})^{\mathsf{Tm}} : F \text{ matches some } \tau \in \mathsf{ABS}\},$$

where F *matches* $\lambda x.\xi(x)$ just in case $\xi(t) \in F(t) \subseteq \xi(t)^{\lhd}$ holds for every $t \in \mathsf{Tm}$. This choice of $\mathcal{D} \subseteq \mathcal{G}(\mathbf{CF})^{\mathsf{Tm}}$ is a logical analogue of Girard's *reducibility candidates* as noticed by Okada.

Given a set substitution $\bullet$ and a valuation $\mathcal{V} : \mathsf{VAR} \longrightarrow \mathcal{D}$, we say that $\mathcal{V}$ *matches* $\bullet$ if $\mathcal{V}(X)$ matches $X^{\bullet} \in \mathsf{ABS}$ for every $X \in \mathsf{VAR}$. That is, $X^{\bullet}(t) \in \mathcal{V}(X)(t) \subseteq X^{\bullet}(t)^{\lhd}$ holds for every $X \in \mathsf{VAR}$ and $t \in \mathsf{Tm}$. The following is what Okada [28] calls his *main lemma* (Appendix B.6).

▶ **Lemma 15.** *Let* $\bullet : \mathsf{VAR} \longrightarrow \mathsf{ABS}$ *be a substitution and* $\mathcal{V}$ *be a valuation that matches* $\bullet$. *Then for every* $\varphi \in \mathsf{FM}$,

$$\varphi^{\bullet} \in \mathcal{V}(\varphi) \subseteq \varphi^{\bullet\lhd}.$$

As a consequence, $\mathcal{V}(\tau) \in \mathcal{D}$ for every $\tau \in \mathsf{ABS}$ (recall that $\mathcal{V}(\lambda x.\xi(x))(t) := \mathcal{V}(\xi(t))$). That is, $\mathcal{CF}$ is a Heyting structure. For another consequence, define a valuation $\mathcal{I}$ by $\mathcal{I}(X)(t) := \gamma(X(t))$, that matches the identity substitution. Then we have $\varphi \in \mathcal{I}(\varphi) \subseteq \varphi^{\lhd}$. More generally, for every sequent $\Gamma \Rightarrow \Pi$ we have $\Gamma \in \mathcal{I}(\Gamma)$ (by closure under weakening and $\mathcal{I}(\Gamma) = \bigcap\{\mathcal{I}(\varphi) : \varphi \in \Gamma\}$) and $\mathcal{I}(\Pi) \subseteq \Pi^{\lhd}$.

▶ **Theorem 16** (Completeness and cut elimination)**.** *For every sequent* $\Gamma \Rightarrow \Pi$*, the following are equivalent.*

1. $\Gamma \Rightarrow \Pi$ *is provable in* $\mathbf{LI2}$.
2. $\Gamma \Rightarrow \Pi$ *is valid in all Heyting structures.*
3. $\Gamma \Rightarrow \Pi$ *is cut-free provable in* $\mathbf{LI2}$.

Proof. (1. $\Rightarrow$ 2.) holds by Lemma 9, and (2. $\Rightarrow$ 3.) by $\Gamma \in \mathcal{I}(\Gamma) \subseteq \mathcal{I}(\Pi) \subseteq \Pi^{\lhd}$ in $\mathcal{CF}$. ◀

Recall that the frame $\mathbf{CF}$ is defined by referring to cut-free provability in $\mathbf{LI2}$. But the above theorem states that it coincides with provability. As a consequence, we have $\gamma(\varphi) = \varphi^{\lhd}$ for every formula φ, so that there is exactly one closed set X such that $\varphi \in X \subseteq \varphi^{\lhd}$. Hence the complete algebra $\mathbf{CF}^+$ can be restricted to a subalgebra $\mathbf{CF}^+_0$ with underlying set $\{\gamma(\varphi) : \varphi \in \mathsf{FM}\}$. It is easy to see that $\mathbf{CF}^+_0$ is isomorphic to the Lindenbaum algebra for $\mathbf{LI2}$ (defined analogously to $\mathbf{L}$ in Section 5) and $\mathbf{CF}^+$ is the MacNeille completion of $\mathbf{CF}^+_0$. To sum up:

▶ **Proposition 17.** $\mathbf{CF}^+$ *is the MacNeille completion of the Lindenbaum algebra for* $\mathbf{LI2}$.

Thus it turns out *a fortiori* that the essence of Maehara and Okada's proof lies in "MacNeille completion + Girard's reducibility candidates."

6.3 Algebraic cut elimination for LIP_{n+1}

We now proceed to an algebraic proof of cut elimination for $\mathbf{LIP}_{n+1}$ ($n \geq -1$). Although we have already shown cut elimination for $\mathbf{LIP}_{n+1}$ in Section 3, the proof does not formalize in $\mathbf{ID}_{n+1}$ but only in $\mathbf{ID}_{n+2}$. Our goal here is to give another proof that locally formalizes in $\mathbf{ID}_{n+1}$. To this end, we combine the algebraic argument in the previous subsection with the Ω-interpretation technique discussed in Section 4.

Define a Heyting frame by

$$\mathbf{CF}_n \;:=\; \langle \wp_{\mathsf{fin}}(\mathsf{FMP}_{n+1} \cup \overline{\mathsf{FMP}}_n), \mathsf{SEQ}_n, \Rightarrow^{cf}_n, \cup, \emptyset, \setminus\!\!\setminus \rangle,$$

where SEQ_n consists of sequents $\Gamma \Rightarrow \Pi$ with $\Gamma \cup \Pi \subseteq \mathsf{FMP}_{n+1} \cup \overline{\mathsf{FMP}}_n$, and $\Gamma \Rightarrow^{cf}_n (\Sigma \Rightarrow \Pi)$ holds just in case $\mathbf{LI}\Omega_n \vdash^{cf} \Gamma, \Sigma \Rightarrow \Pi$. This yields a *full* Heyting structure $\mathcal{CF}_n := \langle \mathbf{CF}^+_n, \mathsf{Tm}, \mathcal{G}(\mathbf{CF}_n)^{\mathsf{Tm}}, L^{\mathcal{CF}_n} \rangle$, where $p^{\mathcal{CF}_n}(\vec{t}) := \gamma(p(\vec{t}))$.

Let $\mathcal{I} : \mathsf{VAR} \longrightarrow \mathcal{G}(\mathbf{CF}_n)^{\mathsf{Tm}}$ be the identity valuation given by $\mathcal{I}(X)(t) := \gamma(X(t))$. It can be extended to $\mathcal{I} : \mathsf{FMP}_{n+1} \longrightarrow \mathcal{G}(\mathbf{CF}_n)$ as in Section 5, except that

$$\begin{array}{rcl} \mathcal{I}(\forall X.\varphi) & := & \gamma(\{\Delta : \Delta \Rightarrow^{cf}_n \overline{\varphi}(Y) \text{ for some } Y \notin \mathsf{FV}(\Delta)\}), \\ \mathcal{I}(\exists X.\varphi) & := & \{(\Delta \Rightarrow \Lambda) : \overline{\varphi}(Y), \Delta \Rightarrow^{cf}_n \Lambda \text{ for some } Y \notin \mathsf{FV}(\Delta, \Lambda)\}^{\lhd}. \end{array}$$

This interpretation avoids use of second order quantifiers at the meta-level, that is what we have called the Ω-interpretation in Section 5. Notice the use of overlining. The main lemma nevertheless holds with respect to $\mathcal{I}$ (see Appendix B.8):

▶ **Lemma 18.** *$\overline{\varphi} \in \mathcal{I}(\varphi) \subseteq \overline{\varphi}^{\lhd}$ for every $\varphi \in \mathsf{FMP}_n$. $\varphi \in \mathcal{I}(\varphi) \subseteq \varphi^{\lhd}$ for every $\varphi \in \mathsf{FMP}_{n+1}$.*

The following lemma is the hardest part of the proof (see Appendix B.9).

▶ **Lemma 19.** *Suppose that $F \in \mathcal{G}(\mathbf{CF}_n)^{\mathsf{Tm}}$ satisfies $\tau(t) \in F(t) \subseteq \tau(t)^{\lhd}$ for some $\tau(x) \in \mathsf{FMP}_{n+1}$. Then $\mathcal{I}(\forall X.\varphi) \subseteq \mathcal{I}[F/X](\varphi)$ and $\mathcal{I}[F/X](\varphi) \subseteq \mathcal{I}(\exists X.\varphi)$ for every $\forall X.\varphi, \exists X.\varphi \in \mathsf{FMP}_{n+1}$.*

A consequence of the above lemma is that the Ω-interpretation employed here coincides with the ordinary interpretation employed in Section 5. Once the hardest lemma has been proved, the rest is an easy soundness argument.

▶ **Lemma 20.** *If $\mathbf{LIP}_{n+1} \vdash \Gamma \Rightarrow \Pi$, then $\mathcal{I}(\Gamma^\circ) \subseteq \mathcal{I}(\Pi^\circ)$ holds for every term substitution $\circ$.*

Proof. We assume $\circ = id$ for simplicity. The proof proceeds by induction on the derivation.

Suppose that it ends with ($\forall X$ left) with main formula $\forall X.\varphi$ and minor formula $\varphi(\tau)$. Define $F \in \mathcal{G}(\mathbf{CF}_n)^{\mathsf{Tm}}$ by $F(t) = \mathcal{I}(\tau(t))$. By Lemma 18, this F satisfies the precondition of Lemma 19. Hence $\mathcal{I}(\forall X.\varphi) \subseteq \mathcal{I}[F/X](\varphi) = \mathcal{I}(\varphi(\tau))$, where the last equation can be shown by induction on φ. That is sufficient to show soundness of ($\forall X$ left).

Suppose that the derivation ends with:

$$\frac{\Gamma \Rightarrow \varphi(Y)}{\Gamma \Rightarrow \forall X.\varphi}\ (\forall X\ \mathsf{right})$$

Let $\Delta \in \mathcal{I}(\Gamma)$. We may assume that $Y \notin \mathsf{FV}(\Delta)$, since otherwise we can rename Y to a new set variable. By the induction hypothesis and Lemma 18, we have $\Delta \in \mathcal{I}(\varphi(Y)) \subseteq \varphi(Y)^{\lhd}$. Hence $\Delta \in \mathcal{I}(\forall X.\varphi)$. The other cases are similar. ◀

▶ **Lemma 21.** *If $\mathbf{LIP}_{n+1} \vdash \Gamma \Rightarrow \Pi$, then $\mathbf{LI}\Omega_n \vdash^{cf} \Gamma \Rightarrow \Pi$.*

Proof. $\Gamma \in \mathcal{I}(\Gamma) \subseteq \mathcal{I}(\Pi) \subseteq \Pi^{\lhd}$ by Lemmas 20 and 18. ◀

Combining it with Lemma 7 (where FMP is restricted to FMP_{n+1}), we obtain:

▶ **Theorem 22** (Completeness and cut elimination). *The following are equivalent.*

1. $\Gamma \Rightarrow \Pi$ *is provable in* $\mathbf{LIP}_{n+1}$.
2. $\Gamma \Rightarrow \Pi$ *is valid in all Heyting-valued structures.*
3. $\Gamma \Rightarrow \Pi$ *is valid in all full Heyting-valued structures.*
4. $\Gamma \Rightarrow \Pi$ *is cut-free provable in* $\mathbf{LIP}_{n+1}$.

A difference from Theorem 16 is that completeness holds with respect to full structures. It holds because we have managed to avoid use of reducibility candidates. Now a natural question arises: which fragment of **LI2** admits completeness with respect to full structures? Is it related to the proof-theoretic strength? We do not have any answer for the moment.

As before, the algebra $\mathbf{CF}_n^+$ coincides with the MacNeille completion of the Lindenbaum algebra for $\mathbf{LI}\Omega_n$. Hence our proof can be described as "MacNeille completion + Ω-rule + Ω-valuation" in contrast to Maehara and Okada's proof.

What is the gain of an algebraic proof compared with the syntactic one in Section 4? In order to prove Lemma 21, we have only employed provability predicate $\mathbf{LI}\Omega_n$, that is definable in $\mathbf{ID}_{n+1}$. Thus we have saved one inductive definition. Furthermore the above argument can be localized (see Appendix B.11, B.12). This implies a folkloric result:

$$\mathbf{I\Sigma}_1 \vdash \mathsf{CE}(\mathbf{LIP}_n) \leftrightarrow \mathsf{1CON}(\mathbf{ID}_n), \qquad \mathbf{I\Sigma}_1 \vdash \mathsf{CE}(\mathbf{LIP}) \leftrightarrow \mathsf{1CON}(\mathbf{ID}_{<\omega}).$$

To our knowledge, the idea of combining the Ω-rule with a semantic argument to save one inductive definition is due to Aehlig [1], where Tait's computability predicate is used instead of the MacNeille completion. He works on the parameter-free, *negative* fragments of second order Heyting arithmetic without induction, and proves a weak form of cut elimination in the matching **ID**-theories. That is comparable with our result, but ours concerns with the *full* cut elimination theorem for a logical system with the *full* set of connectives (recall that the second order definitions of positive connectives are not available).

Conclusion. In this paper we have brought the Ω-rule into the logical setting, and studied it from an algebraic perspective. We have found an intimate connection with the MacNeille completion (Theorem 10), that is important in two ways. First, it provides a link between syntactic and algebraic approaches to cut elimination. Second, it leads to an algebraic form of the Ω-rule, called the Ω-interpretation, that augments a partial observation by Altenkirch and Coquand [6]. These considerations have led to Theorem 22, the intuitionistic analogue of Takeuti's fundamental cut elimination theorem [32], proved (partly) algebraically.

We prefer the algebraic approach, since it provides a *uniform* perspective to the complicated situation in nonclassical logics. Recall that there is a limitation on MacNeille completions: it does not work for proper intermediate logics (Theorem 2). On the other hand:

- There are infinitely many substructural logics such that the corresponding varieties of algebras are closed under MacNeille completions. As a consequence, these logics, when suitably formalized as sequent calculi, admit an algebraic proof of cut elimination [15, 16].
- There are infinitely many intermediate logics for which *hyper*-MacNeille completions work. As a consequence, these logics, when suitably formalized as *hyper*-sequent calculi, admit an algebraic proof of cut elimination [15, 17].

Thus proving cut elimination amounts to finding a suitable notion of algebraic completion. Although this paper has focused on the easiest case of parameter-free intuitionistic logics, we hope that our approach will eventually lead to an algebraic understanding of *hard* results in proof theory.

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A Definitions of sequent calculi

A.1 Sequent calculi LI2, LIP and LIP$_n$

Sequents of **LI2** consist of formulas in FM. Inference rules are as follows:

$$\frac{}{\Gamma, \varphi \Rightarrow \varphi}\ (\mathsf{id}) \qquad \frac{\Gamma \Rightarrow \varphi \quad \varphi, \Gamma \Rightarrow \Pi}{\Gamma \Rightarrow \Pi}\ (\mathsf{cut})$$

$$\frac{\varphi_i, \Gamma \Rightarrow \Pi}{\varphi_1 \wedge \varphi_2, \Gamma \Rightarrow \Pi}\ (\wedge\ \mathsf{left}) \qquad \frac{\Gamma \Rightarrow \varphi_1 \quad \Gamma \Rightarrow \varphi_2}{\Gamma \Rightarrow \varphi_1 \wedge \varphi_2}\ (\wedge\ \mathsf{right})$$

$$\frac{\varphi_1, \Gamma \Rightarrow \Pi \quad \varphi_2, \Gamma \Rightarrow \Pi}{\varphi_1 \vee \varphi_2, \Gamma \Rightarrow \Pi}\ (\vee\ \mathsf{left}) \qquad \frac{\Gamma \Rightarrow \varphi_i}{\Gamma \Rightarrow \varphi_1 \vee \varphi_2}\ (\vee\ \mathsf{right})$$

$$\frac{\Gamma \Rightarrow \varphi_1 \quad \varphi_2, \Gamma \Rightarrow \Pi}{\varphi_1 \to \varphi_2, \Gamma \Rightarrow \Pi}\ (\to\ \mathsf{left}) \qquad \frac{\varphi_1, \Gamma \Rightarrow \varphi_2}{\Gamma \Rightarrow \varphi_1 \to \varphi_2}\ (\to\ \mathsf{right})$$

$$\frac{\varphi(t), \Gamma \Rightarrow \Pi}{\forall x.\varphi(x), \Gamma \Rightarrow \Pi}\ (\forall x\ \mathsf{left}) \qquad \frac{\Gamma \Rightarrow \varphi(y) \quad y \notin \mathsf{Fv}(\Gamma)}{\Gamma \Rightarrow \forall x.\varphi(x)}\ (\forall x\ \mathsf{right})$$

$$\frac{\varphi(y), \Gamma \Rightarrow \Pi \quad y \notin \mathsf{Fv}(\Gamma, \Pi)}{\exists x.\varphi(x), \Gamma \Rightarrow \Pi}\ (\exists x\ \mathsf{left}) \qquad \frac{\Gamma \Rightarrow \varphi(t)}{\Gamma \Rightarrow \exists x.\varphi(x)}\ (\exists x\ \mathsf{right})$$

$$\frac{\varphi(\tau), \Gamma \Rightarrow \Pi}{\forall X.\varphi(X), \Gamma \Rightarrow \Pi}\ (\forall X\ \mathsf{left}) \qquad \frac{\Gamma \Rightarrow \varphi(Y) \quad Y \notin \mathsf{FV}(\Gamma)}{\Gamma \Rightarrow \forall X.\varphi(X)}\ (\forall X\ \mathsf{right})$$

$$\frac{\varphi(Y), \Gamma \Rightarrow \Pi \quad Y \notin \mathsf{FV}(\Gamma, \Pi)}{\exists X.\varphi(X), \Gamma \Rightarrow \Pi}\ (\exists X\ \mathsf{left}) \qquad \frac{\Gamma \Rightarrow \varphi(\tau)}{\Gamma \Rightarrow \exists X.\varphi(X)}\ (\exists X\ \mathsf{right})$$

LIP (resp. **LIP**$_n$ with $n \geq -1$) is obtained by restricting the formulas to FMP (resp. FMP$_n$).

A.2 Sequent calculi $\mathbf{LI}\Omega_n$

$\mathbf{LI}\Omega_{-1}$ is just **LIP** where cut formulas are restricted to $\mathsf{FMP}_{-1} = \overline{\mathsf{FMP}}_{-1} = \mathsf{Fm}$.

For $n \geq 0$, sequents of $\mathbf{LI}\Omega_n$ consist of formulas in $\mathsf{FMP} \cup \overline{\mathsf{FMP}}_n$ (we will further restrict formulas to $\mathsf{FMP}_{n+1} \cup \overline{\mathsf{FMP}}_n$ in Section 6, but it only has a marginal effect). Inference rules are as follows (where ϑ stands for a formula in $\overline{\mathsf{FMP}}_n$):

$$\frac{}{\Gamma, \varphi \Rightarrow \varphi}\ (\mathsf{id}) \qquad \frac{\Gamma \Rightarrow \vartheta \quad \vartheta, \Gamma \Rightarrow \Pi}{\Gamma \Rightarrow \Pi}\ (\mathsf{cut})$$

$$\frac{\varphi_i, \Gamma \Rightarrow \Pi}{\varphi_1 \wedge \varphi_2, \Gamma \Rightarrow \Pi}\ (\wedge\ \mathsf{left}) \qquad \frac{\Gamma \Rightarrow \varphi_1 \quad \Gamma \Rightarrow \varphi_2}{\Gamma \Rightarrow \varphi_1 \wedge \varphi_2}\ (\wedge\ \mathsf{right})$$

$$\frac{\varphi_1, \Gamma \Rightarrow \Pi \quad \varphi_2, \Gamma \Rightarrow \Pi}{\varphi_1 \vee \varphi_2, \Gamma \Rightarrow \Pi}\ (\vee\ \mathsf{left}) \qquad \frac{\Gamma \Rightarrow \varphi_i}{\Gamma \Rightarrow \varphi_1 \vee \varphi_2}\ (\vee\ \mathsf{right})$$

$$\frac{\Gamma \Rightarrow \varphi_1 \quad \varphi_2, \Gamma \Rightarrow \Pi}{\varphi_1 \to \varphi_2, \Gamma \Rightarrow \Pi}\ (\to\ \mathsf{left}) \qquad \frac{\varphi_1, \Gamma \Rightarrow \varphi_2}{\Gamma \Rightarrow \varphi_1 \to \varphi_2}\ (\to\ \mathsf{right})$$

$$\frac{\varphi(t), \Gamma \Rightarrow \Pi}{\forall x.\varphi(x), \Gamma \Rightarrow \Pi}\ (\forall x\ \mathsf{left}) \qquad \frac{\{\ \Gamma \Rightarrow \varphi(t)\ \}_{t \in \mathsf{Tm}}}{\Gamma \Rightarrow \forall x.\varphi(x)}\ (\omega\ \mathsf{right})$$

$$\frac{\{\ \varphi(t), \Gamma \Rightarrow \Pi\ \}_{t \in \mathsf{Tm}}}{\exists x.\varphi(x), \Gamma \Rightarrow \Pi}\ (\omega\ \mathsf{left}) \qquad \frac{\Gamma \Rightarrow \varphi(t)}{\Gamma \Rightarrow \exists x.\varphi(x)}\ (\exists x\ \mathsf{right})$$

$$\frac{\varphi(\tau), \Gamma \Rightarrow \Pi}{\forall X.\varphi(X), \Gamma \Rightarrow \Pi}\ (\forall X\ \mathsf{left}) \qquad \frac{\Gamma \Rightarrow \varphi(Y) \quad Y \notin \mathsf{FV}(\Gamma)}{\Gamma \Rightarrow \forall X.\varphi(X)}\ (\forall X\ \mathsf{right})$$

$$\frac{\varphi(Y), \Gamma \Rightarrow \Pi \quad Y \notin \mathsf{FV}(\Gamma, \Pi)}{\exists X.\varphi(X), \Gamma \Rightarrow \Pi}\ (\exists X\ \mathsf{left}) \qquad \frac{\Gamma \Rightarrow \varphi(\tau)}{\Gamma \Rightarrow \exists X.\varphi(X)}\ (\exists X\ \mathsf{right})$$

$$\frac{\vartheta(Y), \Gamma \Rightarrow \Pi \quad Y \notin \mathsf{FV}(\Gamma, \Pi)}{\overline{\exists} X.\vartheta(X), \Gamma \Rightarrow \Pi}\ (\overline{\exists} X\ \mathsf{left}) \qquad \frac{\Gamma \Rightarrow \vartheta(Y) \quad Y \notin \mathsf{FV}(\Gamma)}{\Gamma \Rightarrow \overline{\forall} X.\vartheta(X)}\ (\overline{\forall} X\ \mathsf{right})$$

$$\frac{\{\ \Delta, \Gamma \Rightarrow \Pi\ \}_{\Delta \in |\overline{\forall} X.\vartheta|}}{\overline{\forall} X.\vartheta, \Gamma \Rightarrow \Pi}\ (\Omega_k\ \mathsf{left}) \qquad \frac{\Gamma \Rightarrow \vartheta(Y) \quad \{\ \Delta, \Gamma \Rightarrow \Pi\ \}_{\Delta \in |\overline{\forall} X.\vartheta|}}{\Gamma \Rightarrow \Pi}\ (\tilde{\Omega}_k\ \mathsf{left})$$

$$\frac{\{\ \Gamma, \Delta \Rightarrow \Lambda\ \}_{(\Delta \Rightarrow \Lambda) \in |\overline{\exists} X.\vartheta|}}{\Gamma \Rightarrow \overline{\exists} X.\vartheta}\ (\Omega_k\ \mathsf{right}) \qquad \frac{\{\ \Gamma, \Delta \Rightarrow \Lambda\ \}_{(\Delta \Rightarrow \Lambda) \in |\overline{\exists} X.\vartheta|} \quad \vartheta(Y), \Gamma \Rightarrow \Pi}{\Gamma \Rightarrow \Pi}\ (\tilde{\Omega}_k\ \mathsf{right})$$

where $k = 0, \dots, n$, which is determined by the level of the main formula $\overline{Q} X.\vartheta$. Rules $(\tilde{\Omega}_k\ \mathsf{left})$ and $(\tilde{\Omega}_k\ \mathsf{right})$ are subject to the eigenvariable condition ($Y \notin \mathsf{FV}(\Gamma, \Pi)$). Index sets are defined by:

$$\begin{array}{lcl} |\overline{\forall} X.\vartheta(X)| & := & \{\Delta : \mathbf{LI}\Omega_{k-1} \vdash^{cf} \Delta \Rightarrow \vartheta(Y) \text{ for some } Y \notin \mathsf{FV}(\Delta)\} \\ |\overline{\exists} X.\vartheta(X)| & := & \{(\Delta \Rightarrow \Lambda) : \mathbf{LI}\Omega_{k-1} \vdash^{cf} \vartheta(Y), \Delta \Rightarrow \Lambda \text{ for some } Y \notin \mathsf{FV}(\Delta, \Lambda)\}. \end{array}$$

B Proofs

B.1 Two lemmas concerning overlines and substitutions

We first begin with a lemma stating that overlines can be freely removed (though cannot be freely added).

▶ **Lemma 23.** $\mathbf{LI}\Omega_n$ *proves both* $\varphi \Rightarrow \overline{\varphi}$ *and* $\overline{\varphi} \Rightarrow \varphi$ *for every* $\varphi \in \mathsf{FMP}_n$.

Proof. By induction on the structure of φ.

If $\varphi = \forall X.\psi$, then $\varphi \Rightarrow \overline{\varphi}$ follows from the induction hypothesis $\psi \Rightarrow \overline{\psi}$ by rules ($\forall X$ left) and ($\overline{\forall} X$ right). To show the other, we employ rule (Ω_k left) with $k \leq n$:

$$\frac{\{\ \Delta \Rightarrow \forall X.\psi\ \}_{\Delta \in |\overline{\forall} X.\overline{\psi}|}}{\overline{\forall} X.\overline{\psi} \Rightarrow \forall X.\psi}\ (\Omega_k\ \mathsf{left})$$

Let $\Delta \in |\overline{\forall} X.\overline{\psi}|$, that is, $\mathbf{LI}\Omega_{k-1} \vdash^{cf} \Delta \Rightarrow \overline{\psi}(Y)$ with $Y \notin \mathsf{FV}(\Delta)$. By applying ($\mathsf{cut}$) with induction hypothesis $\overline{\psi}(Y) \Rightarrow \psi(Y)$, we obtain $\Delta \Rightarrow \psi(Y)$ in $\mathbf{LI}\Omega_n$. So $\Delta \Rightarrow \forall X.\psi$ by rule ($\forall X$ right). Hence $\overline{\forall} X.\overline{\psi} \Rightarrow_n \forall X.\psi$ by rule (Ω_k left).

The case $\varphi = \exists X.\psi$ is completely dual, and the other cases are straightforward. ◀

We next proceed to a substitution lemma.

Let $\mathsf{ABS}_n := \{\lambda x.\varphi : \varphi \in \mathsf{FMP}_n\}$ and $\overline{\mathsf{ABS}}_n := \{\lambda x.\vartheta : \vartheta \in \overline{\mathsf{FMP}}_n\}$. Given a set substitution $\bullet : \mathsf{VAR} \longrightarrow \overline{\mathsf{ABS}}_n$, we obtain another substitution $\bullet' : \mathsf{VAR} \longrightarrow \mathsf{ABS}_n$ such that $X^\bullet = \overline{\tau}$ implies $X^{\bullet'} = \tau$. Given $\Gamma \subseteq \mathsf{FMP}$, we write Γ' instead of $\Gamma^{\bullet'}$.

▶ **Lemma 24.** *Let* $\bullet : \mathsf{VAR} \longrightarrow \overline{\mathsf{ABS}}_n$ *be a set substitution. Let* $\Gamma_1, \Gamma_2 \Rightarrow \Pi_1 \cup \Pi_2$ *be a sequent such that* $\Gamma_1 \cup \Pi_1 \subseteq \overline{\mathsf{FMP}}_n$, $\Gamma_2 \cup \Pi_2 \subseteq \mathsf{FMP}$ *and either* Π_1 *or* Π_2 *is empty. If* $\mathbf{LI}\Omega_n \vdash \Gamma_1, \Gamma_2 \Rightarrow \Pi_1 \cup \Pi_2$, *then* $\mathbf{LI}\Omega_n \vdash \Gamma_1^\bullet, \Gamma_2' \Rightarrow \Pi_1^\bullet \cup \Pi_2'$.

Proof. By induction on n and the derivation.

(1) The derivation consists of an initial sequent $\Gamma_1, \varphi, \Gamma_2 \Rightarrow \varphi$. If $\varphi \in \mathsf{FMP}_{-1} \subseteq \mathsf{FMP} \cap \overline{\mathsf{FMP}}_n$, there are four possibilities:

$$\Gamma_1^\bullet, \varphi^\bullet, \Gamma_2' \Rightarrow \varphi^\bullet, \qquad \Gamma_1^\bullet, \varphi^\bullet, \Gamma_2' \Rightarrow \varphi', \qquad \Gamma_1^\bullet, \varphi', \Gamma_2' \Rightarrow \varphi^\bullet, \qquad \Gamma_1^\bullet, \varphi', \Gamma_2' \Rightarrow \varphi',$$

which are all provable due to Lemma 23 (notice $\varphi^\bullet = \overline{\varphi}'$). If not, the claim is obvious.

(2) The derivation ends with:

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \vartheta \quad \vartheta, \Gamma_1, \Gamma_2 \Rightarrow \Pi_1 \cup \Pi_2}{\Gamma_1, \Gamma_2 \Rightarrow \Pi_1 \cup \Pi_2}\ (\mathsf{cut})$$

We have:

$$\frac{\Gamma_1^\bullet, \Gamma_2' \Rightarrow \vartheta^\bullet \quad \vartheta^\bullet, \Gamma_1^\bullet, \Gamma_2' \Rightarrow \Pi_1^\bullet \cup \Pi_2'}{\Gamma_1^\bullet, \Gamma_2' \Rightarrow \Pi_1^\bullet \cup \Pi_2'}\ (\mathsf{cut})$$

(3) The derivation ends with:

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \vartheta(Y)}{\Gamma_1, \Gamma_2 \Rightarrow \overline{\forall} X.\vartheta}\ (\overline{\forall} X\ \mathsf{right})$$

By updating $\bullet$ so that $Y^\bullet := Z$ (fresh variable), we have:

$$\frac{\Gamma_1^\bullet, \Gamma_2' \Rightarrow \vartheta(Z)}{\Gamma_1^\bullet, \Gamma_2' \Rightarrow \overline{\forall} X.\vartheta}\ (\overline{\forall} X\ \mathsf{right})$$

(4) The derivation ends with:

$$\frac{\Gamma_1, \varphi(\tau), \Gamma_2 \Rightarrow \Pi_1 \cup \Pi_2}{\Gamma_1, \forall X.\varphi, \Gamma_2 \Rightarrow \Pi_1 \cup \Pi_2}\ (\forall X\ \mathsf{left})$$

We have:

$$\frac{\Gamma_1^\bullet, \varphi(\tau'), \Gamma_2' \Rightarrow \Pi_1^\bullet \cup \Pi_2'}{\Gamma_1^\bullet, \forall X.\varphi, \Gamma_2' \Rightarrow \Pi_1^\bullet \cup \Pi_2'} \ (\forall X \ \mathsf{left})$$

(5) The derivation ends with:

$$\frac{\{\ \Delta, \Gamma_1, \Gamma_2 \Rightarrow \Pi_1 \cup \Pi_2\ \}_{\Delta \in |\overline{\forall} X.\vartheta|}}{\overline{\forall} X.\vartheta, \Gamma_1, \Gamma_2 \Rightarrow \Pi_1 \cup \Pi_2} \ (\Omega_k \ \mathsf{left})$$

Let $\Delta \in |\overline{\forall} X.\vartheta|$. We write $\Delta = \Delta(X_1, \ldots, X_m)$ indicating all occurrences of free set variables. Let $\Sigma = \Delta(Z_1, \ldots, Z_m)$, where variables $Z_1, \ldots, Z_m$ are fresh, so that Σ is invariant under the substitutions $(\bullet, \bullet')$. We still have $\Sigma \in |\overline{\forall} X.\vartheta|$ by the induction hypothesis on n. Hence $\Sigma, \Gamma_1, \Gamma_2 \Rightarrow \Pi_1 \cup \Pi_2$ is among the premises. Now update the substitution $\bullet$ by letting $Z_i^\bullet := X_i$ for $i = 1, \ldots, m$. We then have $\Delta, \Gamma_1^\bullet, \Gamma_2' \Rightarrow \Pi_1^\bullet \cup \Pi_2'$ by the induction hypothesis. Finally rule (Ω_k left) gives us $\overline{\forall} X.\vartheta, \Gamma_1^\bullet, \Gamma_2' \Rightarrow \Pi_1^\bullet \cup \Pi_2'$.

The other rules are similarly treated. ◀

B.2 Lemma 5

We prove a stronger statement to make induction works.

▶ **Lemma 25.** *Let $\Gamma_1, \Gamma_2 \Rightarrow \Pi_1 \cup \Pi_2$ be a sequent such that $\Gamma_1 \cup \Gamma_2 \cup \Pi_1 \cup \Pi_2 \subseteq \mathsf{FMP}_n$ and either Π_1 or Π_2 is empty. If $\mathbf{LIP}_n \vdash \Gamma_1, \Gamma_2 \Rightarrow \Pi_1 \cup \Pi_2$, then $\mathbf{LI}\Omega_n \vdash \overline{\Gamma}_1^\circ, \Gamma_2^\circ \Rightarrow \overline{\Pi}_1^\circ \cup \Pi_2^\circ$ for every term substitution $\circ : \mathsf{Var} \longrightarrow \mathsf{Tm}$.*

Proof. By induction on the derivation.

(1) The derivation consists of an initial sequent $\Gamma_1, \varphi, \Gamma_2 \Rightarrow \varphi$. We have

$$\overline{\Gamma}_1^\circ, \overline{\varphi}^\circ, \Gamma_2^\circ \Rightarrow \overline{\varphi}^\circ, \qquad \overline{\Gamma}_1^\circ, \overline{\varphi}^\circ, \Gamma_2^\circ \Rightarrow \varphi^\circ, \qquad \overline{\Gamma}_1^\circ, \varphi^\circ, \Gamma_2^\circ \Rightarrow \overline{\varphi}^\circ, \qquad \overline{\Gamma}_1^\circ, \varphi^\circ, \Gamma_2^\circ \Rightarrow \varphi^\circ,$$

all provable in $\mathbf{LI}\Omega_n$ by Lemme 23.

(2) The derivation ends with

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \varphi \quad \varphi, \Gamma_1, \Gamma_2 \Rightarrow \Pi_1 \cup \Pi_2}{\Gamma_1, \Gamma_2 \Rightarrow \Pi_1 \cup \Pi_2} \ (\mathsf{cut})$$

We have:

$$\frac{\overline{\Gamma}_1^\circ, \Gamma_2^\circ \Rightarrow \overline{\varphi}^\circ \quad \overline{\varphi}^\circ, \overline{\Gamma}_1^\circ, \Gamma_2^\circ \Rightarrow \overline{\Pi}_1^\circ \cup \Pi_2^\circ}{\overline{\Gamma}_1^\circ, \Gamma_2^\circ \Rightarrow \overline{\Pi}_1^\circ \cup \Pi_2^\circ} \ (\mathsf{cut})$$

(3) The derivation ends with

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \varphi(y)}{\Gamma_1, \Gamma_2 \Rightarrow \forall x.\varphi} \ (\forall x \ \mathsf{right})$$

By the induction hypothesis, we have $\overline{\Gamma}_1^\circ, \Gamma_2^\circ \Rightarrow \overline{\varphi}^\circ(t)$ for every $t \in \mathsf{Tm}$. Hence we obtain $\overline{\Gamma}_1^\circ, \Gamma_2^\circ \Rightarrow \forall x.\overline{\varphi}^\circ$ by rule (ω right).

$\overline{\Gamma}_1^\circ, \Gamma_2^\circ \Rightarrow \forall x.\varphi^\circ$ can be similarly obtained.

From now on we assume $\circ = id$ for simplicity.

(4) The derivation ends with

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \varphi(Y)}{\Gamma_1, \Gamma_2 \Rightarrow \forall X.\varphi} \ (\forall X \ \mathsf{right})$$

We have:

$$\frac{\overline{\Gamma}_1, \Gamma_2 \Rightarrow \overline{\varphi}(Y)}{\overline{\Gamma}_1, \Gamma_2 \Rightarrow \overline{\forall} X.\overline{\varphi}} \ (\overline{\forall} X \ \mathsf{right})$$

Proving $\overline{\Gamma}_1, \Gamma_2 \Rightarrow \forall X.\varphi$ is straightforward.

(4) The derivation ends with

$$\frac{\varphi(\tau), \Gamma_1, \Gamma_2 \Rightarrow \Pi_1 \cup \Pi_2}{\forall X.\varphi, \Gamma_1, \Gamma_2 \Rightarrow \Pi_1 \cup \Pi_2} \ (\forall X \ \mathsf{left})$$

We are going to use (Ω_k left), where k is the level of $\overline{\forall} X.\overline{\varphi}$. So let $\Delta \in |\overline{\forall} X.\overline{\varphi}|$, that is, $\mathbf{LI}\Omega_{k-1} \vdash \Delta \Rightarrow \overline{\varphi}(Y)$ for some $Y \notin \mathsf{FV}(\Delta)$. We have $\Delta \Rightarrow \overline{\varphi}(Y)$ in $\mathbf{LI}\Omega_n$ too, so $\Delta \Rightarrow \overline{\varphi}(\overline{\tau})$ by Lemma 24. Hence we obtain:

$$\frac{\dfrac{\{\ \Delta \Rightarrow \overline{\varphi}(\overline{\tau})\ \}_{\Delta \in |\overline{\forall} X.\overline{\varphi}|}}{\overline{\forall} X.\overline{\varphi} \Rightarrow \overline{\varphi}(\overline{\tau})} \ (\Omega_k \ \mathsf{left}) \qquad \overline{\varphi}(\overline{\tau}), \overline{\Gamma}_1, \Gamma_2 \Rightarrow \overline{\Pi}_1 \cup \Pi_2}{\overline{\forall} X.\overline{\varphi}, \overline{\Gamma}_1, \Gamma_2 \Rightarrow \overline{\Pi}_1 \cup \Pi_2} \ (\mathsf{cut})$$

Proving $\forall X.\varphi, \overline{\Gamma}_1, \Gamma_2 \Rightarrow \overline{\Pi}_1 \cup \Pi_2$ is straightforward.

The other rules are treated similarly. ◀

B.3 Lemma 6

We define the *rank* $\mathsf{rank}(\vartheta) < \omega$ for each $\vartheta \in \overline{\mathsf{FMP}}_n$ as follows:

- $\mathsf{rank}(\bot) = \mathsf{rank}(X(t)) = \mathsf{rank}(p(\vec{t})) = \mathsf{rank}(\overline{\forall} X.\xi) = \mathsf{rank}(\overline{\exists} X.\xi) := 0$,
- $\mathsf{rank}(\vartheta \star \xi) := \max\{\mathsf{rank}(\vartheta), \mathsf{rank}(\xi)\} + 1 \ (\star \in \{\wedge, \vee, \to\})$,
- $\mathsf{rank}(\forall x.\vartheta) = \mathsf{rank}(\exists x.\vartheta) := \mathsf{rank}(\vartheta) + 1$.

Given $\alpha \leq \omega$, we write $\vdash_\alpha \Gamma \Rightarrow \Pi$ if $\Gamma \Rightarrow \Pi$ has a derivation in $\mathbf{LI}\Omega_n$ in which all cut formulas are of rank strictly less than α. Thus $\vdash_0 \Gamma \Rightarrow \Pi$ means that the sequent is cut-free provable in $\mathbf{LI}\Omega_n$.

▶ **Lemma 26.** *Let $m < \omega$. If $\vdash_m \Gamma \Rightarrow \vartheta$ and $\vdash_m \vartheta, \Gamma \Rightarrow \Pi$ with $\mathsf{rank}(\vartheta) \leq m$, then $\vdash_m \Gamma \Rightarrow \Pi$.*

Proof. Let d_l be a derivation of $\Gamma \Rightarrow \vartheta$ and d_r be that of $\vartheta, \Gamma \Rightarrow \Pi$. The proof proceeds by double induction on d_l and d_r. Since it is fairly standard, we only verify a few cases.

(1) d_l ends with:

$$\frac{\{\ \Delta, \Gamma \Rightarrow \vartheta\ \}_{\Delta \in |\overline{\forall} X.\xi|}}{\overline{\forall} X.\xi, \Gamma \Rightarrow \vartheta} \ (\Omega_k \ \mathsf{left})$$

and d_r has conclusion $\vartheta, \overline{\forall} X.\xi, \Gamma \Rightarrow \Pi$. By the induction hypothesis (and weakening), we have $\vdash_m \Delta, \overline{\forall} X.\xi, \Gamma \Rightarrow \Pi$ for every $\Delta \in |\overline{\forall} X.\xi|$. Hence $\vdash_m \overline{\forall} X.\xi, \Gamma \Rightarrow \Pi$ by rule (Ω_k left).

(2) d_l and d_r respectively end with:

$$\frac{\{\ \Gamma \Rightarrow \vartheta(t)\ \}_{t \in \mathsf{Tm}}}{\Gamma \Rightarrow \forall x.\vartheta}\ (\omega\ \mathsf{right}) \qquad \frac{\vartheta(t), \forall x.\vartheta, \Gamma \Rightarrow \Pi}{\forall x.\vartheta, \Gamma \Rightarrow \Pi}\ (\forall x\ \mathsf{left})$$

By the induction hypothesis, we have $\vdash_m \vartheta(t), \Gamma \Rightarrow \Pi$. By noting that $\mathsf{rank}(\vartheta(t)) < m$, we obtain $\vdash_m \Gamma \Rightarrow \Pi$.

(3) d_l and d_r respectively end with:

$$\frac{\Gamma \Rightarrow \vartheta(Y)}{\Gamma \Rightarrow \overline{\forall} X.\vartheta}\ (\overline{\forall} X\ \mathsf{right}) \qquad \frac{\{\ \Delta, \overline{\forall} X.\vartheta, \Gamma \Rightarrow \Pi\ \}_{\Delta \in |\overline{\forall} X.\xi|}}{\overline{\forall} X.\vartheta, \Gamma \Rightarrow \Pi}\ (\Omega_k\ \mathsf{left})$$

By the induction hypothesis (and weakening), we have $\vdash_m \Delta, \Gamma \Rightarrow \Pi$ for every $\Delta \in |\overline{\forall} X.\xi|$. Hence we may apply:

$$\frac{\Gamma \Rightarrow \vartheta(Y) \quad \{\ \Delta, \Gamma \Rightarrow \Pi\ \}_{\Delta \in |\overline{\forall} X.\xi|}}{\Gamma \Rightarrow \Pi}\ (\tilde{\Omega}_k\ \mathsf{left})$$

to conclude $\vdash_m \Gamma \Rightarrow \Pi$. ◀

▶ **Lemma 27.** *Let $m < \omega$. If $\vdash_{m+1} \Gamma \Rightarrow \Pi$, then $\vdash_m \Gamma \Rightarrow \Pi$.*

Proof. By induction on the derivation. Suppose that it ends with:

$$\frac{\Gamma \Rightarrow \vartheta \quad \vartheta, \Gamma \Rightarrow \Pi}{\Gamma \Rightarrow \Pi}\ (\mathsf{cut})$$

By the induction hypothesis, we have $\vdash_m \Gamma \Rightarrow \vartheta$ and $\vdash_m \vartheta, \Gamma \Rightarrow \Pi$. Moreover, $\mathsf{rank}(\vartheta) \leq m$, so $\vdash_m \Gamma \Rightarrow \Pi$ by Lemma 26. ◀

▶ **Lemma 28.** *If $\vdash_\omega \Gamma \Rightarrow \Pi$, then $\vdash_0 \Gamma \Rightarrow \Pi$.*

Proof. By induction on the derivation. Suppose that it ends with:

$$\frac{\Gamma \Rightarrow \vartheta \quad \vartheta, \Gamma \Rightarrow \Pi}{\Gamma \Rightarrow \Pi}\ (\mathsf{cut})$$

By the induction hypothesis, we have $\vdash_0 \Gamma \Rightarrow \vartheta$ and $\vdash_0 \vartheta, \Gamma \Rightarrow \Pi$. Let $\mathsf{rank}(\vartheta) = m$, then we have $\vdash_{m+1} \Gamma \Rightarrow \Pi$. Hence applying Lemma 27 $m + 1$ times, we obtain $\vdash_0 \Gamma \Rightarrow \Pi$. ◀

This completes the proof of Lemma 6.

B.4 Lemma 13

Proof. First of all, observe that any $X \in \mathcal{G}(\mathbf{W})$ is closed under (e), (w) and (c), that is, the following inferences are all valid:

$$\frac{x \circ y \in X}{y \circ x \in X}\ (e) \qquad \frac{x \in X}{x \circ y \in X}\ (w) \qquad \frac{x \circ x \in X}{x \in X}\ (c)$$

We only verify (w). Suppose that $x \in X$ and $z \in X^{\triangleright}$. Then $x\ R\ z$, i.e., $x \circ \varepsilon\ R\ z$. So $\varepsilon\ R\ x \backslash\!\!\backslash z$ and $y\ R\ x \backslash\!\!\backslash z$ by (w). Hence $x \circ y\ R\ z$. Since this holds for every $z \in X^{\triangleright}$, we conclude $x \circ y \in X^{\triangleright\triangleleft} = X$.

Next, we show that $X \to Y \in \mathcal{G}(\mathbf{W})$ whenever $Y \in \mathcal{G}(\mathbf{W})$. This can be shown by proving

$$X \to Y = (X \backslash\!\!\backslash Y^{\rhd})^{\lhd}$$

where $X \backslash\!\!\backslash Y^{\rhd} := \{x \backslash\!\!\backslash z \in W' : x \in X, z \in Y^{\rhd}\}$.

For the forward direction, let $y \in X \to Y$, $x \in X$ and $z \in Y^{\rhd}$. Then $x \circ y \in Y$, so $x \circ y \; R \; z$, hence $y \; R \; x \backslash\!\!\backslash z$. Since this holds for every $x \backslash\!\!\backslash z \in X \backslash\!\!\backslash Y^{\rhd}$, we conclude $y \in (X \backslash\!\!\backslash Y^{\rhd})^{\lhd}$.

For the backward direction, let $y \in (X \backslash\!\!\backslash Y^{\rhd})^{\lhd}$, $x \in X$ and $z \in Y^{\rhd}$. Then we have $y \; R \; x \backslash\!\!\backslash z$, so $x \circ y \; R \; z$. Since this holds for every $z \in Y^{\rhd}$, we have $x \circ y \in Y^{\rhd\lhd} = Y$. Since this holds for every $x \in X$, we conclude $y \in X \to Y$.

We now prove that

$$X \cap Y \subseteq Z \quad \iff \quad X \subseteq Y \to Z$$

holds for every $X, Y, Z \in \mathcal{G}(\mathbf{W})$. For the forward direction, let $x \in X$ and $y \in Y$. Then $x \circ y \in X \cap Y$ by (e) and (w), so $x \circ y \in Z$ by assumption. Since this holds for every $y \in Y$, we have $X \subseteq Y \to Z$.

For the backward direction, let $x \in X \cap Y$. Then $x \circ x \in Z$ by assumption, so $x \in Z$ by (c). This proves $X \cap Y \subseteq Z$. ◀

B.5 Theorem 14

Proof. Let us only verify that the completion is MacNeille. Let $X \in \mathcal{G}(\mathbf{W_A})$. For $\bigvee$-density, we have $X = \gamma\left(\bigcup\{\gamma(a) : a \in X\}\right) = \bigvee\{\gamma(a) : a \in X\}$. For $\bigwedge$-density, notice that $X = \bigcap\{a^{\lhd} : a \in X^{\rhd}\}$ and $\gamma(a) = a^{\lhd}$. ◀

B.6 Lemma 15

We split the lemma into three parts so that the first and second can be reused later. We write $\Rightarrow^{cf}$ for $\Rightarrow^{cf}_{\mathbf{LI2}}$ for short.

▶ **Lemma 29.** *Let $X, Y \in \mathcal{G}(\mathbf{CF})$ and $\varphi, \psi \in \mathsf{FM}$.*

1. *If $\varphi \in X$ and $\psi \in Y$, then $\varphi \wedge \psi \in X \cap Y$.*
2. *If $\varphi \in X$ and $\psi \in Y$, then $\varphi \vee \psi \in \gamma(X \cup Y)$.*
3. *If $X \subseteq \varphi^{\lhd}$ and $\psi \in Y$, then $\varphi \to \psi \in X \to Y$.*
4. *If $X \subseteq \varphi^{\lhd}$ and $Y \subseteq \psi^{\lhd}$, then $X \cap Y \subseteq (\varphi \wedge \psi)^{\lhd}$.*
5. *If $X \subseteq \varphi^{\lhd}$ and $Y \subseteq \psi^{\lhd}$, then $\gamma(X \cup Y) \subseteq (\varphi \vee \psi)^{\lhd}$.*
6. *If $\varphi \in X$ and $Y \subseteq \psi^{\lhd}$, then $X \to Y \subseteq (\varphi \to \psi)^{\lhd}$.*

Proof. (1) We claim that $\varphi \wedge \psi \in X$. Let $(\Gamma \Rightarrow \Pi) \in X^{\rhd}$. Then we have $\varphi, \Gamma \Rightarrow^{cf} \Pi$ by assumption, so $\varphi \wedge \psi, \Gamma \Rightarrow^{cf} \Pi$ by rule ($\wedge$ left). That is, $\varphi \wedge \psi \in X^{\rhd\lhd} = X$. Likewise, $\varphi \wedge \psi \in Y$. So it belongs to $X \cap Y$.

(2) Let $(\Gamma \Rightarrow \Pi) \in (X \cup Y)^{\rhd}$. Then we have $\varphi, \Gamma \Rightarrow^{cf} \Pi$ and $\psi, \Gamma \Rightarrow^{cf} \Pi$ by assumption. Hence $\varphi \vee \psi, \Gamma \Rightarrow^{cf} \Pi$ by rule ($\vee$ left). That is, $\varphi \vee \psi \in (X \cup Y)^{\rhd\lhd} = \gamma(X \cup Y)$.

(3) Let $\Sigma \in X$ and $(\Gamma \Rightarrow \Pi) \in Y^{\rhd}$. Then $\Sigma \Rightarrow^{cf} \varphi$ and $\psi, \Gamma \Rightarrow^{cf} \Pi$ by assumption. Hence $\Sigma, \varphi \to \psi, \Gamma \Rightarrow^{cf} \Pi$ by rule ($\to$ left). Since this holds for any $(\Gamma \Rightarrow \Pi) \in Y^{\rhd}$, we have $\Sigma, \varphi \to \psi \in Y^{\rhd\lhd} = Y$. Since this holds for any $\Sigma \in X$, we conclude $\varphi \to \psi \in X \to Y$.

(4) Let $\Gamma \in X \cap Y$. Then we have $\Gamma \Rightarrow^{cf} \varphi$ and $\Gamma \Rightarrow^{cf} \psi$ by assumption. So $\Gamma \Rightarrow^{cf} \varphi \wedge \psi$ by rule ($\wedge$ right). That is, $\Gamma \in (\varphi \wedge \psi)^{\lhd}$.

(5) Let $\Gamma \in X \cup Y$, say $\Gamma \in X$. Then $\Gamma \Rightarrow^{cf} \varphi$ by assumption. Hence $\Gamma \Rightarrow^{cf} \varphi \vee \psi$ by rule ($\vee$ right). That is, $\Gamma \in (\varphi \vee \psi)^{\lhd}$. This proves that $\gamma(X \cup Y) \subseteq (\varphi \vee \psi)^{\lhd}$.

(6) Let $\Gamma \in X \to Y$. Since $\varphi \in X$ and $Y \subseteq \psi^{\lhd}$ by assumption, we have $\varphi, \Gamma \Rightarrow^{cf} \psi$. Hence $\Gamma \Rightarrow^{cf} \varphi \to \psi$ by rule ($\to$ right). That is, $\Gamma \in (\varphi \to \psi)^{\lhd}$. ◀

▶ **Lemma 30.** *Let $F \in \mathcal{G}(\mathbf{CF})^{\mathsf{Tm}}$ and $\varphi(x) \in \mathsf{FM}$.*

1. *If $\varphi(t) \in F(t)$ for every $t \in \mathsf{Tm}$, then $\forall x.\varphi \in \bigcap_{t \in \mathsf{Tm}} F(t)$.*
2. *If $\varphi(t) \in F(t)$ for every $t \in \mathsf{Tm}$, then $\exists x.\varphi \in \gamma\left(\bigcup_{t \in \mathsf{Tm}} F(t)\right)$.*
3. *If $F(t) \subseteq \varphi(t)^{\lhd}$ for every $t \in \mathsf{Tm}$, then $\bigcap_{t \in \mathsf{Tm}} F(t) \subseteq (\forall x.\varphi)^{\lhd}$.*
4. *If $F(t) \subseteq \varphi(t)^{\lhd}$ for every $t \in \mathsf{Tm}$, then $\gamma\left(\bigcup_{t \in \mathsf{Tm}} F(t)\right) \subseteq (\exists x.\varphi)^{\lhd}$.*

Proof. (1) We claim that $\forall x.\varphi \in F(t)$ for every $t \in \mathsf{Tm}$. Let $(\Gamma \Rightarrow \Pi) \in F(t)^{\rhd}$. Then we have $\varphi(t), \Gamma \Rightarrow^{cf} \Pi$ by assumption, so $\forall x.\varphi, \Gamma \Rightarrow^{cf} \Pi$ by rule ($\forall x$ left). That is, $\forall x.\varphi \in F(t)^{\rhd\lhd} = F(t)$. So it belongs to $\bigcap_{t \in \mathsf{Tm}} F(t)$.

(2) Let $(\Gamma \Rightarrow \Pi) \in \left(\bigcup_{t \in \mathsf{Tm}} F(t)\right)^{\rhd}$. Since $\varphi(y) \in \bigcup_{t \in \mathsf{Tm}} F(t)$ by assumption, where $y \notin \mathsf{Fv}(\Gamma, \Pi)$, we have $\varphi(y), \Gamma \Rightarrow^{cf} \Pi$. Hence $\exists x.\varphi, \Gamma \Rightarrow^{cf} \Pi$ by rule ($\exists x$ left). That is, $\exists x.\varphi \in \left(\bigcup_{t \in \mathsf{Tm}} F(t)\right)^{\rhd\lhd} = \gamma\left(\bigcup_{t \in \mathsf{Tm}} F(t)\right)$.

(3) Let $\Gamma \in \bigcap_{t \in \mathsf{Tm}} F(t)$. Then in particular, it belongs to $F(y)$ with $y \notin \mathsf{Fv}(\Gamma)$. We have $\Gamma \Rightarrow^{cf} \varphi(y)$ by assumption, so $\Gamma \Rightarrow^{cf} \forall x.\varphi$ by rule ($\forall x$ right). That is, $\Gamma \in (\forall x.\varphi)^{\lhd}$.

(4) Let $\Gamma \in \bigcup_{t \in \mathsf{Tm}} F(t)$, say $\Gamma \in F(t)$. Then $\Gamma \Rightarrow^{cf} \varphi(t)$ by assumption. Hence $\Gamma \Rightarrow^{cf} \exists x.\varphi$ by rule ($\exists x$ right). That is, $\Gamma \in (\exists x.\varphi)^{\lhd}$. This proves that $\gamma\left(\bigcup_{t \in \mathsf{Tm}} F(t)\right) \subseteq (\exists x.\varphi)^{\lhd}$. ◀

▶ **Lemma 31.** *Suppose that $\varphi_0[\tau/X] \in \mathcal{V}[F/X](\varphi) \subseteq \varphi_0[\tau/X]^{\lhd}$ for every $F \in \mathcal{D}$ and $\tau \in \mathsf{ABS}$ that match each other, i.e., $\tau(t) \in F(t) \subseteq \tau(t)^{\lhd}$. We then have:*

1. *$\forall X.\varphi_0 \in \mathcal{V}(\forall X.\varphi) \subseteq (\forall X.\varphi_0)^{\lhd}$.*
2. *$\exists X.\varphi_0 \in \mathcal{V}(\exists X.\varphi) \subseteq (\exists X.\varphi_0)^{\lhd}$.*

Proof. (1.) We claim that $\forall X.\varphi_0 \in \mathcal{V}[F/X](\varphi)$ for every $F \in \mathcal{D}$ (that matches τ). Let $(\Gamma \Rightarrow \Pi) \in \mathcal{V}[F/X](\varphi)^{\rhd}$. Then we have $\varphi_0[\tau/X], \Gamma \Rightarrow^{cf} \Pi$ by assumption, so $\forall X.\varphi_0, \Gamma \Rightarrow^{cf} \Pi$ by rule ($\forall X$ left). That is, $\forall X.\varphi_0 \in \mathcal{V}[F/X](\varphi)$ for every $F \in \mathcal{D}$. So it belongs to $\mathcal{V}(\forall X.\varphi)$.

Let $\Gamma \in \mathcal{V}(\forall X.\varphi)$. Then in particular, it belongs to $\mathcal{V}[F_Y/X](\varphi)$, where $Y \notin \mathsf{FV}(\Gamma)$ and $F_Y(t) := \gamma(Y(t))$. Since F_Y matches substitution $[Y/X]$, i.e., $Y(t) \in F_Y(t) \subseteq Y(t)^{\lhd}$, we have $\Gamma \in \varphi_0[Y/X]^{\lhd}$, so $\Gamma \Rightarrow^{cf} \varphi_0[Y/X]$. Hence $\Gamma \Rightarrow^{cf} \forall X.\varphi_0$ by rule ($\forall X$ right). That is, $\Gamma \in (\forall X.\varphi_0)^{\lhd}$.

(2.) Let $(\Gamma \Rightarrow \Pi) \in \left(\bigcup_{F \in \mathcal{D}} \mathcal{V}[F/X](\varphi)\right)^{\rhd}$. Since $\varphi_0[Y/X] \in \mathcal{V}[F_Y/X](\varphi)$ by assumption, where $Y \notin \mathsf{FV}(\Gamma, \Pi)$, we have $\varphi_0[Y/X], \Gamma \Rightarrow^{cf} \Pi$. Hence $\exists X.\varphi_0, \Gamma \Rightarrow^{cf} \Pi$ by rule ($\exists X$ left). That is, $\exists X.\varphi_0 \in \left(\bigcup_{F \in \mathcal{D}} \mathcal{V}[F/X](\varphi)\right)^{\rhd\lhd} = \mathcal{V}(\exists X.\varphi)$.

Let $\Gamma \in \bigcup_{F \in \mathcal{D}} \mathcal{V}[F/X](\varphi)$, say $\Gamma \in V[F/X](\varphi)$ for $F \in \mathcal{D}$ that matches τ. Then $\Gamma \Rightarrow^{cf} \varphi_0[\tau/X]$ by assumption. Hence $\Gamma \Rightarrow^{cf} \exists X.\varphi_0$ by rule ($\exists X$ right). That is, $\Gamma \in (\exists X.\varphi_0)^{\lhd}$. This proves that $\mathcal{V}(\exists X.\varphi) \subseteq (\exists X.\varphi_0)^{\lhd}$. ◀

Now Lemma 15 can be proved as follows.

Proof. By induction on φ. If $\varphi = X(t)$, then $\varphi^{\bullet} = X^{\bullet}(t)$ and we have $X^{\bullet}(t) \in \mathcal{V}(X(t)) \subseteq X^{\bullet}(t)^{\lhd}$ by assumption.

If $\varphi = \psi \star \xi$ with $\star \in \{\vee, \wedge, \to\}$, we have $\psi^\bullet \in \mathcal{V}(\psi) \subseteq \psi^{\bullet\lhd}$ and $\xi^\bullet \in \mathcal{V}(\xi) \subseteq \xi^{\bullet\lhd}$ by the induction hypothesis. Hence $\varphi^\bullet \in \mathcal{V}(\varphi) \subseteq \varphi^{\bullet\lhd}$ by Lemma 29.

If $\varphi = Qx.\psi$ with $Q \in \{\forall, \exists\}$, we have $\psi^\bullet(t) \in \mathcal{V}(\psi(t)) \subseteq \psi^\bullet(t)^\lhd$ by the induction hypothesis. Hence applying Lemma 30 with $F(t) := \mathcal{V}(\psi(t))$, we obtain $\varphi^\bullet \in \mathcal{V}(\varphi) \subseteq \varphi^{\bullet\lhd}$.

If $\varphi = QX.\psi$ with $Q \in \{\forall, \exists\}$, we have $\psi^\bullet[\tau/X] \in \mathcal{V}[F/X](\psi) \subseteq \psi^\bullet[\tau/X]^\lhd$ for any $F \in \mathcal{D}$ that matches τ by the induction hypothesis. Here a notational convention is that $\psi^\bullet[\tau/X]$ is a formula obtained from ψ by replacing X with τ and any other set variable Y with $Y^\bullet$. This new substitution indeed matches $\mathcal{V}[F/X]$. Hence by Lemma 31, we obtain $\varphi^\bullet \in \mathcal{V}(\varphi) \subseteq \varphi^{\bullet\lhd}$. ◀

B.7 An update of Lemma 30

Since we change the proof system from **LI2** to $\mathbf{LI}\Omega_n$, we need to update Lemmas 29 and 30. While the former can be proved for $\mathbf{LI}\Omega_n$ exactly as before, the latter has to be rechecked again, since the rules ($\forall x$ right) and ($\exists x$ left) are replaced by rules (ω right) and (ω left).

▶ **Lemma 32.** *Let $F \in \mathcal{G}(\mathbf{CF}_n)^{\mathsf{Tm}}$ and $\varphi(x) \in \mathsf{FMP}_{n+1} \cup \overline{\mathsf{FMP}}_n$.*

1. *If $\varphi(t) \in F(t)$ for every $t \in \mathsf{Tm}$, then $\forall x.\varphi \in \bigcap_{t\in\mathsf{Tm}} F(t)$.*
2. *If $\varphi(t) \in F(t)$ for every $t \in \mathsf{Tm}$, then $\exists x.\varphi \in \gamma\left(\bigcup_{t\in\mathsf{Tm}} F(t)\right)$.*
3. *If $F(t) \subseteq \varphi(t)^\lhd$ for every $t \in \mathsf{Tm}$, then $\bigcap_{t\in\mathsf{Tm}} F(t) \subseteq (\forall x.\varphi)^\lhd$.*
4. *If $F(t) \subseteq \varphi(t)^\lhd$ for every $t \in \mathsf{Tm}$, then $\gamma\left(\bigcup_{t\in\mathsf{Tm}} F(t)\right) \subseteq (\exists x.\varphi)^\lhd$.*

Proof. (1) and (4) can be shown exactly in the same way as before.

(2) We have $\varphi(t) \in \bigcup_{t\in\mathsf{Tm}} F(t)$ for every $t \in \mathsf{Tm}$. Hence if $(\Gamma \Rightarrow \Pi) \in \left(\bigcup_{t\in\mathsf{Tm}} F(t)\right)^\rhd$, we have $\{\ \varphi(t), \Gamma \Rightarrow_n^{cf} \Pi\ \}_{t\in\mathsf{Tm}}$. Therefore $\exists x.\varphi, \Gamma \Rightarrow_n^{cf} \Pi$ by rule (ω left).

(3) We have $\bigcap_{t\in\mathsf{Tm}} F(t) \subseteq \varphi(t)^\lhd$ for every $t \in \mathsf{Tm}$. Hence if $\Gamma \in \bigcap_{t\in\mathsf{Tm}} F(t)$, we have $\{\ \Gamma \Rightarrow_n^{cf} \varphi(t)\ \}_{t\in\mathsf{Tm}}$. Therefore $\Gamma \Rightarrow_n^{cf} \forall x.\varphi$ by rule (ω right). ◀

B.8 Lemma 18

Let X_0 be a fixed variable. From now on, we fix a function $F \in \mathcal{G}(\mathbf{CF}_n)^{\mathsf{Tm}}$ that satisfies the precondition of Lemma 19, that is, $\tau(t) \in F(t) \subseteq \tau(t)^\lhd$ holds for some $\tau(x) \in \mathsf{FMP}_{n+1}$. Let $\bullet$ be the set substitution defined by $X_0^\bullet(t) := \tau(t)$ and $X^\bullet(t) := X(t)$ for $X \neq X_0$.

Define a valuation $\mathcal{J} : \mathsf{VAR} \longrightarrow \mathcal{G}(\mathbf{CF}_n)^{\mathsf{Tm}}$ by $\mathcal{J}(X_0)(t) := F(t)$ and $\mathcal{J}(X)(t) := \gamma(X(t))$ for $X \neq X_0$. This can be extended to an interpretation $\mathcal{J} : \overline{\mathsf{FMP}}_n \longrightarrow \mathcal{G}(\mathbf{CF}_n)$ as in Section 5 except the treatment of overlined quantifiers. Let us write it down for convenience:

$$
\begin{array}{lcl}
\mathcal{J}(p(\vec{t})) & := & \gamma(p(\vec{t})) \\
\mathcal{J}(X_0(t)) & := & F(t) \\
\mathcal{J}(X(t)) & := & \gamma(X(t)), \qquad (X \neq X_0) \\
\mathcal{J}(\bot) & := & \gamma(\emptyset) \\
\mathcal{J}(\varphi \star \psi) & := & \mathcal{J}(\varphi) \star \mathcal{J}(\psi) \\
\mathcal{J}(\forall x.\varphi(x)) & := & \bigcap_{t\in\mathsf{Tm}} \mathcal{J}(\varphi(t)) \\
\mathcal{J}(\exists x.\varphi(x)) & := & \gamma\left(\bigcup_{t\in\mathsf{Tm}} \mathcal{J}(\varphi(t))\right) \\
\mathcal{J}(\overline{\forall} X.\vartheta) & := & \gamma(\{\Delta : \Delta \Rightarrow_n^{cf} \vartheta(Y) \text{ for some } Y \notin \mathsf{FV}(\Delta)\}, \\
\mathcal{J}(\overline{\exists} X.\vartheta) & := & \{(\Delta \Rightarrow \Lambda) : \vartheta(Y), \Delta \Rightarrow_n^{cf} \Lambda \text{ for some } Y \notin \mathsf{FV}(\Delta, \Lambda)\}^\lhd.
\end{array}
$$

Notice that $\mathcal{I}[F/X_0](\varphi) = \mathcal{J}(\overline{\varphi})$ holds for $\varphi \in \mathsf{FMP}_n$, where $\mathcal{I}$ is the identity valuation defined in Subsection 6.3. In particular, $\mathcal{I}(\varphi) = \mathcal{J}(\overline{\varphi})$ whenever $X_0 \notin \mathsf{FV}(\varphi)$.

Hence Lemma 18 amounts to the following:

▶ **Lemma 33.** **1.** $\vartheta \in \mathcal{J}(\vartheta) \subseteq \vartheta^{\lhd}$ *for every* $\vartheta \in \overline{\mathsf{FMP}}_n$ *with* $X_0 \notin \mathsf{FV}(\vartheta)$.
2. $\mathcal{J}(\vartheta) = \gamma(\vartheta) = \vartheta^{\lhd}$ *for every* $\vartheta \in \overline{\mathsf{FMP}}_n$ *with* $X_0 \notin \mathsf{FV}(\varphi)$.
3. $\varphi \in \mathcal{I}(\varphi) \subseteq \varphi^{\lhd}$ *for every* $\varphi \in \mathsf{FMP}_{n+1}$ *with* $X_0 \notin \mathsf{FV}(\varphi)$.

Proof. (1) By induction on the structure of ϑ. If it is $X(t)$, the claim is obvious. If it is of the form $\vartheta_1 \star \vartheta_2$ with $\star \in \{\wedge, \vee, \to\}$, apply (an analogue of) Lemma 29. If it is of the form $Qx.\xi$ with $Q \in \{\forall, \exists\}$, apply Lemma 32.

Now suppose that $\vartheta = \overline{\forall} X.\xi$. We have $\overline{\forall} X.\xi \Rightarrow_n^{cf} \xi(Y)$, hence $\vartheta \in \mathcal{J}(\vartheta)$. Moreover, if $\Delta \Rightarrow_n^{cf} \xi(Y)$ for some $Y \notin \mathsf{FV}(\Delta)$, then $\Delta \Rightarrow_n^{cf} \overline{\forall} X.\xi$. Hence $\mathcal{J}(\vartheta) \subseteq \vartheta^{\lhd}$.

If $\vartheta = \overline{\exists} X.\xi$, we have $\xi(Y) \Rightarrow_n^{cf} \overline{\exists} X.\xi$, hence $\mathcal{J}(\vartheta) \subseteq \vartheta^{\lhd}$. Moreover, if $\xi(Y), \Delta \Rightarrow_n^{cf} \Lambda$ with $Y \notin \mathsf{FV}(\Delta, \Lambda)$, then $\overline{\exists} X.\xi(X), \Delta \Rightarrow_n^{cf} \Lambda$. Hence $\vartheta \in \mathcal{J}(\vartheta)$.

(2) This is a consequence of (1) and Lemma 6. It just suffices to show that $\vartheta^{\lhd} \subseteq \gamma(\vartheta)$. So suppose that $\Gamma \in \vartheta^{\lhd}$ and $(\Delta \Rightarrow \Lambda) \in \vartheta^{\rhd}$. Then we have $\Gamma \Rightarrow_n^{cf} \vartheta$ and $\vartheta, \Delta \Rightarrow_n^{cf} \Lambda$. By rule ($\mathsf{cut}$) and Lemma 6, we obtain $\Gamma, \Delta \Rightarrow_n^{cf} \Lambda$.

(3) By induction on the structure of φ. Suppose that $\varphi = \forall X.\psi$. We have $\forall X.\psi \Rightarrow_n^{cf} \overline{\psi}(Y)$ by Lemma 23 and rule ($\forall X$ left), so $\varphi \in \mathcal{I}(\varphi)$. Moreover, if $\Delta \Rightarrow_n^{cf} \overline{\psi}(Y)$ with $Y \notin \mathsf{FV}(\Delta)$, then $\Delta \Rightarrow_n^{cf} \psi(Y)$ by Lemma 23, rule (cut) and Lemma 6. Finally rule ($\forall X$ right) yields $\Delta \Rightarrow_n^{cf} \forall X.\psi$. The other cases are similar. ◀

A small variant of the above lemma is needed later.

▶ **Lemma 34.** $\varphi^{\bullet} \in \mathcal{J}(\varphi) \subseteq \varphi^{\bullet\lhd}$ *for every* $\varphi \in \overline{\mathsf{FMP}}_{-1} = \mathsf{FMP}_{-1}$.

It can be proved exactly as in the proof of Lemma 15.

We end with an easy but very crucial lemma that lies at the bottleneck of the whole argument.

▶ **Lemma 35.** *For every* $\overline{\forall} X.\vartheta, \overline{\exists} X.\vartheta \in \overline{\mathsf{FMP}}_n$,

$$\mathcal{J}(\overline{\forall} X.\vartheta) = \gamma(|\overline{\forall} X.\vartheta|), \qquad \mathcal{J}(\overline{\exists} X.\vartheta) = |\overline{\exists} X.\vartheta|^{\lhd}.$$

Proof. Clearly $\gamma(|\overline{\forall} X.\vartheta|)$ is a subset of $\mathcal{J}(\overline{\forall} X.\vartheta)$. To show the other inclusion, Let $\Sigma \Rightarrow_n^{cf} \vartheta(Y)$ with $Y \notin \mathsf{FV}(\Sigma)$ and $(\Gamma \Rightarrow \Pi) \in |\overline{\forall} X.\vartheta|^{\rhd}$. The latter means that we have $\{\ \Delta, \Gamma \Rightarrow_n^{cf} \Pi\ \}_{\Delta \in |\overline{\forall} X.\vartheta|}$. Hence $\Sigma, \Gamma \Rightarrow_n^{cf} \Pi$ by rule ($\tilde{\Omega}_k$ left), where k is the level of $\overline{\forall} X.\vartheta$.

For the existential quantifier, proving the forward inclusion is easy. To show the backward one, let $\Gamma \in |\overline{\exists} X.\vartheta|^{\lhd}$ and $\vartheta(Y), \Sigma \Rightarrow_n^{cf} \Xi$ with $Y \notin \mathsf{FV}(\Sigma, \Xi)$. The former means that we have $\{\ \Gamma, \Delta \Rightarrow_n^{cf} \Lambda\ \}_{(\Delta \Rightarrow \Lambda) \in |\overline{\exists} X.\vartheta|}$. Hence $\Gamma, \Sigma \Rightarrow_n^{cf} \Xi$ by rule ($\tilde{\Omega}_k$ right). ◀

B.9 Lemma 19

Let $X_0, F, \tau, \bullet, \mathcal{J}$ be fixed as in the previous subsection. We now define *asymmetric interpretations* $\mathcal{J}^l, \mathcal{J}^r : \mathsf{FMP}_{n+1} \cup \overline{\mathsf{FMP}}_n \longrightarrow \mathcal{G}(\mathbf{CF}_n)$ as follows.

- $\mathcal{J}^l(\vartheta) = \mathcal{J}^r(\vartheta) := \mathcal{J}(\vartheta)$ if $\vartheta \in \overline{\mathsf{FMP}}_n$,
- $\mathcal{J}^l(\varphi) := \gamma(\varphi^{\bullet})$, $\mathcal{J}^r(\varphi) := \varphi^{\bullet\lhd}$ otherwise.

▶ **Lemma 36** (Asymmetric soundness). *If* $\mathbf{LI}\Omega_n \vdash^{cf} \Gamma \Rightarrow \Pi$*, then* $\mathcal{J}^l(\Gamma) \subseteq \mathcal{J}^r(\Pi)$.

Proof. By induction on the derivation.
(1) The derivation consists of an initial sequent $\Gamma, \varphi \Rightarrow \varphi$. If $\varphi \in \overline{\mathsf{FMP}}_n$, then $\mathcal{J}^l(\Gamma) \cap \mathcal{J}(\varphi) \subseteq \mathcal{J}(\varphi)$ holds obviously. If not, $\mathcal{J}^l(\Gamma) \cap \gamma(\varphi^{\bullet}) \subseteq \varphi^{\bullet\lhd}$ is again obvious.

(2) The derivation ends with:

$$\frac{\{\ \Gamma \Rightarrow \varphi(t)\ \}_{t \in \mathsf{Tm}}}{\Gamma \Rightarrow \forall x.\varphi}\ (\omega\ \mathsf{right})$$

If $\forall x.\varphi \in \overline{\mathsf{FMP}}_n$, we have $\mathcal{J}^l(\Gamma) \subseteq \mathcal{J}(\varphi(t))$ for every $t \in \mathsf{Tm}$ by the induction hypothesis. Hence $\mathcal{J}^l(\Gamma) \subseteq \bigcap_{t \in \mathsf{Tm}} \mathcal{J}(\varphi(t)) = \mathcal{J}(\forall x.\varphi)$. If not, apply Lemma 32 (3) with $F(t) := \mathcal{J}^l(\Gamma)$. Similarly for the case of (ω left).

(3) Suppose that the derivation ends with:

$$\frac{\varphi(t), \Gamma \Rightarrow \Pi}{\forall x.\varphi, \Gamma \Rightarrow \Pi}\ (\forall x\ \mathsf{left})$$

If $\forall x.\varphi \in \overline{\mathsf{FMP}}_n$, it amounts to $\mathcal{J}(\forall x.\varphi) \subseteq \mathcal{J}(\varphi(t))$, that holds by definition. If not, apply Lemma 32 (2) with $F(t) := \mathcal{J}^l(\Gamma) \to \mathcal{J}^r(\Pi)$. Similarly for the case of ($\exists x$ right).

(4) Suppose that the derivation ends with:

$$\frac{\varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}\ (\to\ \mathsf{right})$$

If $\varphi \to \psi \in \overline{\mathsf{FMP}}_n$, then the induction hypothesis $\mathcal{J}(\varphi) \cap \mathcal{J}^l(\Gamma) \subseteq \mathcal{J}(\psi)$ implies $\mathcal{J}^l(\Gamma) \subseteq \mathcal{J}(\varphi) \to \mathcal{J}(\psi)$. If not, we have $\mathcal{J}^l(\varphi) \cap \mathcal{J}^l(\Gamma) \subseteq \mathcal{J}^r(\psi)$ by the induction hypothesis. So $\gamma(\varphi^\bullet) \cap \mathcal{J}^l(\Gamma) \subseteq \psi^{\bullet\lhd}$, using Lemma 34 if necessary. Hence $\mathcal{J}^l(\Gamma) \subseteq \gamma(\varphi^\bullet) \to \psi^{\bullet\lhd} \subseteq (\varphi \to \psi)^{\bullet\lhd}$ by Lemma 29 (6) with $X := \gamma(\varphi^\bullet)$ and $Y := \psi^{\bullet\lhd}$. Other rules for propositional connectives $\{\wedge, \vee, \to\}$ are treated all similarly.

(5) Suppose that the derivation ends with:

$$\frac{\{\ \Delta, \Gamma \Rightarrow \Pi\ \}_{\Delta \in |\overline{\forall} X.\vartheta|}}{\overline{\forall} X.\vartheta, \Gamma \Rightarrow \Pi}\ (\Omega_k\ \mathsf{left})$$

We use Lemma 35. So let $\Delta = \Delta(X_0) \in |\overline{\forall} X.\vartheta|$, $\Gamma_0 \in \mathcal{J}^l(\Gamma)$ and $\Pi_0 \in \mathcal{J}^r(\Pi)^\rhd$. Our goal is to show that $\Delta(X_0), \Gamma_0 \Rightarrow_n^{cf} \Pi_0$, that implies $\mathcal{J}(\overline{\forall} X.\vartheta) \cap \mathcal{J}^l(\Gamma) \subseteq \mathcal{J}^r(\Pi)$ as required.

First, we have $\Delta(Z) \in |\overline{\forall} X.\vartheta|$ with Z fresh. Hence $\Delta(Z), \Gamma \Rightarrow \Pi$ is among the premises. By the induction hypothesis, $\mathcal{J}^l(\Delta(Z)) \cap \mathcal{J}^l(\Gamma) \subseteq \mathcal{J}^r(\Pi)$. Hence by Lemma 33, $\Delta(Z), \Gamma_0 \Rightarrow_n^{cf} \Pi_0$. Finally we obtain $\Delta(X_0), \Gamma_0 \Rightarrow_n^{cf} \Pi_0$ by substituting X_0 for Z.

(6) Suppose that the derivation ends with:

$$\frac{\{\ \Gamma, \Delta \Rightarrow \Lambda\ \}_{(\Delta \Rightarrow \Lambda) \in |\overline{\exists} X.\vartheta|}}{\Gamma \Rightarrow \overline{\exists} X.\vartheta}\ (\Omega_k\ \mathsf{right})$$

We use Lemma 35. So let $(\Delta \Rightarrow \Lambda) \in |\overline{\exists} X.\vartheta|$ and $\Gamma_0 \in \mathcal{J}^l(\Gamma)$.

We may assume that $X_0 \notin \mathsf{FV}(\Delta, \Lambda)$, since otherwise it can be renamed by a fresh variable as in (5) above. By the induction hypothesis, we have $\mathcal{J}^l(\Gamma) \cap \mathcal{J}^l(\Delta) \subseteq \mathcal{J}^r(\Lambda)$. By Lemma 33, we obtain $\Gamma_0, \Delta \Rightarrow_n^{cf} \Lambda$. This shows that $\mathcal{J}^l(\Gamma) \subseteq |\overline{\exists} X.\vartheta|^\lhd = \mathcal{J}(\overline{\exists} X.\vartheta)$.

(7) Suppose that the derivation ends with:

$$\frac{\Gamma \Rightarrow \vartheta(Y)}{\Gamma \Rightarrow \overline{\forall} X.\vartheta}\ (\overline{\forall} X\ \mathsf{right})$$

Let $\Gamma_0 \in \mathcal{J}^l(\Gamma)$. We may assume that $Y \notin \Gamma_0$, as otherwise Y can be renamed. By the induction hypothesis and Lemma 33, $\Gamma_0 \in \mathcal{J}(\vartheta(Y)) \subseteq \vartheta(Y)^{\lhd}$. Hence $\Gamma_0 \Rightarrow_n^{cf} \vartheta(Y)$, so $\Gamma_0 \in \mathcal{J}(\overline{\forall} X.\vartheta)$.

The cases of ($\forall X$ right), ($\overline{\exists} X$ left) and ($\exists X$ left) are all similar.

(8) Suppose that the derivation ends with:

$$\frac{\varphi(\tau), \Gamma \Rightarrow \Pi}{\forall X.\varphi, \Gamma \Rightarrow \Pi}\ (\forall X \text{ left})$$

If $\varphi(\tau) \notin \overline{\mathsf{FMP}}_n$, then $\mathcal{J}^l(\varphi(\tau)) = \gamma(\varphi(\tau^\bullet))$. Hence It suffices to show that $\gamma(\forall X.\varphi) \subseteq \gamma(\varphi(\tau^\bullet))$. Let $(\Delta \Rightarrow \Lambda) \in \varphi(\tau^\bullet)^{\rhd}$. Then $\varphi(\tau^\bullet), \Delta \Rightarrow_n^{cf} \Lambda$, so $\forall X.\varphi, \Delta \Rightarrow_n^{cf} \Lambda$. $\forall X.\varphi \in \gamma(\varphi(\tau))$. If $\varphi(\tau) \in \overline{\mathsf{FMP}}_n$, i.e. if i $\varphi(\tau) \in \overline{\mathsf{FMP}}_{-1}$, then $\gamma(\varphi(\tau^\bullet)) \subseteq \mathcal{J}(\varphi(\tau))$ by Lemma 34. Hence it reduces to the above.

The case of ($\exists X$ right) is similar.

(9) Notice that rule ($\tilde{\Omega}_k$ left) is a combination of (Ω_k left), ($\overline{\forall} X$ right) and (cut) (on a formula in $\overline{\mathsf{FMP}}_n$), which are all sound. Hence there is no need to deal with it separately. The same applies to ($\tilde{\Omega}_k$ right). ◀

Now Lemma 19 is proved as follows.

Proof. Suppose that $F \in \mathcal{G}(\mathbf{CF}_n)^{\mathsf{Tm}}$ satisfies the precondition of Lemma 19, that is, $\tau(t) \in F(t) \subseteq \tau(t)^{\lhd}$ holds for some $\tau(x) \in \mathsf{FMP}_{n+1}$. Our goal is to show that $\mathcal{I}(\forall X.\varphi) \subseteq \mathcal{I}[F/X_0](\varphi(X_0))$ and $\mathcal{I}[F/X_0](\varphi(X_0)) \subseteq \mathcal{I}(\exists X.\varphi)$ for every $\varphi \in \mathsf{FMP}_n$.

For the former, suppose that $\Delta \Rightarrow_n^{cf} \overline{\varphi}(X_0)$ with $X_0 \notin \mathsf{FV}(\Delta)$. Then by Lemmas 33 and 36, $\Delta \in \mathcal{J}^l(\Delta) \subseteq \mathcal{J}(\overline{\varphi}(X_0))$. Moreover, the last set is equivalent to $\mathcal{I}[F/X_0](\varphi(X_0))$. This proves that $\mathcal{I}(\forall X.\varphi) \subseteq \mathcal{I}[F/X_0](\varphi(X_0))$. The latter inclusion can be shown similarly. ◀

B.10 Supplement for Lemma 19

Due to space limitation, we were not able to include the case of second order existential quantifer in the proof of Lemma 19. It is as follows.

Suppose that the derivation ends with ($\exists X$ right) with main formula $\exists X.\varphi$ and minor formula $\varphi(\tau)$. Define $F \in \mathcal{G}(\mathbf{CF}_n)^{\mathsf{Tm}}$ by $F(t) = \mathcal{I}(\tau(t))$. By Lemma 18, this F satisfies the precondition of Lemma 19. Hence $\mathcal{I}(\varphi(\tau)) = \mathcal{I}[F/X](\varphi) \subseteq \mathcal{I}(\exists X.\varphi)$, where the first equation can be shown by induction on φ. That is sufficient to show soundness of ($\exists X$ right).

Suppose that the derivation ends with:

$$\frac{\varphi(Y), \Gamma \Rightarrow \Pi}{\exists X.\varphi, \Gamma \Rightarrow \Pi}\ (\exists X \text{ left})$$

Assume $\Gamma = \emptyset$ just for simplicity. It suffices to show that $\mathcal{I}(\Pi)^{\rhd} \subseteq \mathcal{I}(\exists X.\varphi)^{\rhd}$. So let $(\Delta \Rightarrow \Lambda) \in \mathcal{I}(\Pi)^{\rhd}$. We may assume that $Y \notin \mathsf{FV}(\Delta, \Lambda)$, since otherwise we can rename Y to a new set variable. By (the contrapositions of) the induction hypothesis and Lemma 18, we have $(\Delta \Rightarrow \Lambda) \in \mathcal{I}(\Pi)^{\rhd} \subseteq \mathcal{I}(\varphi(Y))^{\rhd} \subseteq \overline{\varphi}(Y)^{\rhd}$, that is, $\overline{\varphi}(Y), \Delta \Rightarrow_n^{cf} \Lambda$. This proves $(\Delta \Rightarrow \Lambda) \in \mathcal{I}(\exists X.\varphi)^{\rhd}$. The other cases are similar.

B.11 A sketch of local formalization in ID_{n+1} (1)

We here outline how to formalize our proof of cut elmination for $\mathbf{LIP}_{n+1}$ within $\mathbf{ID}_{n+1}$. It should be stressed that the argument below is just a sketch, not intended to be a rigorous formalization.

Our proof in Section 6.3 relies on provability predicates $\mathbf{LI}\Omega_k$ for $k = -1, \dots, n$, which are definable in $\mathbf{ID}_{n+1}$. Concretely, there are formulas $\mathsf{LI}\Omega_k(x)$ whose intended meaning is

$$\mathsf{LI}\Omega_k(\ulcorner\Gamma \Rightarrow \Pi\urcorner) \iff \mathbf{LI}\Omega_k \vdash \Gamma \Rightarrow \Pi,$$

where $\ulcorner\cdot\urcorner$ is a suitable coding function. What does it mean by "intended meaning"? Establishing the equivalence at the meta-level (i.e., outside $\mathbf{ID}_{n+1}$) is neither necessary nor sufficient. What we have to do is to prove intensional properties of $\mathsf{LI}\Omega_k(x)$ within $\mathbf{ID}_{n+1}$ as many as needed.

Likewise, we are given formulas $\mathsf{LI}\Omega_k^{cf}$, LIP_{n+1} and LIP_{n+1}^{cf} with obvious meanings. Notice that LIP_{n+1} and LIP_{n+1}^{cf} are actually Σ_1^0 formulas of $\mathbf{PA}$, since the proof system is finitary. We have claimed in Section 4 that

$$\mathbf{ID}_{n+2} \vdash \forall x.\mathsf{LIP}_{n+1}(x) \to \mathsf{LIP}_{n+1}^{cf}(x).$$

The same cannot be proved in $\mathbf{ID}_{n+1}$, since cut elimination for $\mathbf{LIP}_{n+1}$ implies 1-consistency of $\mathbf{ID}_{n+1}$ as explained in Section 3. We are thus led to a local formalization of cut elimination.

Suppose that a derivation d of $\Gamma \Rightarrow \Pi$ in $\mathbf{LIP}_{n+1}$ is given. The core of the proof in Subsection 6.3 is a semantic argument based on the identity valuation $\mathcal{I}$. For each $\mathbf{LIP}$-formula φ, we define an $\mathbf{ID}$-formula $\mathsf{Id}_\varphi(x)$ such that

$$\mathsf{Id}_\varphi(\ulcorner\Delta\urcorner) \iff \Delta \in \mathcal{I}(\varphi).$$

This is possible at all because of the Ω-interpretation technique. Since there are only finitely many formulas occurring in d, we obtain a single formula $\mathsf{Id}(x, y)$ such that

$$\mathsf{Id}(\ulcorner\Delta\urcorner, \ulcorner\Gamma\urcorner) \iff \Delta \in \mathcal{I}(\Gamma).$$

Now the hardest part is to build a derivation of soundness (Lemma 20):

$$\mathbf{ID}_{n+1} \vdash \forall x. \forall\ulcorner\circ\urcorner.\, \mathsf{Id}(x, \ulcorner\Gamma^\circ\urcorner) \to \mathsf{Id}(x, \ulcorner\Pi^\circ\urcorner)$$

primitive recursively in d. By formalizing Lemma 33, we would then be able to obtain a derivation of $\mathbf{ID}_{n+1} \vdash \mathsf{LIP}_{n+1}^{cf}(\ulcorner\Gamma \Rightarrow \Pi\urcorner)$. If $\mathbf{ID}_{n+1}$ is 1-consistent, then $\mathbf{LIP}_{n+1} \vdash^{cf} \Gamma \Rightarrow \Pi$ follows. That is why cut elimination for $\mathbf{LIP}_{n+1}$ is equivalent to 1-consistency of $\mathbf{ID}_{n+1}$.

In formalizing soundness, the main obstacle is Lemma 19, which in turn relies on Lemma 36. We will argue that it is indeed formalizable in the next subsection.

B.12 A sketch of local formalization in ID_{n+1} (2)

Before addressing Lemma 36, a bit of preliminary is needed.

We fix a variable X_0 and a formula $\varphi(\tau)$ that occurs in d. Lemma 36 is invoked by letting $F := \mathcal{I}(\tau)$ and by considering a cut-free derivation of $\Delta \Rightarrow \overline{\varphi}(X_0)$ in $\mathbf{LI}\Omega_n$ with $X_0 \notin \mathsf{FV}(\Delta)$ (actually there is also a dual case, but let us forget about it for simplicity). So, there is a concrete $\mathbf{ID}$-formula $\mathsf{F}(x, y)$ whose intended meaning is that $\mathsf{F}(\ulcorner\Delta\urcorner, \ulcorner t\urcorner)$ holds iff $\Delta \in F(t)$ iff $\Delta \in \mathcal{I}(\tau(t))$.

Given a set variable X, let

$$\mathsf{Gamma}(\ulcorner\Delta\urcorner, X) \ := \ \forall\ulcorner\Gamma \Rightarrow \Pi\urcorner.\,[\forall\ulcorner\Sigma\urcorner \in X.\mathsf{LI}\Omega_n^{cf}(\ulcorner\Sigma, \Gamma \Rightarrow \Pi\urcorner)] \to \mathsf{LI}\Omega_n^{cf}(\ulcorner\Delta, \Gamma \Rightarrow \Pi\urcorner).$$

The intended meaning is that $\mathsf{Gamma}(\ulcorner\Delta\urcorner, X)$ iff $\Delta \in \gamma(X)$.

We define the set of *skeletons* by the following grammar:

$$\alpha, \beta ::= X_0 \mid \Box \mid \alpha \wedge \beta \mid \alpha \vee \beta \mid \alpha \to \beta \mid \forall\alpha \mid \exists\alpha.$$

Each formula $\vartheta \in \overline{\mathsf{FMP}}_n$ is translated to a skeleton ϑ^{sk} as follows:

$$\begin{array}{lcll} \vartheta^{sk} & := & \Box & \text{if } X_0 \notin \mathsf{FV}(\vartheta), \\ (X_0(t))^{sk} & := & X_0, & \\ (\vartheta \star \xi)^{sk} & := & \vartheta^{sk} \star \xi^{sk}, & \\ (Qx.\vartheta)^{sk} & := & Q\vartheta^{sk}. & \end{array}$$

where $\star \in \{\wedge, \vee, \to\}$ and $Q \in \{\forall, \exists\}$.

For each skeleton α, we construct an **ID**-formula $\mathsf{J}_\alpha(x, y)$ such that

$$\mathsf{J}_\alpha(\ulcorner\Delta\urcorner, \ulcorner\vartheta\urcorner) \quad \Longleftrightarrow \quad \vartheta^{sk} = \alpha \text{ and } \Delta \in \mathcal{J}(\vartheta).$$

The construction proceeds by induction on α.

$$\begin{array}{lcl} \mathsf{J}_\Box(\ulcorner\Delta\urcorner, \ulcorner\vartheta\urcorner) & \Leftrightarrow & \mathsf{Gamma}(\ulcorner\Delta\urcorner, (\lambda x.x = \ulcorner\vartheta\urcorner)), \\ \mathsf{J}_{X_0}(\ulcorner\Delta\urcorner, \ulcorner X_0(t)\urcorner) & \Leftrightarrow & \mathsf{F}(\ulcorner\Delta\urcorner, \ulcorner t\urcorner), \\ \mathsf{J}_{\alpha\wedge\beta}(\ulcorner\Delta\urcorner, \ulcorner\vartheta \wedge \xi\urcorner) & \Leftrightarrow & \mathsf{J}_\alpha(\ulcorner\Delta\urcorner, \ulcorner\vartheta\urcorner) \wedge \mathsf{J}_\beta(\ulcorner\Delta\urcorner, \ulcorner\xi\urcorner), \\ \mathsf{J}_{\alpha\vee\beta}(\ulcorner\Delta\urcorner, \ulcorner\vartheta \vee \xi\urcorner) & \Leftrightarrow & \mathsf{Gamma}(\ulcorner\Delta\urcorner, \lambda x.\mathsf{J}_\alpha(x, \ulcorner\vartheta\urcorner) \vee \mathsf{J}_\beta(x, \ulcorner\xi\urcorner)), \\ \mathsf{J}_{\alpha\to\beta}(\ulcorner\Delta\urcorner, \ulcorner\vartheta \to \xi\urcorner) & \Leftrightarrow & \forall\ulcorner\Sigma\urcorner.\mathsf{J}_\alpha(\ulcorner\Sigma\urcorner, \ulcorner\vartheta\urcorner) \to \mathsf{J}_\beta(\ulcorner\Sigma, \Delta\urcorner, \ulcorner\xi\urcorner), \\ \mathsf{J}_{\forall\alpha}(\ulcorner\Delta\urcorner, \ulcorner\forall x.\vartheta\urcorner) & \Leftrightarrow & \forall\ulcorner t\urcorner.\mathsf{J}_\alpha(\ulcorner\Delta\urcorner, \ulcorner\vartheta(t)\urcorner), \\ \mathsf{J}_{\exists\alpha}(\ulcorner\Delta\urcorner, \ulcorner\exists x.\vartheta\urcorner) & \Leftrightarrow & \mathsf{Gamma}(\ulcorner\Delta\urcorner, \lambda x.\exists\ulcorner t\urcorner.\mathsf{J}_\alpha(x, \ulcorner\vartheta(t)\urcorner)). \end{array}$$

The first line states that $\mathcal{J}(\vartheta) = \gamma(\{\vartheta\})$ whenever $X_0 \notin \mathsf{FV}(\vartheta)$. This is just as shown in Lemma 33 (2).

A set ϑ of skeletons is *saturated* if it contains $\Box$ and is closed under subexpressions. If ϑ is finite, then there is a single formula $\mathsf{J}(x, y)$ whose intended meaning is

$$\mathsf{J}(\ulcorner\Delta\urcorner, \ulcorner\vartheta\urcorner) \quad \Longleftrightarrow \quad \vartheta^{sk} \in \Phi \text{ and } \Delta \in \mathcal{J}(\vartheta),$$

Moreover, J can be easily extended to J^l and J^r that cover all formulas ψ in FMP_{n+1} (recall that $\mathcal{J}^l(\psi) = \gamma(\psi^\bullet)$ and $\mathcal{J}^r(\psi) = \psi^{\bullet\lhd}$).

Below is a crucial observation:

► **Lemma 37.** *Let Φ be a saturated set of skeletons which contains $\overline{\varphi}(X_0)^{sk}$. For any Δ with $X_0 \notin \mathsf{FV}(\Delta)$, for any cut-free derivation of $\Delta \Rightarrow \overline{\varphi}(X_0)$ in $\mathbf{LI}\Omega_n$, and for any formula $\vartheta \in \overline{\mathsf{FMP}}_n$ occurring in it, the skeleton ϑ^{sk} belongs to Φ.*

Actually the above lemma needs a proviso. In rule ($\tilde{\Omega}_k$ left), the index set $|\overline{\forall}X.\vartheta|$ can be restricted to those Δ such that $X_0 \notin \mathsf{FV}(\Delta)$. This fact is implicitly used in the proof of Lemma 36. Similarly for ($\tilde{\Omega}_k$ right). The above lemma holds with this modification.

Let Φ be the least saturated set containing $\overline{\varphi}(X_0)^{sk}$. It is a finite set, and it gives us formulas J, J^l and J^r in $\mathbf{ID}_{n+1}$ that work for all relevant derivations in $\mathbf{LI}\Omega_n$. Now our task, formalization of Lemma 19, amounts to proving:

$$\mathbf{ID}_{n+1} \vdash \forall\ulcorner\Delta\urcorner.\, \mathsf{LI}\Omega_n(\ulcorner\Delta \Rightarrow \overline{\varphi}(X_0)\urcorner) \wedge (X_0 \notin \mathsf{FV}(\Delta)) \to \forall x.\mathsf{J}^l(x, \ulcorner\Delta\urcorner) \to \mathsf{J}^r(x, \ulcorner\overline{\varphi}(X_0)\urcorner).$$

That would be an extremely tedious work.