THE JOURNAL OF SYMBOLIC LOGIC
Volume 00, Number 0, XXX 0000

WHICH STRUCTURAL RULES ADMIT CUT ELIMINATION? — AN ALGEBRAIC CRITERION

KAZUSHIGE TERUI

Abstract. Consider a general class of structural inference rules such as exchange, weakening, contraction and their generalizations. Among them, some are harmless but others do harm to cut elimination. Hence it is natural to ask under which condition cut elimination is preserved when a set of structural rules is added to a structure-free logic. The aim of this work is to give such a condition by using algebraic semantics.

We consider full Lambek calculus (**FL**), i.e., intuitionistic logic without any structural rules, as our basic framework. Residuated lattices are the algebraic structures corresponding to **FL**. In this setting, we introduce a criterion, called the propagation property, that can be stated both in syntactic and algebraic terminologies. We then show that, for any set $\mathcal{R}$ of structural rules, the cut elimination theorem holds for **FL** enriched with $\mathcal{R}$ if and only if $\mathcal{R}$ satisfies the propagation property.

As an application, we show that any set $\mathcal{R}$ of structural rules can be "completed" into another set $\mathcal{R}^\star$, so that the cut elimination theorem holds for **FL** enriched with $\mathcal{R}^\star$, while the provability remains the same.

§1. Introduction. Gentzen's sequent calculus has been playing a central role in proof theory and logic in computer science. Its main advantage is the *cut elimination theorem*, which not only yields a lot of important corollaries, but also is the main target of research by itself in the proofs-as-programs paradigm of computation. Another remarkable feature of sequent calculus is that the inference rules concerning the structure of hypotheses/conclusions can be distinguished from those concerning the use of logical connectives. The former are called *structural rules*. One can therefore obtain various logical systems by selecting a suitable set of structural rules, while keeping the logical connectives and the associated inference rules unchanged. Studies of structural rules have a long history, but it is relatively recently that their relevance to computation has been pointed out (especially in the framework of *linear logic* [**??**, **??**]) and a systematic study of such logics with selected structural rules has been undertaken (under the name *substructural logics* [**??**, **??**]).

Gentzen's original sequent calculus contains three structural rules:

0022-4812/00/0000-0000/$00.00

Partially supported by Grant-in-Aid for Scientific Research, MEXT, Japan.

Exchange: Weakening: Contraction:

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Gamma, \beta, \alpha, \Delta \Rightarrow \gamma}\ \mathbf{e} \qquad \frac{\Gamma, \Delta \Rightarrow \gamma}{\Gamma, \alpha, \Delta \Rightarrow \gamma}\ \mathbf{w} \qquad \frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \gamma}{\Gamma, \alpha, \Delta \Rightarrow \gamma}\ \mathbf{c}$$

where α, β and γ stand for formulas, and Γ and Δ for sequences of formulas (we only consider intuitionistic sequents in this paper). In addition, one can also consider other non-standard structural rules such as:

Expansion (cf. [**??**]): Mingle (cf. [**??**]):

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \gamma}{\Gamma, \alpha, \alpha, \Delta \Rightarrow \gamma}\ \mathbf{exp} \qquad \frac{\Gamma, \Sigma, \Delta \Rightarrow \gamma \quad \Gamma, \Theta, \Delta \Rightarrow \gamma}{\Gamma, \Sigma, \Theta, \Delta \Rightarrow \gamma}\ \mathbf{min}$$

(See also [**??**, **??**] for a detailed account.) Among them, some are harmless but others cause failure of cut elimination. In fact, cut elimination is very sensitive to the choice of structural rules:

- In general, sequent calculi with Contraction but without Exchange do not enjoy cut elimination. One way to recover cut elimination is to generalize Contraction to the one for *sequences* of formulas:

$$\frac{\Gamma, \Sigma, \Sigma, \Delta \Rightarrow \gamma}{\Gamma, \Sigma, \Delta \Rightarrow \gamma}\ \mathbf{seq\text{-}c}$$

- Expansion and Mingle are derivable from each other. However, Mingle admits cut elimination whereas Expansion does not.

In view of these intricacies, it is natural to look for some general criteria for a set of structural rules to admit cut elimination. Such criteria, if given on a suitable level of abstraction, would also help us understand the nature of cut elimination from a deeper point of view. The aim of this paper is to give a criterion for cut elimination by using algebraic semantics.

Our criterion, called the *propagation property*, originates in Girard's *naturality test*. In Appendix C.4 of [**??**], Girard proposes a test for naturality of logical principles (i.e., structural rules). Roughly, a principle (structural rule) R passes Girard's test if the following is true for every phase space $(M, \bot)$: whenever R holds for all *atomic facts* $\{x\}^{\bot\bot}$, it also holds for arbitrary *facts* $X^{\bot\bot}$ (here, $x \in M$ and $X \subseteq M$). In other words, a principle passes the test when it *propagates* from atomic facts to all facts. Based on this test, Weakening and Contraction are justified (in the presence of Exchange) whereas Expansion is abandoned (which is called a *Broccoli*, a completely artificial construct). Moreover, a connection between this test and cut elimination is hinted (see footnote 36 in Appendix C.4). Our propagation property is obtained by making Girard's test more precise and more general.

Actually, the propagation property we propose has two equivalent forms, one syntactic and the other semantic. A set of rules satisfies the syntactic propagation property if, roughly, it propagates from propositional variables to their disjunctions and fusions (i.e. multiplicative conjunctions). Similarly, it satisfies the semantic propagation property if it propagates from an arbitrary set of elements to their (infinite) joins and multiplications in all residuated lattices. Our main contribution is then the following characterization of cut elimination, which clarifies and confirms Girard's idea:

- The cut elimination theorem holds for a structure-free sequent calculus enriched with a set $\mathcal{R}$ of structural rules iff $\mathcal{R}$ satisfies the syntactic propagation property iff $\mathcal{R}$ satisfies the semantic propagation property.

As an application, we show that every set of structural rules can be *completed* into another set that enjoys cut elimination without changing provability.

In Section 2, we review *full Lambek calculus* [**??**, **??**, **??**], i.e., intuitionistic logic without any structural rules, as our basic system. More precisely, we consider its 0-free fragment, denoted by $\mathbf{FL}^+$. We then introduce (additive) structural rules on $\mathbf{FL}^+$ in a general format. In Section 3, we introduce a syntactic version of the propagation property, and show that cut elimination implies the syntactic propagation property. In Section 4, we review the *residuated lattices* (see [**??**, **??**]), that is, algebraic structures for $\mathbf{FL}^+$, and introduce a semantic version of the propagation property. We then show that the syntactic propagation property implies the semantic one. In Section 5, we consider the *phase structures* (see for instance [**??**, **??**, **??**]), a particular class of residuated lattices, and describe a useful construction of phase structures due to Okada [**??**, **??**, **??**] in which the validity of a formula directly implies its cut-free provability. If our choice $\mathcal{R}$ of structural rules satisfies the semantic propagation property, then Okada's phase structure, defined on the basis of $\mathcal{R}$, becomes a model of $\mathcal{R}$. Therefore, together with the soundness theorem, we obtain the cut elimination theorem. Section 6 is devoted to the completion of structural rules as mentioned above.

Related work. Sufficient conditions on cut elimination have been considered in [**??**, **??**, **??**]; in particular [**??**] and [**??**] discuss various structural rules. The papers [**??**, **??**, **??**] give necessary and sufficient conditions for logics with various logical connectives, but only those with full structural rules.

After submission of this paper, we have extended our analysis to a more general class of structural rules, and moreover to a certain class of logical connectives too [**??**]. Our recent work [**??**] also discusses quantification rules (in the setting of multiple conclusion sequent calculi with Exchange) and gives a syntactic criterion.

§2. Full Lambek Calculus and Structural Rules. The *formulas of* $\mathbf{FL}^+$ are built from propositional variables $p, q, r, \dots$ and constants 1 (unit), $\top$ (true) and $\bot$ (false) by using binary logical connectives $\cdot$ (fusion), $\backslash$ (left implication), $/$ (right implication), $\wedge$ (conjunction) and $\vee$ (disjunction). The set of formulas is denoted by $\mathcal{F}$. Small Greek letters $\alpha, \beta, \dots$ range over $\mathcal{F}$. For simplicity, we do not consider negation and 0 in this paper; it is, however, easy to incorporate them if one wishes. We use $\rightarrow$ as synonym for $\backslash$, because it can be read as implication more naturally. The other implication $/$ is not much used in this paper. For the convenience of the reader who is familiar with linear logic, the correspondence with (intuitionistic noncommutative) linear logic connectives is given in Table 1. Notice in particular that $\bot$ of $\mathbf{FL}^+$ corresponds to $\mathbf{0}$ of linear logic.

A *sequent of* $\mathbf{FL}^+$ is of the form $\alpha_1, \dots, \alpha_n \Rightarrow \beta$. In the sequel, $\Gamma, \Delta, \dots$ stand for finite sequences of formulas, and $\emptyset$ for the empty sequence.

$\mathbf{FL}^+$	1	$\top$	$\bot$	$\cdot$	$\backslash\ (\to)$	$/$	$\wedge$	$\vee$
Linear Logic	$\mathbf{1}$	$\top$	$\mathbf{0}$	$\otimes$	$\multimap$	$\circ\!-$	$\&$	$\oplus$

TABLE 1. Correspondence with linear logic connectives

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta_1, \alpha, \Delta_2 \Rightarrow \gamma}{\Delta_1, \Gamma, \Delta_2 \Rightarrow \gamma}\ \mathbf{cut} \qquad \frac{}{\alpha \Rightarrow \alpha}\ \mathbf{init} \qquad \frac{}{\Rightarrow 1}\ 1r$$

$$\frac{\Gamma_1, \alpha, \beta, \Gamma_2 \Rightarrow \gamma}{\Gamma_1, \alpha \cdot \beta, \Gamma_2 \Rightarrow \gamma}\ \cdot l \qquad \frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta}\ \cdot r \qquad \frac{\Gamma_1, \Gamma_2 \Rightarrow \delta}{\Gamma_1, 1, \Gamma_2 \Rightarrow \delta}\ 1l$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta_1, \beta, \Delta_2 \Rightarrow \delta}{\Delta_1, \Gamma, \alpha\backslash\beta, \Delta_2 \Rightarrow \delta}\ \backslash l \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\backslash\beta}\ \backslash r \qquad \frac{}{\Gamma_1, \bot, \Gamma_2 \Rightarrow C}\ \bot l$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta_1, \beta, \Delta_2 \Rightarrow \delta}{\Delta_1, \beta/\alpha, \Gamma, \Delta_2 \Rightarrow \delta}\ /l \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta/\alpha}\ /r \qquad \frac{}{\Gamma \Rightarrow \top}\ \top r$$

$$\frac{\Gamma_1, \alpha, \Gamma_2 \Rightarrow \delta \quad \Gamma_1, \beta, \Gamma_2 \Rightarrow \delta}{\Gamma_1, \alpha \vee \beta, \Gamma_2 \Rightarrow \delta}\ \vee l \qquad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ \vee r_1 \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ \vee r_2$$

$$\frac{\Gamma_1, \alpha, \Gamma_2 \Rightarrow \delta}{\Gamma_1, \alpha \wedge \beta, \Gamma_2 \Rightarrow \delta}\ \wedge l_1 \qquad \frac{\Gamma_1, \beta, \Gamma_2 \Rightarrow \delta}{\Gamma_1, \alpha \wedge \beta, \Gamma_2 \Rightarrow \delta}\ \wedge l_2 \qquad \frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ \wedge r$$

FIGURE 1. Inference Rules of $\mathbf{FL}^+$

Given a set Φ of formulas, we say that a sequent $\Gamma \Rightarrow \alpha$ is *deducible* from Φ in $\mathbf{FL}^+$ and write $\Phi \vdash_{\mathbf{FL}^+} \Gamma \Rightarrow \alpha$, if it can be derived from sequents of the form $\Rightarrow \beta$ with $\beta \in \Phi$ by using the inference rules in Figure 1. We also say that α is *deducible* from Φ in $\mathbf{FL}^+$ and write $\Phi \vdash_{\mathbf{FL}^+} \alpha$, if Γ is empty. Furthermore, α is simply said to be *provable* in $\mathbf{FL}^+$, if Φ is also empty. (See [**??**, **??**] for more information).

When it is necessary to indicate variables $p_1, \dots, p_m$ that might possibly occur in a formula α, we shall use the notation $\alpha[p_1, \dots, p_m]$, or $\alpha[\vec{p}]$ for short. The formula obtained from $\alpha[p_1, \dots, p_m]$ by substituting β_i for each p_i is denoted by $\alpha[\beta_1, \dots, \beta_m]$, or $\alpha[\vec{\beta}]$. Similar notation is used for sequences of formulas (and structural rules introduced below).

For $\Sigma \equiv \alpha_1, \dots, \alpha_n$ $(n \geq 1)$, define

$$*\Sigma \equiv \alpha_1 \cdot \dots \cdot \alpha_n,$$
$$\bigvee \Sigma \equiv \alpha_1 \vee \dots \vee \alpha_n.$$

Notice here that parentheses are omitted; it is justified by the associativity of $\cdot$ and $\vee$ in $\mathbf{FL}^+$. If $\Sigma \equiv \emptyset$, we define $*\Sigma \equiv 1$ and $\bigvee \Sigma \equiv \bot$.

Listed below are some elementary facts used in this paper: for any set Φ of formulas,

- $\Phi \vdash_{\mathbf{FL}^+} *\Sigma, \Gamma \Rightarrow \gamma$ iff $\Phi \vdash_{\mathbf{FL}^+} \Sigma, \Gamma \Rightarrow \gamma$.
- $\Phi \vdash_{\mathbf{FL}^+} \Gamma \Rightarrow \alpha \rightarrow \beta$ iff $\Phi \vdash_{\mathbf{FL}^+} \alpha, \Gamma \Rightarrow \beta$.
- $\Phi \vdash_{\mathbf{FL}^+} \bigvee \Sigma, \Gamma \Rightarrow \beta$ iff $\Phi \vdash_{\mathbf{FL}^+} \alpha, \Gamma \Rightarrow \beta$ for every $\alpha \in \Sigma$.

$\mathbf{FL}^+$ does not have structural rules at all. Various systems of substructural logics are obtained by enriching it with a suitable set of structural rules.

DEFINITION 2.1. A *structural rule* R is an $n+1$ tuple $(\Theta_0 \lhd \Theta_1; \ldots ; \Theta_n)$, where $n \geq 1$ and each Θ_i is a finite sequence of variables, that satisfies the following condition:

(*) any variable occurring in $\Theta_1, \ldots, \Theta_n$ also occurs in Θ_0.

The last condition will be referred to as the *non-erasing condition*.

Let $R[\vec{p}]$ be a structural rule $(\Theta_0[\vec{p}] \lhd \Theta_1[\vec{p}]; \ldots ; \Theta_n[\vec{p}])$, and $\vec{\beta}$ a sequence of formulas. Then the result of substitution $R[\vec{\beta}] = (\Theta_0[\vec{\beta}] \lhd \Theta_1[\vec{\beta}]; \ldots ; \Theta_n[\vec{\beta}])$, is called an *instance* of R. In particular, when $\vec{\beta}$ consists of propositional variables, $R[\vec{\beta}]$ is called a *variable instance* of R. Each instance $R[\vec{\beta}]$ codifies an inference scheme of the form:

$$\frac{\Gamma, \Theta_1[\vec{\beta}], \Delta \Rightarrow \gamma \quad \cdots \quad \Gamma, \Theta_n[\vec{\beta}], \Delta \Rightarrow \gamma}{\Gamma, \Theta_0[\vec{\beta}], \Delta \Rightarrow \gamma}$$

with Γ, Δ and γ arbitrary.

For example, the structural rules mentioned in the introduction can be formally specified as follows:

- **e**: $(q, p \lhd p, q)$
- **w**: $(p \lhd \emptyset)$
- **c**: $(p \lhd p, p)$
- **exp**: $(p, p \lhd p)$
- **min**: $\{(p_1, \ldots, p_k, q_1, \ldots, q_l \lhd p_1, \ldots, p_k; q_1, \ldots, q_l) \mid k, l \geq 0\}$
- **seq-c**: $\{(p_1, \ldots, p_k \lhd p_1, \ldots, p_k, p_1, \ldots, p_k) \mid k \geq 0\}$

Notice that **min** and **seq-c** are specified by a countable set of structural rules.

Some remarks on the generality of our definition are in order. First, the structural rules considered in this paper are *additive*, in the sense that they are applicable in an arbitrary context Γ, Δ, γ and the context must be identical in all the upper sequents. Non-additive structural rules are considered in [**??**]. Structural rules that affect the right hand side of a sequent, such as right weakening, are also considered in the latter paper.

Second, our structural rules are supposed to respect the non-erasing condition. We need this condition because a structural rule might cause a disastrous effect without it. In particular, any unary rule $(\Theta_0 \lhd \Theta_1)$ that violates the non-erasing condition causes logical inconsistency. On the other hand, all structural rules that respect the non-erasing condition are admissible in intuitionistic logic.

Given a set $\mathcal{R}$ of structural rules, the system $\mathbf{FL}^+(\mathcal{R})$ is defined to be $\mathbf{FL}^+$ enriched with *all* instances of the additional structural rules $\mathcal{R}$. For instance, $\mathbf{FL}^+(\{\mathbf{e}\})$ amounts to $\mathbf{FL}^+_{\mathbf{e}}$ (intuitionistic linear logic without modality) and $\mathbf{FL}^+(\{\mathbf{e}, \mathbf{w}, \mathbf{c}\})$ is nothing but intuitionistic logic.

Due to the non-erasing condition, our structural rules satisfy the following property: any formula occurring in the upper sequents of a structural rule also occurs in the lower sequent. It follows that the cut elimination theorem always implies the subformula property.

More precisely, for each formula α, define the set $Pos(\alpha)$ of *positive subformulas* and the set $Neg(\alpha)$ of *negative subformulas* as follows. When $\dagger$ is either a variable or a constant, $Pos(\dagger) = \{\dagger\}$; $Neg(\dagger) = \emptyset$. When $\star \in \{\wedge, \vee, \cdot\}$, $Pos(\alpha \star \beta) = Pos(\alpha) \cup Pos(\beta) \cup \{\alpha \star \beta\}$; $Neg(\alpha \star \beta) = Neg(\alpha) \cup Neg(\beta)$. For residuals, $Pos(\alpha\backslash\beta) = Pos(\beta) \cup Neg(\alpha) \cup \{\alpha\backslash\beta\}$; $Neg(\alpha\backslash\beta) = Neg(\beta) \cup Pos(\alpha)$. Similarly for $/$. Finally, for a sequent $S \equiv \beta_1, \dots, \beta_k \Rightarrow \alpha$, $Pos(S) = Neg(\beta_1) \cup \dots \cup Neg(\beta_k) \cup Pos(\alpha)$ and $Neg(S) = Pos(\beta_1) \cup \dots \cup Pos(\beta_k) \cup Neg(\alpha)$. We then have:

LEMMA 2.2. *Let $\mathcal{R}$ be a set of structural rules. Suppose that $\mathbf{FL}^+(\mathcal{R})$ enjoys cut elimination. Then it satisfies the (polarized) subformula property: if a sequent $\Gamma \Rightarrow \alpha$ is provable in $\mathbf{FL}^+(\mathcal{R})$, then it has a derivation π in which only subformulas of $\Gamma \Rightarrow \alpha$ occur. Moreover, any formula of occurring on the left (right, resp.) hand side of a sequent in π is a negative (positive, resp.) subformula of $\Gamma \Rightarrow \alpha$.*

To study the properties of structural rules, it is convenient to represent them as formulas. Given a structural rule $R = (\Theta_0 \lhd \Theta_1; \dots; \Theta_n)$, define its *axiomatic form* $\hat{R}$ by

$$\hat{R} \equiv *\Theta_0 \to (*\Theta_1 \vee \cdots \vee *\Theta_n).$$

For instance, $\hat{\mathbf{e}} \equiv q \cdot p \to p \cdot q$ and $\hat{\mathbf{w}} \equiv p \to 1$. The axiomatic form of $\mathbf{min}_1 = (p, q \lhd p; q)$ is $p \cdot q \to p \vee q$.

When $\mathcal{R}$ is a set of structural rules, $\widehat{\mathcal{R}}$ denotes the set $\{\hat{R} \mid R \in \mathcal{R}\}$.

As expected, there is an instance-wise correspondence between structural rules and their axiomatic forms:

LEMMA 2.3. *Let $R[\vec{p}]$ be a structural rule. Then an instance $R[\vec{\alpha}]$ is derivable from $\hat{R}[\vec{\alpha}]$ in $\mathbf{FL}^+$ and vice versa.*

PROOF. For simplicity of notation, let us just consider a binary structural rule $R = (\Theta_0 \lhd \Theta_1; \Theta_2)$ and an instance given by the identity substitution. Now R can be derived from $\hat{R}$ as follows:

$$\dfrac{\dfrac{\Rightarrow *\Theta_0 \to (*\Theta_1 \vee *\Theta_2)}{\Theta_0 \Rightarrow *\Theta_1 \vee *\Theta_2} \qquad \dfrac{\dfrac{\Gamma, \Theta_1, \Delta \Rightarrow \gamma}{\Gamma, *\Theta_1, \Delta \Rightarrow \gamma} \quad \dfrac{\Gamma, \Theta_2, \Delta \Rightarrow \gamma}{\Gamma, *\Theta_2, \Delta \Rightarrow \gamma}}{\Gamma, *\Theta_1 \vee *\Theta_2, \Delta \Rightarrow \gamma}}{\Gamma, \Theta_0, \Delta \Rightarrow \gamma}$$

Conversely, $\hat{R}$ can be proved by using R as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Theta_1 \Rightarrow *\Theta_1}}{\Theta_1 \Rightarrow *\Theta_1 \vee *\Theta_2} \quad \dfrac{\dfrac{}{\Theta_2 \Rightarrow *\Theta_2}}{\Theta_2 \Rightarrow *\Theta_1 \vee *\Theta_2}}{\Theta_0 \Rightarrow *\Theta_1 \vee *\Theta_2}\, R}{*\Theta_0 \Rightarrow *\Theta_1 \vee *\Theta_2}}{\Rightarrow *\Theta_0 \to (*\Theta_1 \vee *\Theta_2)}$$

$\dashv$

§3. Syntactic Propagation. Let us now introduce a syntactic version of the propagation property. To motivate the notion, consider the contrast between $\mathbf{FL}^+(\{\mathbf{c}\})$ and $\mathbf{FL}^+(\textbf{seq-c})$. As is mentioned in the introduction, the former does not enjoy cut elimination. For instance, the cut below cannot be eliminated:

$$\dfrac{\dfrac{\dfrac{}{\alpha\Rightarrow\alpha}\quad\dfrac{}{\beta\Rightarrow\beta}}{\alpha,\beta\Rightarrow\alpha\cdot\beta}\qquad \dfrac{\dfrac{\dfrac{}{\alpha\cdot\beta\Rightarrow\alpha\cdot\beta}\quad\dfrac{}{\alpha\cdot\beta\Rightarrow\alpha\cdot\beta}}{\alpha\cdot\beta,\alpha\cdot\beta\Rightarrow(\alpha\cdot\beta)\cdot(\alpha\cdot\beta)}\,\mathbf{c}}{\alpha\cdot\beta\Rightarrow(\alpha\cdot\beta)\cdot(\alpha\cdot\beta)}}{\alpha,\beta\Rightarrow(\alpha\cdot\beta)\cdot(\alpha\cdot\beta)}\,\mathbf{cut}$$

On the other hand, if $\mathbf{c}$ is generalized to **seq-c**, the above cut can be easily eliminated:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\alpha\Rightarrow\alpha}\quad\dfrac{}{\beta\Rightarrow\beta}}{\alpha,\beta\Rightarrow\alpha\cdot\beta}\qquad\dfrac{\dfrac{}{\alpha\Rightarrow\alpha}\quad\dfrac{}{\beta\Rightarrow\beta}}{\alpha,\beta\Rightarrow\alpha\cdot\beta}}{\alpha,\beta,\alpha,\beta\Rightarrow(\alpha\cdot\beta)\cdot(\alpha\cdot\beta)}}{\alpha,\beta\Rightarrow(\alpha\cdot\beta)\cdot(\alpha\cdot\beta)}\,\textbf{seq-c}$$

Now our question is this: is it possible to distinguish $\mathbf{c}$ and **seq-c** without mentioning cut elimination? It is certainly possible. The idea is to replace the variable p in $\hat{\mathbf{c}} = \hat{\mathbf{c}}[p]$ by a fusion $p_1 \cdot p_2$ of two variables. Then the resulting instance $\hat{\mathbf{c}}[p_1 \cdot p_2]$ is not deducible from the variable instances of $\hat{\mathbf{c}}$. Namely,

$$\{\ p_1 \to p_1 \cdot p_1,\ p_2 \to p_2 \cdot p_2\ \} \not\vdash_{\mathbf{FL}^+} (p_1 \cdot p_2) \to (p_1 \cdot p_2) \cdot (p_1 \cdot p_2).$$

We say that $\mathbf{c}$ *does not propagate with respect to fusion*. In contrast, when we replace a variable p in $\hat{R}[p] \in \widehat{\textbf{seq-c}}$ by a fusion $p_1 \cdot p_2$, the resulting instance $\hat{R}[p_1 \cdot p_2]$ is always deducible from the variable instances of rules in $\widehat{\textbf{seq-c}}$. For instance,

$$\{\ p_1 \cdot p_2 \to p_1 \cdot p_2 \cdot p_1 \cdot p_2\ \} \vdash_{\mathbf{FL}^+} (p_1 \cdot p_2) \to (p_1 \cdot p_2) \cdot (p_1 \cdot p_2),$$

where the left formula $p_1 \cdot p_2 \to p_1 \cdot p_2 \cdot p_1 \cdot p_2$ is a variable instance of a rule in $\widehat{\textbf{seq-c}}$ of length two, while the right formula is a (fusion) instance of a rule in $\widehat{\textbf{seq-c}}$ of length one. Therefore, **seq-c** *propagates with respect to fusion.*

It is worth noting that $\mathbf{c}$ propagates with respect to fusion *in the presence of* $\mathbf{e}$:

$$\left\{\begin{array}{c} p_1 \to p_1 \cdot p_1,\ p_2 \to p_2 \cdot p_2 \\ p_1 \cdot p_2 \to p_2 \cdot p_1,\ p_2 \cdot p_1 \to p_1 \cdot p_2 \end{array}\right\} \vdash_{\mathbf{FL}^+} (p_1 \cdot p_2) \to (p_1 \cdot p_2) \cdot (p_1 \cdot p_2).$$

and indeed $\mathbf{FL}^+(\{\mathbf{e},\mathbf{c}\})$ admits cut elimination.

Next, consider the contrast between $\mathbf{FL}^+(\{\mathbf{exp}\})$ and $\mathbf{FL}^+(\mathbf{min})$. The former does not enjoy cut elimination, as witnessed by:

$$\dfrac{\dfrac{\dfrac{}{\beta\Rightarrow\beta}}{\beta\Rightarrow\alpha\vee\beta}\qquad\dfrac{\dfrac{\dfrac{}{\alpha\Rightarrow\alpha}}{\alpha\Rightarrow\alpha\vee\beta}\quad\dfrac{\dfrac{}{\alpha\vee\beta\Rightarrow\alpha\vee\beta}}{\alpha\vee\beta,\alpha\vee\beta\Rightarrow\alpha\vee\beta}\,\mathbf{exp}}{\alpha,\alpha\vee\beta\Rightarrow\alpha\vee\beta}\,\mathbf{cut}}{\alpha,\beta\Rightarrow\alpha\vee\beta}\,\mathbf{cut}$$

Notice that one cannot obtain a cut-free proof even if $\mathbf{exp}$ is generalized to a sequence version as above. On the other hand, when $\mathbf{exp}$ is replaced with $\mathbf{min}$, a cut-free proof can be obtained:

$$\dfrac{\dfrac{\overline{\alpha \Rightarrow \alpha}}{\alpha \Rightarrow \alpha \vee \beta} \quad \dfrac{\overline{\beta \Rightarrow \beta}}{\beta \Rightarrow \alpha \vee \beta}}{\alpha, \beta \Rightarrow \alpha \vee \beta}\ \mathbf{min}$$

Here, we may again ask whether it is possible to distinguish **exp** and **min** without mentioning cut elimination. This time, our answer is that **min** *propagates with respect to disjunction*, while **exp** does not. Namely, when we replace a variable p in $R[p] \in \mathbf{min}$ by a disjunction $p_1 \vee p_2$, the resulting instance $\hat{R}[p_1 \vee p_2]$ is deducible from the variable instances of $\widehat{\mathbf{min}}$ in $\mathbf{FL}^+$. For instance,

$$\{\ p_1 \cdot q \to p_1 \vee q,\ p_2 \cdot q \to p_2 \vee q\ \} \vdash_{\mathbf{FL}^+} (p_1 \vee p_2) \cdot q \to (p_1 \vee p_2) \vee q.$$

This property does not hold for **exp**:

$$\{\ p_1 \cdot p_1 \to p_1,\ p_2 \cdot p_2 \to p_2\ \} \not\vdash_{\mathbf{FL}^+} (p_1 \vee p_2) \cdot (p_1 \vee p_2) \to (p_1 \vee p_2).$$

These observations lead us to the following definition.

Definition 3.1. Given a set $\mathcal{R}$ of structural rules, let $VarI(\mathcal{R})$ be the set of variable instances of the rules in $\mathcal{R}$. We say that $\mathcal{R}$ satisfies the *syntactic propagation property* if the following holds:

- For any $R[p_1, \ldots, p_m] \in \mathcal{R}$ and any finite sequences $\Sigma_1, \ldots, \Sigma_m$ of propositional variables, the formulas $\hat{R}[*\Sigma_1, \ldots, *\Sigma_m]$ and $\hat{R}[\bigvee \Sigma_1, \ldots, \bigvee \Sigma_m]$ are deducible from $\widehat{VarI(\mathcal{R})}$ in $\mathbf{FL}^+$.

In view of Lemma 2.3, this is equivalent to say that

- the formulas $\hat{R}[*\Sigma_1, \ldots, *\Sigma_m]$ and $\hat{R}[\bigvee \Sigma_1, \ldots, \bigvee \Sigma_m]$ are provable in $\mathbf{FL}^+$ enriched with variable instances $VarI(\mathcal{R})$ of structural rules $\mathcal{R}$.

Another thing to be noted is that it is actually sufficient to consider only those sequences $\Sigma_1, \ldots, \Sigma_m$ with $|\Sigma_i| \leq 2$ for every $1 \leq i \leq m$ in the above definition. One can then easily prove by induction that the rules satisfy the syntactic propagation property.

Proposition 3.2. *Let $\mathcal{R}$ be a set of structural rules. If $\mathbf{FL}^+(\mathcal{R})$ enjoys cut elimination, then $\mathcal{R}$ satisfies the syntactic propagation property.*

Proof. Let $R[p_1, \ldots, p_m] = (\Theta_0 \lhd \Theta_1; \ldots; \Theta_n) \in \mathcal{R}$. Then $\hat{R}$ is of the form $*\Theta_0 \to *\Theta_1 \vee \cdots \vee *\Theta_n$. One can show that for any sequences $\Sigma_1, \ldots, \Sigma_m$ of propositional variables, the formula $\hat{R}[\bigvee \Sigma_1, \ldots, \bigvee \Sigma_m]$ is equivalent in $\mathbf{FL}^+$ to a formula $\hat{R}_\vee$ such that

- for any implication subformula $\alpha \to \beta$ occurring in $\hat{R}_\vee$, α is a propositional variable.

This is true because we have the following logical equivalences in $\mathbf{FL}^+$:

$$\begin{aligned}
\alpha \cdot \sigma \to \delta &\leftrightarrow \sigma \to (\alpha \to \delta) \\
\alpha \vee \beta \to \delta &\leftrightarrow (\alpha \to \delta) \wedge (\beta \to \delta) \\
1 \to \delta &\leftrightarrow \delta
\end{aligned}$$

Now, Lemma 2.3 ensures that $\hat{R}[\bigvee \Sigma_1, \ldots, \bigvee \Sigma_m]$ is provable in $\mathbf{FL}^+(\mathcal{R})$, hence so is $\hat{R}_\vee$. By assumption, $\mathbf{FL}^+(\mathcal{R})$ enjoys cut elimination. Hence Lemma 2.2 entails that $\hat{R}_\vee$ has a derivation in which only propositional variables occur on the left hand side of a sequent, because $\hat{R}_\vee$ has only propositional variables as negative subformulas. This means that only variable instances of structural rules $\mathcal{R}$ are used in the derivation. Therefore $\hat{R}_\vee$, and so $\hat{R}[\bigvee\Sigma_1,\dots,\bigvee\Sigma_m]$, are deducible from $\widehat{VarI(\mathcal{R})}$ in $\mathbf{FL}^+$ by Lemma 2.3.

It can be shown similarly that $\hat{R}[*\Sigma_1,\dots,*\Sigma_m]$ is deducible from $\widehat{VarI(\mathcal{R})}$. ⊣

§4. Residuated lattices and semantic propagation. Let us now move on to algebraic semantics for $\mathbf{FL}^+$ and find out a semantic analogue of the syntactic propagation property.

An algebra $\mathbf{P} = \langle P, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ is called a *(bounded) residuated lattice* if

1. $\langle P, \wedge, \vee\rangle$ is a lattice with the greatest element $\top$ and the least element $\bot$.
2. $\langle P, \cdot, 1\rangle$ is a monoid.
3. The operations $\backslash$ and $/$ are left and right residuals of $\cdot$. Namely, for any $x, y, z \in P$,
$$x \cdot y \leq z \iff y \leq x\backslash z \iff x \leq z/y.$$

(See [**??**, **??**] for general introductions to residuated lattices.)

A *valuation* f on $\mathbf{P}$ maps each propositional variable to an element of P. Given a set $X \subseteq P$, f is called an *X-valuation* if the range is a subset of X. As usual, f can be extended to a map from the formulas $\mathcal{F}$ to P as follows:

$$\begin{array}{rcll} f(\dagger) & = & \dagger & \text{for } \dagger \in \{\top, \bot, 1\}, \\ f(\alpha\star\beta) & = & f(\alpha)\star f(\beta) & \text{for } \star \in \{\wedge, \vee, \cdot, \backslash, /\}. \end{array}$$

A formula α is said to be *true* under valuation f in $\mathbf{P}$ if $f(\alpha) \geq 1$. In particular, $\alpha \to \beta$, i.e., $\alpha\backslash\beta$ is true iff $f(\alpha) \leq f(\beta)$. A formula α is *valid* (*X-valid*, resp.) in $\mathbf{P}$ if it is true under all valuations (X-valuations, resp.) on $\mathbf{P}$.

The residuated lattices are algebraic models of $\mathbf{FL}^+$. In particular, the following strong form of soundness holds:

Lemma 4.1. *Let $\mathbf{P}$ be a residuated lattice and f a valuation on it. If α is deducible from Φ and all formulas in Φ are true under f in $\mathbf{P}$, then α is also true under f.*

Given a set $\mathcal{R}$ of structural rules, an *$\mathcal{R}$-residuated lattice* is a residuated lattice in which all formulas in $\widehat{\mathcal{R}}$ are valid. By the previous lemma, any formula provable in $\mathbf{FL}^+(\mathcal{R})$ is valid in all $\mathcal{R}$-residuated lattices.

Coming back to the residuated lattices in general, we may observe that the monoid multiplication $\cdot$ is *continuous* in the following sense:

Lemma 4.2. *Let $c_0, \dots, c_m \in P$ and let*
$$\delta(x_1,\dots,x_m) = c_0 \cdot x_1 \cdot c_1 \cdots c_{m-1} \cdot x_m \cdot c_m,$$
for any $x_1,\dots,x_m \in P$. Let also $\tilde{\delta}(x) = \delta(x,\dots,x)$. Suppose that X is a subset of P for which $\bigvee X$ exists. We then have:
$$\tilde{\delta}(\bigvee X) = \bigvee_{Y\subseteq_{fin} X} \tilde{\delta}(\bigvee Y),$$
where $Y \subseteq_{fin} X$ holds iff Y is a finite subset of X.

PROOF. Note that, by distributivity of $\bigvee$ over $\cdot$,

$$\tilde{\delta}(\bigvee X) = \bigvee_{x_1 \in X} \cdots \bigvee_{x_m \in X} \delta(x_1, \ldots, x_m).$$

Now, for any $x_1, \ldots, x_m \in X$ and $x = x_1 \vee \cdots \vee x_m$, we have

$$\begin{aligned} \delta(x_1, \ldots, x_m) &\leq \delta(x, \ldots, x) \\ &= \tilde{\delta}(x) \\ &\leq \bigvee_{Y \subseteq_{fin} X} \tilde{\delta}(\bigvee Y). \end{aligned}$$

Therefore, one direction holds. The other direction can be shown easily. ⊣

Given $X \subseteq P$, the *multiplication closure* $\prod(X)$, the *join closure* $\coprod(X)$ and the *finite join closure* $\coprod_{fin}(X)$ are defined by

$$\begin{aligned} \prod(X) &= \{x_1 \cdots x_n \mid n \geq 0, x_1, \ldots, x_n \in X\}, \\ \coprod(X) &= \{\bigvee Y \mid Y \subseteq X, \bigvee Y \text{ exists}\}, \\ \coprod_{fin}(X) &= \{\bigvee Y \mid Y \subseteq_{fin} X\}. \end{aligned}$$

We are now ready to define a semantic analogue of the syntactic propagation property.

DEFINITION 4.3. A set $\mathcal{R}$ of structural rules satisfies the *semantic propagation property* if for any residuated lattice $\mathbf{P}$ and $X \subseteq P$, the following holds:

- if all formulas in $\widehat{\mathcal{R}}$ are X-valid, then they are also $\coprod(\prod(X))$-valid.

We then have:

PROPOSITION 4.4. *If a set $\mathcal{R}$ of structural rules satisfies the syntactic propagation property, it also satisfies the semantic propagation property.*

PROOF. Let $\mathbf{P}$ be a residuated lattice and $X \subseteq P$. Suppose that all formulas in $\widehat{\mathcal{R}}$ are X-valid. It is then easy to see that all formulas in $\widehat{VarI(\mathcal{R})}$ are also X-valid.

Let $R[p_1, \ldots, p_m] = (\Theta_0 \lhd \Theta_1; \ldots; \Theta_n)$ be a structural rule in $\mathcal{R}$. Our first claim is that $\hat{R}$ is $\prod(X)$-valid. So suppose that f is a $\prod(X)$-valuation which in particular assigns to each p_i ($1 \leq i \leq m$) a finite multiplication $f(p_i) = *Y_i$ with $Y_i \subseteq_{fin} X$. One can consider sequences $\Sigma_1, \ldots, \Sigma_m$ of propositional variables and a X-valuation g such that $g(*\Sigma_i) = *Y_i = f(p_i)$ for every $1 \leq i \leq m$. In particular,

$$f(\hat{R}) = g(\hat{R}[*\Sigma_1, \ldots, *\Sigma_m]).$$

Since g is an X-valuation, all formulas in $\widehat{VarI(\mathcal{R})}$ are true under g. Moreover, the syntactic propagation property ensures that the formula $\hat{R}[*\Sigma_1, \ldots, *\Sigma_m]$ is deducible from $\widehat{VarI(\mathcal{R})}$. Hence by the strong soundness (Lemma 4.1), $\hat{R}[*\Sigma_1, \ldots, *\Sigma_m]$ is true under g, and so $\hat{R}$ is true under f.

Thus we have shown that $\hat{R}$ is $\prod(X)$-valid. Similarly, one can show that $\hat{R}$ is $\coprod_{fin}(\prod(X))$-valid.

Finally, to show that $\hat{R}$ is $\coprod(\prod(X))$-valid, let f be a $\coprod(\prod(X))$-valuation that assigns to each variable p_i a join $\bigvee Y_i$ with $Y_i \subseteq \prod(X)$. We write $g \subseteq_{fin} f$ if g is a $\coprod_{fin}(\prod(X))$-valuation which assigns to each p_i a finite join $\bigvee Z_i$ with $Z_i \subseteq_{fin} Y_i$.

Recall that $*\Theta_0$ is a fusion of variables in $\{p_1, \ldots, p_m\}$ possibly with some repetitions. By applying Lemma 4.2 on each of $p_1, \ldots, p_n$ and by using the fact that $\hat{R}$ is $\coprod_{fin}(\prod(X))$-valid (and hence $g(*\Theta_0) \leq g(*\Theta_1 \vee \cdots \vee *\Theta_n)$ whenever $g \subseteq_{fin} f$), we obtain:

$$\begin{aligned} f(*\Theta_0) &= \bigvee_{g \subseteq_{fin} f} g(*\Theta_0) \\ &\leq \bigvee_{g \subseteq_{fin} f} g(*\Theta_1 \vee \cdots \vee *\Theta_n) \\ &= f(*\Theta_1 \vee \cdots \vee *\Theta_n) \end{aligned}$$

Therefore, $\hat{R}$ is true under f. ⊣

§5. Phase structures and semantic cut elimination. We now introduce a special class of residuated lattices, called (intuitionistic noncommutative) *phase structures* (see [**??**, **??**, **??**]). Let $\mathbf{M} = \langle M, \cdot, 1 \rangle$ be a monoid. Denote the powerset of M by $\wp(M)$, and define for $X, Y \in \wp(M)$,

$$X \bullet Y = \{x \cdot y \mid x \in X, y \in Y\}.$$

A function $C : \wp(M) \longrightarrow \wp(M)$ is said to be a *closure operator* on $\wp(M)$ if for all $X, Y \in \wp(M)$,

1. $X \subseteq C(X)$,
2. $C(C(X)) \subseteq C(X)$,
3. $X \subseteq Y$ implies $C(X) \subseteq C(Y)$,
4. $C(X) \bullet C(Y) \subseteq C(X \bullet Y)$.

A set $X \in \wp(M)$ is *closed* if $X = C(X)$. The set of all closed sets in $\wp(M)$ is denoted by $\mathcal{C}_M$. Define for any closed sets $X, Y \in \mathcal{C}_M$ and for any family $\mathcal{X}$ of closed sets,

$$\begin{aligned} X \cup_C Y &= C(X \cup Y), \\ \textstyle\bigcup_C \mathcal{X} &= C(\textstyle\bigcup \mathcal{X}), \\ X \bullet_C Y &= C(X \bullet Y), \\ X \mathbin{\backslash\!\!\backslash} Y &= \{y \mid \forall x \in X, x \cdot y \in Y\}, \\ Y /\!\!/ X &= \{y \mid \forall x \in X, y \cdot x \in Y\}. \end{aligned}$$

We then have:

Lemma 5.1. *If* $\mathbf{M}$ *is a monoid and* C *is a closure operator on* $\wp(M)$*, then the algebra*

$$\mathbf{C_M} = \langle \mathcal{C}_M, \cap, \cup_C, \bullet_C, \mathbin{\backslash\!\!\backslash}, /\!\!/, C(\{1\}) \rangle$$

is a complete residuated lattice with infinite join $\bigcup_C$.

In every phase structure, the following hold:

1. $C(\{x \cdot y\}) = C(\{x\}) \bullet_C C(\{y\})$ for any $x, y \in M$,

2. $C(X) = \bigcup_{C\ x \in X} C(\{x\})$ for any $X \subseteq M$.

As a consequence, phase structures satisfy the following remarkable property which plays a key role in connecting the semantic propagation property to cut elimination:

LEMMA 5.2. *Suppose that* $\mathbf{M}$ *is finitely generated by a set* A*, i.e., any element* x *of* M *can be written as* $y_1 \cdots y_n$ *for some* $y_1, \dots, y_n \in A$*. Let* $\mathcal{C}'_A = \{C(\{y\}) \mid y \in A\}$*. Then we have* $\mathcal{C}_M = \coprod(\prod(\mathcal{C}'_A))$*.*

PROOF. By 1 above, we have

$$C(\{x\}) = C(\{y_1\}) \bullet_C \cdots \bullet_C C(\{y_n\}),$$

for any $x = y_1 \cdots y_n$. Thus $C(\{x\}) \in \prod(\mathcal{C}'_A)$. Now the lemma immediately follows by 2 above. ⊣

It follows that the semantic propagation property roughly corresponds to Girard's naturality test [**??**] as far as the phase structures are concerned. More precisely, let us say that a set $\mathcal{R}$ of structural rules passes the *modified naturality test* if in every phase structure whose underlying monoid is finitely generated by A, the $\mathcal{C}'_A$-validity of $\widehat{\mathcal{R}}$ implies its validity. Then we see that any $\mathcal{R}$ with the semantic propagation property passes the modified naturality test by virtue of Lemma 5.2.

We now describe a specific construction of a phase structure due to [**??**, **??**] (and slightly remedied by [**??**]), which is quite useful for proving the cut elimination theorem. (See also [**??**], where Okada's construction is reformulated as algebraic *quasi-completion* and *quasi-embedding*.)

Let $\mathcal{F}^*$ be the free monoid generated by the formulas $\mathcal{F}$ of $\mathbf{FL}^+$; the elements of $\mathcal{F}^*$ are sequences of formulas, the monoid multiplication is concatenation, and the unit element is the empty sequence $\emptyset$.

Let us fix a set $\mathcal{R}$ of structural rules. The operator C is defined on the basis of *cut-free* provability in $\mathbf{FL}^+(\mathcal{R})$:

$$\begin{aligned} [\![\Gamma\_\Delta \Rightarrow \gamma]\!] &= \{\Sigma \mid \Gamma, \Sigma, \Delta \Rightarrow \gamma \text{ is } \textit{cut-free} \text{ provable in } \mathbf{FL}^+(\mathcal{R})\}, \\ \mathcal{D} &= \{[\![\Gamma\_\Delta \Rightarrow \gamma]\!] \mid \Gamma, \Delta, \gamma \text{ arbitrary}\}, \\ C(X) &= \bigcap_{X \subseteq Y \in \mathcal{D}} Y. \end{aligned}$$

Then one can show that C is indeed a closure operator on $\wp(\mathcal{F}^*)$. Hence by Lemma 5.1, the algebra

$$\mathbf{C}_{\mathcal{F}^*} = \langle \mathcal{C}_{\mathcal{F}^*}, \cap, \cup_C, \bullet_C, \backslash\!\backslash, /\!\!/, C(\{\emptyset\}) \rangle$$

is a complete residuated lattice.

Let f_0 be a valuation on $\mathbf{C}_{\mathcal{F}^*}$ defined by $f_0(p) = C(\{p\})$ for every propositional variable p. In this setting, we have *Okada's lemma*:

LEMMA 5.3. *For every formula* α, $\alpha \in f_0(\alpha) \subseteq [\![\_ \Rightarrow \alpha]\!]$*. In particular, for every sequent* $\Gamma \Rightarrow \alpha$*, if* $(*\Gamma) \to \alpha$ *is true under* f_0*, then* $\Gamma \Rightarrow \alpha$ *is cut-free provable in* $\mathbf{FL}^+(\mathcal{R})$*.*

PROOF. The first claim can be proved by induction on the complexity of α (see [**??**]). As for the second claim, let $\Gamma \equiv \beta_1, \ldots, \beta_k$. Then we have $f(\beta_1) \bullet_C \cdots \bullet_C f(\beta_k) \subseteq f(\alpha)$. Since $\beta_i \in f(\beta_i)$ for $1 \leq i \leq k$ and $f(\alpha) \subseteq [\![\_ \Rightarrow \alpha]\!]$, we have $(\beta_1, \ldots, \beta_k) \in [\![\_ \Rightarrow \alpha]\!]$, i.e., $\beta_1, \ldots, \beta_k \Rightarrow \alpha$ is cut-free provable in $\mathbf{FL}^+(\mathcal{R})$. ⊣

It is worth noting that Okada's lemma holds independently of which structural rules $\mathcal{R}$ we adopt. It only concerns with the properties of logical inference rules. What depends on the choice of $\mathcal{R}$ is the following:

LEMMA 5.4. *If $\mathcal{R}$ satisfies the semantic propagation property, then $\mathbf{C}_{\mathcal{F}^*}$ is an $\mathcal{R}$-residuated lattice.*

PROOF. Let $\mathcal{C}'_{\mathcal{F}}$ be $\{C(\{\alpha\}) \mid \alpha \in \mathcal{F}\}$, and let $R = (\Theta_0 \lhd \Theta_1; \ldots; \Theta_n) \in \mathcal{R}$. We first prove that $\hat{R}$ is $\mathcal{C}'_{\mathcal{F}}$-valid.

To show this, suppose that f is a $\mathcal{C}'_{\mathcal{F}}$-valuation, i.e., for any variable p, $f(p)$ is of the form $C(\{\alpha\})$. We can then naturally associate to each variable p a formula $\check{p}$ such that $f(p) = C(\{\check{p}\})$.

Now, recall that $\hat{R}$ is of the form $*\Theta_0 \to (*\Theta_1 \vee \cdots \vee *\Theta_n)$, and each Θ_i is a sequence of the form $p^i_1, \ldots, p^i_{k_i}$ $(0 \leq i \leq n)$. Denote by $\check{\Theta}_i$ the sequence $\check{p}^i_1, \ldots, \check{p}^i_{k_i}$. Then we have:

$$\begin{aligned} f(*\Theta_i) &= f(p^i_1) \bullet_C \cdots \bullet_C f(p^i_{k_i}) \\ &= C(\{\check{p}^i_1\}) \bullet_C \cdots \bullet_C C(\{\check{p}^i_{k_i}\}) \\ &= C(\{\check{p}^i_1, \ldots, \check{p}^i_{k_i}\}) \\ &= C(\{\check{\Theta}_i\}), \\ f(*\Theta_1 \vee \cdots \vee *\Theta_n) &= C(\{\check{\Theta}_1\}) \cup_C \cdots \cup_C C(\{\check{\Theta}_n\}) \\ &= C(\{\check{\Theta}_1; \ldots; \check{\Theta}_n\}), \end{aligned}$$

where $\{\check{\Theta}_1; \ldots; \check{\Theta}_n\}$ is a set of n elements $\check{\Theta}_1, \ldots, \check{\Theta}_n$, each of which is a sequence of formulas. Hence $\hat{R}$ is true under f iff $C(\{\check{\Theta}_0\}) \subseteq C(\{\check{\Theta}_1; \ldots; \check{\Theta}_n\})$ iff $\check{\Theta}_0 \in C(\{\check{\Theta}_1; \ldots; \check{\Theta}_n\})$.

To show the last membership, suppose that $\{\check{\Theta}_1; \ldots; \check{\Theta}_n\} \subseteq [\![\Gamma\_\Delta \Rightarrow \gamma]\!]$. This means that $\Gamma, \check{\Theta}_i, \Delta \Rightarrow \gamma$ is cut-free provable for each $1 \leq i \leq n$. Since $(\check{\Theta}_0 \lhd \check{\Theta}_1; \ldots; \check{\Theta}_n)$ is an instance of R, the sequent $\Gamma, \check{\Theta}_0, \Delta \Rightarrow \gamma$ is cut-free provable in $\mathbf{FL}^+(\mathcal{R})$. Namely, $\check{\Theta}_0 \in [\![\Gamma\_\Delta \Rightarrow \gamma]\!]$. Since it holds for arbitrary Γ, Δ and γ, we have $\check{\Theta}_0 \in C(\{\check{\Theta}_1; \ldots; \check{\Theta}_n\})$, and thus $\hat{R}$ is true under f.

We have shown that $\hat{R}$ is $\mathcal{C}'_{\mathcal{F}}$-valid. Since $\mathcal{R}$ is supposed to satisfy the semantic propagation property, $\hat{R}$ is also valid in $\mathbf{C}_{\mathcal{F}^*}$ by Lemma 5.2. ⊣

PROPOSITION 5.5. *If $\mathcal{R}$ satisfies the semantic propagation property, then $\mathbf{FL}^+(\mathcal{R})$ enjoys cut elimination.*

PROOF. Suppose that $\Gamma \Rightarrow \alpha$ is provable in $\mathbf{FL}^+(\mathcal{R})$. Then by the soundness, $(*\Gamma) \to \alpha$ is valid in all $\mathcal{R}$-residuated lattices. In particular it is valid in $\mathbf{C}_{\mathcal{F}^*}$ by Lemma 5.4. This implies that $\Gamma \Rightarrow \alpha$ is cut-free provable by Lemma 5.3. ⊣

By putting Propositions 3.2, 4.4 and 5.5 together, we obtain our main theorem:

THEOREM 5.6. *Let $\mathcal{R}$ be a set of structural rules. Then the following are equivalent:*

1. $\mathbf{FL}^+(\mathcal{R})$ *enjoys cut elimination.*
2. $\mathcal{R}$ *satisfies the syntactic propagation property.*
3. $\mathcal{R}$ *satisfies the semantic propagation property.*

§6. Completion of Structural Rules. Recall that Contraction **c** can be generalized to its sequence version **seq-c** without changing provability so that the cut elimination theorem holds for $\mathbf{FL}^+(\textbf{seq-c})$. We say that **c** can be *completed* to **seq-c**. Likewise, Expansion **exp** can be completed to Mingle **min**. The completion techniques implicitly used there, which we call *stretching* and *ramification*, are by no means specific to **c** and **exp**. In fact, we show in this section that they are widely applicable, and an *arbitrary* set of structural rules can be completed by stretching and ramification. To show this, our characterization of cut elimination by the syntactic propagation property turns out useful.

To each propositional variable p, we associate a countable set of variables $p^1, p^2, \dots$. The sequence $p^1, \dots, p^k$ is denoted by $\overline{p}(k)$. Let us extend the notion of substitution so that a *sequence* $\overline{p}(k)$ can be substituted for a variable p occurring in a variable sequence Θ. A *stretch* of $R[p_1, \dots, p_m]$ is then of the form $R[\overline{p}_1(k_1), \dots, \overline{p}_m(k_m)]$ for some $k_1, \dots, k_m \geq 0$. Define $Str(\mathcal{R})$ to be the set

$$\{R' \mid R' \text{ is a stretch of some } R \in \mathcal{R}\}.$$

For instance, a stretch of $\mathbf{min}_1 = (p, q \lhd p; q)$ is of the form

$$(\overline{p}(k), \overline{q}(l) \lhd \overline{p}(k); \overline{q}(l)) = (p^1, \dots, p^k, q^1, \dots, q^l \lhd p^1, \dots, p^k; q^1, \dots, q^l),$$

for some $k, l \geq 0$. Hence the set $Str(\{\mathbf{min}_1\})$ is nothing but **min**. Likewise, $Str(\{\mathbf{c}\})$ is nothing but **seq-c**.

Proposition 6.1. *Let $\mathcal{R}$ be a set of structural rules. Then the following hold:*

1. $\mathbf{FL}^+(\mathcal{R})$ *and* $\mathbf{FL}^+(Str(\mathcal{R}))$ *are equivalent.*
2. $Str(\mathcal{R})$ *propagates with respect to fusion. Namely, for any $R_s[p_1, \dots, p_m] \in Str(\mathcal{R})$ and any finite sequences $\Sigma_1, \dots, \Sigma_m$ of propositional variables, $\hat{R}_s[*\Sigma_1, \dots, *\Sigma_m]$ is deducible from $Var\widehat{I(Str}(\mathcal{R}))$ in $\mathbf{FL}^+$.*

Proof. 1. Notice that $R \in \mathcal{R}$ itself is a stretch of R (up to renaming of variables), and thus belongs to $Str(\mathcal{R})$. Conversely, any (axiomatic form of a) stretch $\hat{R}[\overline{p}_1(k_1), \dots, \overline{p}_m(k_m)]$ is deducible from an instance $\hat{R}[*\overline{p}_1(k_1), \dots, *\overline{p}_m(k_m)]$ of $\hat{R}$.

2. $\hat{R}_s[*\Sigma_1, \dots, *\Sigma_m]$ is deducible from $\hat{R}_s[\Sigma_1, \dots, \Sigma_m]$, where $R_s[\Sigma_1, \dots, \Sigma_m]$ is a variable instance of another stretch in $Str(\mathcal{R})$. ⊣

A structural rule $R = (\Theta_0 \lhd \Theta_1; \dots; \Theta_n)$ is said to be *left-linear* if Θ_0 is linear, i.e., each variable in Θ_0 has exactly one occurrence in it. The following lemma provides us with a useful criterion for the propagation with respect to disjunction.

Lemma 6.2. *If a rule $R[p_1, \dots, p_m] = (\Theta_0 \lhd \Theta_1; \dots; \Theta_n)$ is left-linear, then it propagates with respect to disjunction. Namely, for any finite sequences $\Sigma_1, \dots, \Sigma_m$ of propositional variables, $\hat{R}[\bigvee \Sigma_1, \dots, \bigvee \Sigma_m]$ is deducible from $Var\widehat{I(\{R}\})$ in $\mathbf{FL}^+$.*

PROOF. For any $q_1 \in \Sigma_1, \ldots, q_m \in \Sigma_m$, $\hat{R}[q_1, \ldots, q_m]$ is a variable instance of $\hat{R}$. It implies in $\mathbf{FL}^+$ the formula

$$*\Theta_0[q_1, \ldots, q_m] \to (*\Theta_1 \vee \cdots \vee *\Theta_n)[\bigvee \Sigma_1, \ldots, \bigvee \Sigma_m].$$

By left-linearity and distributivity of $\bigvee$ over $\cdot$, we have $\hat{R}[\bigvee \Sigma_1, \ldots, \bigvee \Sigma_m]$ deducible from $\widehat{VarI(\{R\})}$ in $\mathbf{FL}^+$. ⊣

Now let Θ be a sequence of variables and suppose that a variable p has $k > 1$ occurrences in Θ. We denote by $lin_p(\Theta)$ the sequence obtained by replacing the jth occurrence of p in Θ with p^j for every $1 \leq j \leq k$. For instance, if $\Theta \equiv p, q, p$, then $lin_p(\Theta) \equiv p^1, q, p^2$. In addition, we denote by $\langle\Theta\rangle_p^k$ the set of variable sequences Θ' such that Θ' is obtained from Θ by replacing each occurrence of p with one of $p^1, \ldots, p^k$ (not assumed to be chosen distinct). For instance, if $\Theta \equiv p, q, p$ as before, then

$$\langle\Theta\rangle_p^2 \equiv \{(p^1, q, p^1), (p^1, q, p^2), (p^2, q, p^1), (p^2, q, p^2)\}.$$

Now let $R = (\Theta_0 \lhd \Theta_1; \ldots ; \Theta_n)$ and suppose that a variable p has $k > 1$ occurrences in Θ_0. Then the *ramification of R at p* is defined to be:

$$Ram_p(R) = (lin_p(\Theta_0) \lhd \langle\Theta_1\rangle_p^k; \cdots ; \langle\Theta_n\rangle_p^k),$$

where each set $\langle\Theta_i\rangle_p^k$ is ordered according to some fixed ordering. Finally, if $p_1, \ldots, p_m$ are the variables that have more than one occurrences in Θ_0, then the *ramification* of R is defined by

$$Ram(R) = Ram_{p_1} \cdots Ram_{p_m}(R).$$

For instance, $Ram(\mathbf{exp})$ is nothing but $\mathbf{min}_1$, and $Ram(p, p \lhd p, p, p)$ is

$$(\, p^1, p^2 \lhd p^1, p^1, p^1 \; ; \; p^1, p^1, p^2 \; ; \; p^1, p^2, p^1 \; ; \; p^1, p^2, p^2 \; ; \; p^2, p^1, p^1 \; ; \; p^2, p^1, p^2 \; ; \; p^2, p^2, p^1 \; ; \; p^2, p^2, p^2 \,).$$

Let $Ram(\mathcal{R})$ be the set of ramifications of rules in $\mathcal{R}$.

PROPOSITION 6.3. *Let $\mathcal{R}$ be a set of structural rules. Then the following hold:*

1. *$\mathbf{FL}^+(\mathcal{R})$ and $\mathbf{FL}^+(Ram(\mathcal{R}))$ are equivalent.*
2. *$Ram(\mathcal{R})$ propagates with respect to disjunction.*

PROOF. 1. It is sufficient to show that $\mathbf{FL}^+(\{R\})$ and $\mathbf{FL}^+(\{Ram_p(R)\})$ are equivalent for any structural rule R and any variable p. Let R be of the form $R[p, \vec{q}] = (\Theta_0 \lhd \Theta_1; \ldots ; \Theta_n)$ and suppose that the variable p has k occurrences in Θ_0. It is clear that R itself can be seen as an instance of $Ram_p(R)$; if p is substituted for all of $p^1, \ldots, p^k$, every $\Theta_i' \in \langle\Theta_i\rangle_p^k$ becomes identical with Θ_i and $lin_p(\Theta_0)$ identical with Θ_0.

To prove the converse, we show that $\hat{R}[p^1 \vee \cdots \vee p^k, \vec{q}]$ implies $Ram_p(R)$. But it is clear, because $*lin_p(\Theta_0) \to *\Theta_0[p^1 \vee \cdots \vee p^k, \vec{q}]$ and $*\Theta_i[p^1 \vee \cdots \vee p^k, \vec{q}] \to \bigvee\langle\Theta_i\rangle_p^k$ $(1 \leq i \leq n)$ are provable in $\mathbf{FL}^+$.

2. By Lemma 6.2. ⊣

Observe that the stretching operation does not affect left-linearity. We have therefore obtained a general completion result:

THEOREM 6.4. *Given a set $\mathcal{R}$ of structural rules, define $\mathcal{R}^\star$ to be $Str(Ram(\mathcal{R}))$. Then the following hold.*

- $\mathbf{FL}^+(\mathcal{R})$ *and* $\mathbf{FL}^+(\mathcal{R}^\star)$ *are equivalent.*
- $\mathcal{R}^\star$ *satisfies the syntactic propagation property. Hence* $\mathbf{FL}^+(\mathcal{R}^\star)$ *enjoys cut-elimination.*

Note that $Ram(\mathcal{R}) = \mathcal{R}$ when all rules in $\mathcal{R}$ are already left-linear. Moreover, we can identify $Str(Str(\mathcal{R}))$ with $Str(\mathcal{R})$ under suitable renaming of variables. Therefore, we have $\mathcal{R}^{\star\star}$ equivalent to $\mathcal{R}^\star$ in practice. Hence the operation $\star$ can be legitimately called a completion.

§7. Conclusion. Gentzen's sequent calculus comprises two sorts of inference rules: logical inference rules and structural inference rules. Accordingly, it is natural to consider that any proof of cut elimination consists of two parts (even if they are entangled in practice): the logical part and the structural part. Of these two, the former part is semantically dealt with by Okada's construction (at least when substructural logics, linear logic and intuitionistic/classical logics are concerned), which is algebraically reformulated as *quasi-completion* by [**??**]; remember that Okada's lemma is only concerned with the properties of logical connectives, and it holds for any extension of $\mathbf{FL}^+$, no matter which inference rules and/or axioms are added. On the other hand, what we have established is the fact that the structural part is deeply connected with the propagation property; it is only when $\mathcal{R}$ satisfies the propagation property that Okada's phase structure with respect to $\mathcal{R}$ is guaranteed to be an $\mathcal{R}$-residuated lattice, and thus cut elimination is obtained. We believe that the propagation property together with the quasi-completion captures some essence of cut elimination from an algebraic point of view.

Acknowledgement. We are indebted to Jean-Yves Girard, who suggested to the author a possible linkage between his test and cut elimination in 1999, and thus motivated the current work. Our thanks are also due to Shun'ichi Amano, Felix Bou, Agata Ciabattoni, Nikolaos Galatos, Makoto Kanazawa, Mitsuhiro Okada, Hiroakira Ono, Takafumi Sakurai, Kentaro Sato and Hiroki Takamura for various comments and stimulating discussions.

NATIONAL INSTITUTE OF INFORMATICS
2-1-2 HITOTSUBASHI, CHIYODA-KU, TOKYO 101-8430, JAPAN
E-mail: terui@nii.ac.jp