Towards a Semantic Characterization of Cut Elimination

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- Our approach: purely algebraic (phase semantic)
  1. Consider some general class of sequent calculi. Some enjoy cut-elimination, others do not.
  2. Give algebraic criteria for such a sequent calculus to admit cut-elimination.
- Our program:
  1. Structural rules
  2. Logical connectives
  3. Classical sequent calculi
  4. Modalities
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- By-product: completion of structural rules.

  Any set of structural rules can be converted into another set which admits cut-elimination.

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- Show that for any simple sequent calculus $\mathcal{L}$

$$\text{reductive cut-elimination} \quad \Longleftrightarrow \quad \text{propagation and coherence.}$$

- Also give a characterization of axiom expansion. When structural rules satisfy propagation,

$$\text{axiom expansion} \quad \Longleftrightarrow \quad \text{opposite of coherence}$$

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$$\forall x \in \mathbf{M}.\ \{x\}^{\bot\bot} \multimap \{x \cdot x\}^{\bot\bot} \implies \forall X : \text{ fact } (X \multimap X \otimes X).$$

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  - Relationship with cut elimination is hinted, but not proved.
- How is it possible to relate such a semantic criterion to syntactic cut elimination?

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- Okada's phase semantic proof of cut elimination for linear logic (1996).
  - A degenerate version of Tait-Girard's reducibility argument.
  - A powerful technique to prove cut elimination for various logics.
- Girard's test, when suitably modified, gives a sufficient condition for the applicability of Okada's argument.

Non-Commutative Intuitionistic Linear Logic

- Formulas: $A \mathbin{\&} B$, $A \oplus B$, $A \otimes B$, $A \multimap B$, $B \circ\!- A$, $!A$, $\top$, $\mathbf{0}$, $\mathbf{1}$.
- Sequents: $\Gamma \Rightarrow Z$ (Γ: sequence of formulas)
- Use $X, Y, Z, \ldots$ as metavariables to be replaced by formulas.
- (Selected) inference rules:

$$\frac{\Gamma \Rightarrow X \quad \Delta_1, X, \Delta_2 \Rightarrow Z}{\Delta_1, \Gamma, \Delta_2 \Rightarrow C}\ Cut \qquad \frac{}{X \Rightarrow X}\ Identity \qquad \frac{\Gamma \Rightarrow X \quad \Delta \Rightarrow Y}{\Gamma, \Delta \Rightarrow X \otimes Y}\ \otimes r$$

$$\frac{\Gamma \Rightarrow X \quad \Delta_1, Y, \Delta_2 \Rightarrow Z}{\Delta_1, \Gamma, X \multimap Y, \Delta_2 \Rightarrow Z}\ {\multimap} l \qquad \frac{X, \Gamma \Rightarrow Y}{\Gamma \Rightarrow X \multimap Y}\ {\multimap} r \qquad \frac{\Gamma_1, X, Y, \Gamma_2 \Rightarrow Z}{\Gamma_1, X \otimes Y, \Gamma_2 \Rightarrow Z}\ \otimes$$

$$\frac{\Gamma_1, X, \Gamma_2 \Rightarrow Z \quad \Gamma_1, Y, \Gamma_2 \Rightarrow Z}{\Gamma_1, X \oplus Y, \Gamma_2 \Rightarrow Z}\ \oplus l \qquad \frac{\Gamma \Rightarrow X}{\Gamma \Rightarrow X \oplus Y}\ \oplus r_1 \qquad \frac{\Gamma \Rightarrow Y}{\Gamma \Rightarrow X \oplus Y}\ \oplus r_2$$

Basic facts

- $\bigotimes \Sigma, \Gamma \Rightarrow C$ is provable iff $\Sigma, \Gamma \Rightarrow C$ is provable.
- $\Gamma \Rightarrow A \multimap B$ is provable iff $A, \Gamma \Rightarrow B$ is provable.
- $\bigoplus \Phi, \Gamma \Rightarrow B$ is provable iff $A, \Gamma \Rightarrow B$ is provable for every $A \in \Phi$.
- $B \otimes (\bigoplus \Phi) \otimes C \Rightarrow \bigoplus_{A \in \Phi} (B \otimes A \otimes C)$ and the converse are provable.

Structural rules: example

- Gentzen's original rules:

Exchange:

$$\frac{\Gamma, X, Y, \Delta \Rightarrow C}{\Gamma, Y, X, \Delta \Rightarrow C}$$

$$X \otimes Y \multimap Y \otimes X$$

Weakening:

$$\frac{\Gamma, \Delta \Rightarrow C}{\Gamma, X, \Delta \Rightarrow C}$$

$$X \multimap \mathbf{1}$$

Contraction:

$$\frac{\Gamma, X, X, \Delta \Rightarrow C}{\Gamma, X, \Delta \Rightarrow C}$$

$$X \multimap X \otimes X$$

- Additional rules:

Seq-Contraction

$$\frac{\Gamma, \vec{X}, \vec{X}, \Delta \Rightarrow C}{\Gamma, \vec{X}, \Delta \Rightarrow C}$$

$$\{\mathbf{X}_n \multimap \mathbf{X}_n \otimes \mathbf{X}_n \mid n \in N\}$$

$$(\mathbf{X}_n \equiv X_1 \otimes \cdots \otimes X_n)$$

Expansion

(van Benthem 91)

$$\frac{\Gamma, X, \Delta \Rightarrow C}{\Gamma, X, X, \Delta \Rightarrow C}$$

$$X \otimes X \multimap X$$

Mingle

(Ohnishi-Matsumoto 64)

$$\frac{\Gamma, \vec{X}, \Delta \Rightarrow C \quad \Gamma, \vec{Y}, \Delta \Rightarrow C}{\Gamma, \vec{X}, \vec{Y}, \Delta \Rightarrow C}$$

$$\{\mathbf{X}_n \otimes \mathbf{Y}_m \multimap \mathbf{X}_n \oplus \mathbf{Y}_m \mid n, m \in$$

Structural rules: definition

- A structural rule is a scheme of the form:

$$\frac{\Gamma, \vec{X}_1, \Delta \Rightarrow C \quad \cdots \quad \Gamma, \vec{X}_n, \Delta \Rightarrow C}{\Gamma, \vec{X}_0, \Delta \Rightarrow C} \; R$$

such that $\{\vec{X}_1, \ldots, \vec{X}_n\} \subseteq \{\vec{X}_0\}$ (*).

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- When $A_1, A_2, \ldots \in \Phi$, $R[A_1/X_1, A_2/X_2, \ldots]$ is called an Φ-instance.

Motivating Example (1)

- Contraction alone (without Exchange) does not admit cut elimination:

$$
\dfrac{\dfrac{\overline{A \Rightarrow A} \quad \overline{B \Rightarrow B}}{A, B \Rightarrow A \otimes B} \qquad \dfrac{\dfrac{\overline{A \otimes B \Rightarrow A \otimes B} \quad \overline{A \otimes B \Rightarrow A \otimes B}}{A \otimes B, A \otimes B \Rightarrow (A \otimes B) \otimes (A \otimes B)}}{A \otimes B \Rightarrow (A \otimes B) \otimes (A \otimes B)}\, Cntr}{A, B \Rightarrow (A \otimes B) \otimes (A \otimes B)}\, Cut
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- Is it possible to describe the difference between them without using the word ‘cut elimination’?

Motivating Example (1)

- Seq-Contraction propagates from atomic instances to $\otimes$-instances:

$$\mathbf{X}_n \multimap \mathbf{X}_n \otimes \mathbf{X}_n[\alpha_1 \otimes \beta_1/X_1, \ldots, \alpha_n \otimes \beta_n/X_n]$$

is derivable in NCILL from

$$\mathbf{X}_{2n} \multimap \mathbf{X}_{2n} \otimes \mathbf{X}_{2n}[\alpha_1/X_1, \beta_1/X_2, \ldots, \alpha_n/X_{2n-1}, \beta_n/X_{2n}]$$

- On the other hand, Contraction does not propagate from atomic instances to $\otimes$-instances:

$$X \multimap X \otimes X[\alpha \otimes \beta/X]$$

is not derivable from

$$X \multimap X \otimes X[\alpha/X], \quad X \multimap X \otimes X[\beta/X], \quad \text{etc.}$$

Motivating Example (2)

- Expansion does not admit cut elimination:

$$\dfrac{\dfrac{\overline{B \Rightarrow B}}{B \Rightarrow A \oplus B} \quad \dfrac{\dfrac{\overline{A \Rightarrow A}}{A \Rightarrow A \oplus B} \quad \dfrac{\overline{A \oplus B \Rightarrow A \oplus B}}{A \oplus B, A \oplus B \Rightarrow A \oplus B}\,Exp}{A, A \oplus B \Rightarrow A \oplus B}\,Cut}{A, B \Rightarrow A \oplus B}\,Cut$$

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- Expansion $X \otimes X \multimap X$ and Mingle $\{\mathbf{X}_n \otimes \mathbf{Y}_m \multimap \mathbf{X}_n \oplus \mathbf{Y}_m \mid n, m \in N\}$ are equivalent with respect to provability in NCILL.

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- Mingle propagates from atomic instances to $\oplus$-instances: Eg,

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- whereas Expansion does not propagate from atomic instances to $\oplus$-instances:

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  is not derivable from

$$X \otimes X \multimap X[\alpha_1/X], \quad X \otimes X \multimap X[\alpha_2/X], \quad etc.$$

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- Formulas with implications restricted to $\alpha \multimap D$ = "real statements"
- Fact 1: Every $\oplus$-instance (and $\otimes$-instance) of $\hat{R}$ is equivalent (in NCILL) to a real statement C.

$$\begin{aligned}
\hat{R}[\alpha_i \oplus \beta_i / X_i] &\equiv \bigotimes \vec{X}_0 \multimap (\bigotimes \vec{X}_1) \oplus \cdots \oplus (\bigotimes \vec{X}_n)[\alpha_i \oplus \beta_i / X_i] \\
\alpha \otimes \beta \multimap C &\Leftrightarrow \alpha \multimap \beta \multimap C \\
\alpha \oplus \beta \multimap C &\Leftrightarrow (\alpha \multimap C) \,\&\, (\beta \multimap C)
\end{aligned}$$

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Intermezzo

- We have shown

  Cut Elimination $\Longrightarrow$ Syntactic Propagation

- We next show

  Syntactic Propagation $\Longrightarrow$ Semantic Propagation

Phase structures

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- $\mathcal{ATOM}_{\mathbf{P}}$: the set of atomic closed sets $C(\{a\})$ with $a \in A$.

Phase structures

- For any $X, Y \in \mathcal{CL}_{\mathbf{P}}$,

$$\begin{aligned} X \mathbin{\&} Y &= X \cap Y, \\ X \oplus Y &= C(X \cup Y), \\ X \otimes Y &= C(X \bullet Y) = C(\{x \cdot y \mid x \in X, y \in Y\}), \\ X \multimap Y &= \{y \mid \forall x \in X, x \cdot y \in Y\}, \\ Y \circ\!\!- X &= \{y \mid \forall x \in X, y \cdot x \in Y\}. \end{aligned}$$

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- An $\mathcal{R}$-phase structure is a phase structure in which all axiom representations $\hat{R} \in \widehat{\mathcal{R}}$ are valid.
- Soundness: Any formula provable in NCILL+$\mathcal{R}$ is valid in all $\mathcal{R}$-phase structures.

Semantic propagation property

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Key property of phase structures

- Lemma: In every phase structure,

$$\mathcal{CL}_{\mathbf{P}} = \bigoplus \bigotimes_{fin} \mathcal{ATOM}_{\mathbf{P}},$$

i.e., any $X \in \mathcal{CL}_{\mathbf{P}}$ can be decomposed as:

$$\begin{aligned} X &= \bigoplus_{x \in X} C(\{x\}) \\ &= \bigoplus_{x \in X} \bigotimes_{x = a_1 \cdots a_n} C(\{a_i\}) \end{aligned}$$

Intermezzo

- We have shown

$$\text{Cut Elimination} \implies \text{Syntactic Propagation}$$
$$\implies \text{Semantic Propagation}$$

- We finally show

$$\text{Semantic Propagation} \implies \text{Cut Elimination}$$

by employing Okada's construction of phase structures for cut elimination.

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- The operator $Cl : \wp(\mathcal{F}^*) \longrightarrow \wp(\mathcal{F}^*)$ defined by:

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\begin{array}{rcl}
[\![\Gamma\_\Delta \Rightarrow C]\!] & = & \{\Sigma \mid \Gamma, \Sigma, \Delta \Rightarrow C \text{ is cut-free provable in NCILL}+\mathcal{R}\}, \\
\mathcal{BASE} & = & \{[\![\Gamma\_\Delta \Rightarrow C]\!] \mid \Gamma, \Delta, C \text{ arbitrary}\}, \\
Y \in \mathcal{CL}_{\mathbf{P}} & \Leftrightarrow & Y = \bigcap_{i \in \Lambda} [\![\Gamma_i\_\Delta_i \Rightarrow C_i]\!] \\
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- In particular, $\mathcal{ATOM} = \{Cl(\{A\}) \mid A \text{ is a formula}\}$

Semantic cut-elimination

- Meaning of closure operator:

  $\Sigma \in Cl(\{\Lambda\}) \iff$
  Whenever $\Gamma, \Lambda, \Delta \Rightarrow C$ is cut-free derivable, so is $\Gamma, \Sigma, \Delta \Rightarrow C$.

Idea of semantic cut-elimination

- Without structural rules:

"Real" elements:

$$\mathcal{BASE} : \llbracket \Gamma\_\Delta \Rightarrow C \rrbracket$$

codify cut-free provability in NCILL

$\Downarrow$ add

"ideal" elements:

$$\mathcal{CL}_{\mathbf{P}} : \bigcap_{i \in I} \llbracket \Gamma_i\_\Delta_i \Rightarrow C_i \rrbracket$$

model of NCILL with cuts

$$\text{NCILL} \vdash A \quad \overset{\text{Soundness}}{\Longrightarrow} \quad \mathbf{P} \models A$$

$$\overset{\text{Okada's Lemma}}{\Longrightarrow} \quad \text{“} \mathbf{P} \models_{base} A \text{”}$$

$$\Longrightarrow \quad \text{Cut-free NCILL} \vdash A$$

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model of NCILL+$\mathcal{R}$???

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$\Longrightarrow$ Cut-free NCILL $\vdash A$

Semantic cut-elimination

- Lemma: Any $\hat{R} \in \widehat{\mathcal{R}}$ is atomically valid in $\mathbf{P} = (\mathcal{F}^*, C)$.

$$\frac{\Gamma, \vec{X}_1, \Delta \Rightarrow C \quad \cdots \quad \Gamma, \vec{X}_n, \Delta \Rightarrow C}{\Gamma, \vec{X}_0, \Delta \Rightarrow C}\ R$$

- Corollary: If $\mathcal{R}$ satisfies the semantic propagation property, then $\mathbf{P}$ is an $\mathcal{R}$-phase structure.
- Proposition: If $\mathcal{R}$ satisfies the semantic propagation property, then NCILL$+\mathcal{R}$ enjoys cut elimination.

Main result of Part I

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  3. $\mathcal{R}$ satisfies the semantic propagation property.
- Corollary: If NCILL+$\mathcal{R}_1$ and NCILL+$\mathcal{R}_2$ admit cut-elimination, then so does NCILL+$(\mathcal{R}_1 \cup \mathcal{R}_2)$.

Completion of structural rules

- Substitution of sequences yields $\otimes$-syntactic propagation:

$$\frac{\Gamma, X, X, \Delta \Rightarrow C}{\Gamma, X, \Delta \Rightarrow C}\ (Con) \quad \Longrightarrow \quad \frac{\Gamma, \vec{X}, \vec{X}, \Delta \Rightarrow C}{\Gamma, \vec{X}, \Delta \Rightarrow C}\ (Seq - Con)$$

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- Substitution of sequences yields $\otimes$-syntactic propagation:

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  - NCILL+$\mathcal{R}^*$ admits cut elimination.

Remark on Part I

- We only considered uniform structural rules (that work in all contexts Γ, Δ, C):

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- Such a non-uniform rule can be handled in the framework of prephase structures (Ciabattoni-Terui 05).

Part II: Motivating Example (3)

- Consider a new "connective" $A \sqcap B$ (tend) defined by:

$$\dfrac{\Sigma \Rightarrow A \quad \Sigma \Rightarrow B}{\Sigma \Rightarrow A \sqcap B}\ (\sqcap, r) \qquad\qquad \dfrac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, A \sqcap B, \Delta \Rightarrow C}\ (\sqcap, l)$$

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- Without any structural rules, it does not satisfy (reductive) cut-elimination nor axiom expansion:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Sigma \Rightarrow A\end{array} \quad \begin{array}{c}\vdots\\ \Sigma \Rightarrow B\end{array}}{\Sigma \Rightarrow A \sqcap B} \quad \dfrac{\begin{array}{c}\vdots\\ \Gamma, A, B, \Delta \Rightarrow C\end{array}}{\Gamma, A \sqcap B, \Delta \Rightarrow C}}{\Gamma, \Sigma, \Delta \Rightarrow C}\ (Cut) \quad \not\Longrightarrow \quad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \Sigma \Rightarrow A\end{array} \quad \dfrac{\begin{array}{c}\vdots\\ \Sigma \Rightarrow B\end{array} \quad \begin{array}{c}\vdots\\ \Gamma, A, B, \Delta \Rightarrow C\end{array}}{\Gamma, A, \Sigma, \Delta \Rightarrow C}}{\Gamma, \Sigma, \Sigma, \Delta \Rightarrow C}}{\Gamma, \Sigma, \Delta \Rightarrow C}\ (???)$$

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- Rules determine the meaning!

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- Let us associate two interpretations to $\sqcap$ in a phase structure $\mathbf{P}$: for $X, Y \in \mathcal{CL}_{\mathbf{P}}$,

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$$X \sqcap^l Y \;=\; X \otimes Y \subseteq (X \otimes \mathbf{1}) \,\&\, (\mathbf{1} \otimes Y) \;=\; X \,\&\, Y \;=\; X \sqcap^r Y$$

A natural conjecture

- Let $\mathcal{L}$ be an (intuitionistic, propositional) sequent calculus having a logical connective $\star$ and "good" (i.e. propagating) structural rules $\mathcal{R}$.

$\mathcal{L}$ admits reductive cut-elimination $\iff$ $\star^r(\vec{X}) \subseteq \star^l(\vec{X})$ in every $\mathcal{R}$-phase structure

$\mathcal{L}$ admits axiom expansion $\iff$ $\star^l(\vec{X}) \subseteq \star^r(\vec{X})$ in every $\mathcal{R}$-phase structure

Supporting facts

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- If axiom expansion holds, $\star^l(\vec{X}) \Rightarrow \star^r(\vec{X})$ is derived from atomic axioms $X_i \Rightarrow X_i$.
- Thus by soundness, $\star^l(\vec{X}) \subseteq \star^r(\vec{X})$.

Basic notions

- Propositional variables: $\alpha, \beta, \gamma, \ldots$
- Logical connectives (of suitable arity): $\star_1, \star_2, \star_3, \ldots$
- Formulae: propositional variables or $\star(A_1, \ldots, A_m)$ with $A_1, \ldots, A_m$ formulae.
- (Single-conclusion) sequents: $\Gamma \Rightarrow \Delta$ with $|\Delta| \leq 1$.

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$$\frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \cdots \quad \Upsilon_n \Rightarrow \Psi_n}{\Theta_l, \star(\vec{X}), \Theta_r \Rightarrow \Xi}\ (\star, l)_j \qquad \frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \cdots \quad \Upsilon_n \Rightarrow \Psi_n}{\Theta_l \Rightarrow \star(\vec{X})}\ (\star, r)_k$$

with $\{\Upsilon_1, \ldots, \Upsilon_n\} \subseteq \{\Theta_l, \Theta_r, \vec{X}\}$, $\{\Psi_1, \ldots, \Psi_n\} \subseteq \{\Xi, \vec{X}\}$, $\{\Theta_l, \Theta_r\} \cap \{\Xi\} = \emptyset$.

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- $\{\Theta_l, \Theta_r\}$ (left context variables), $\{\Xi\}$ (right context variables), $\{\vec{X}\}$ (active variables) are mutually disjoint.

3 Types of Cuts

1. Cut against a context formula of a structural rule:

$$\dfrac{\Sigma \Rightarrow A \qquad \dfrac{\cdots \quad \Gamma' \Rightarrow C' \quad \cdots}{\Gamma, A, \Delta \Rightarrow C}\,(R)}{\Gamma, \Sigma, \Delta \Rightarrow C}$$

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3. Cut between two principal formulae of logical rules:

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- We want to concentrate on 3. Hence we only consider those logical rules which allow shifting-up of type-2 cuts.

Example

- $\sqcap$ is defined by a countable set of left and right logical rules:

$$\frac{\Theta_i \Rightarrow X \quad \Theta_i \Rightarrow Y}{\Theta_i \Rightarrow X \sqcap Y}\ (\sqcap, r)_i \qquad \frac{\Theta_j, X, Y, \Theta_k \Rightarrow \Xi}{\Theta_j, X \sqcap Y, \Theta_k \Rightarrow \Xi}\ (\sqcap, l)_{jk}$$

for all $i, j, k \in N$. Here

$$\Theta_i \equiv Z_1, \ldots, Z_i$$

(sequence of i distinct metavariables.)

Reductive cut-elimination

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$$\frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \cdots \quad \Upsilon_m \Rightarrow \Psi_m}{Y_1, \ldots, Y_k \Rightarrow \star(\vec{X})}\ (\star, r)_i$$

$\star_{\mathbf{P}}^{r,i}(\vec{X})$ = the maximal value of $C(Y_1 \bullet \cdots \bullet Y_k) \in \mathcal{CL}_{\mathbf{P}}$ such that $\Upsilon_1 \subseteq \Psi_1, \ldots, \Upsilon_m \subseteq \Psi_m$ (with $\vec{Y}$ ranging over $\mathcal{CL}_{\mathbf{P}}$)

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$$\frac{Z_1, X, Y, Z_2 \Rightarrow W}{Z_1, X \sqcap Y, Z_2 \Rightarrow W}\ (\sqcap, l)_1$$

$$\begin{aligned} X \sqcap^{l,1}_{\mathbf{P}} Y &= \bigcap \{ Z_1 \multimap W \circ\!\!- Z_2 \mid Z_1, Z_2, W \in \mathcal{CL}_{\mathbf{P}},\ Z_1 \bullet X \bullet Y \bullet Z_2 \subseteq W \\ &= \mathbf{1} \multimap C(X \bullet Y) \circ\!\!- \mathbf{1} \ = \ C(X \bullet Y) \end{aligned}$$

Interpretation of logical connectives

- When $\{(\star, r)_i\}_{i\in\Lambda_r}$ and $\{(\star, l)_j\}_{j\in\Lambda_l}$ are the right and left logical rules introducing $\star$, define

$$\begin{aligned}\star^{r}_{\mathbf{P}}(\vec{X}) &= \bigoplus_{i\in\Lambda_r} \star^{r,i}_{\mathbf{P}}(\vec{X}) \\ \star^{l}_{\mathbf{P}}(\vec{X}) &= \bigcap_{j\in\Lambda_l} \star^{l,j}_{\mathbf{P}}(\vec{X})\end{aligned}$$

Interpretation of logical connectives

- When $\{(\star, r)_i\}_{i \in \Lambda_r}$ and $\{(\star, l)_j\}_{j \in \Lambda_l}$ are the right and left logical rules introducing $\star$, define

$$\begin{aligned} \star^r_{\mathbf{P}}(\vec{X}) &= \bigoplus_{i \in \Lambda_r} \star^{r,i}_{\mathbf{P}}(\vec{X}) \\ \star^l_{\mathbf{P}}(\vec{X}) &= \bigcap_{j \in \Lambda_l} \star^{l,j}_{\mathbf{P}}(\vec{X}) \end{aligned}$$

- In particular:

$$X \sqcap^r_{\mathbf{P}} Y = X \cap Y \qquad X \sqcap^l_{\mathbf{P}} Y = C(X \bullet Y)$$

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- Note: the existence of a full-valuation is not guaranteed yet!

Coherence and Rigidity

- A logical connective $\star$ is coherent in $\mathcal{L}$ (with structural rules $\mathcal{R}$) if

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- If rigidity holds in addition, then the valuation is uniquely determined by the values of atomic formulae.

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- Proof: By axiom expansion:

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Semantic reading $\Longrightarrow$

$$\dfrac{X_1 \subseteq X_1 \quad \cdots \quad X_n \subseteq X_n}{\star^l_{\mathbf{P}}(\vec{X}) \subseteq \star^r_{\mathbf{P}}(\vec{X})} \vdots \text{ Str. rules + } (\star, l) \text{ + } (\star, r)$$

for any interpretations $\vec{X} \in \mathcal{CL}_{\mathbf{P}}$ of active variables.

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for any interpretations $\vec{X} \in \mathcal{CL}_{\mathbf{P}}$ of active variables.

- Key point: since there is no cut, one can interpret $\star(\vec{X})$ differently, depending on whether it appears on the left or right.

Intermezzo

- We have seen:

$$\text{Reductive cut-elimination} \Longrightarrow \text{Coherence}$$
$$\text{Axiom expansion} \Longrightarrow \text{Rigidity}$$

- We must show:

$$\text{Coherence + Propagation} \Longrightarrow \text{Reductive cut-elimination}$$
$$\text{Rigidity + Propagation} \Longrightarrow \text{Axiom expansion}$$

- In the sequel, we show (non-reductive) cut-elimination instead of reductive cut-elimination.

Syntactic phase structure

- Fix a simple sequent calculus $\mathcal{L}$. As before:
- $\mathcal{F}^*$: free monoid generated by the formulas $\mathcal{F}$ of $\mathcal{L}$.
- The operator $C : \wp(\mathcal{F}^*) \longrightarrow \wp(\mathcal{F}^*)$ defined by:

$$
\begin{array}{rcl}
[\![\Gamma\_\Delta \Rightarrow C]\!] & = & \{\Sigma \mid \Gamma, \Sigma, \Delta \Rightarrow C \text{ is cut-free provable in } \mathcal{L}\}, \\
Y \in \mathcal{CL}_{\mathbf{P}} & \Leftrightarrow & Y = \bigcap_{i \in \Lambda} [\![\Gamma_i\_\Delta_i \Rightarrow C_i]\!]
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  If X_i is a closed set such that $A_i \in X_i \subseteq [\![A_i]\!]$, then

  (1) $\star^r(\vec{X}) \subseteq [\![\star(\vec{A})]\!]$,
  (2) $\star(\vec{A}) \in \star^l(\vec{X})$.

Coherence recovers One Okada

- Lemma: If all the logical connectives in $\mathcal{L}$ are coherent, then there is a valuation f_0 such that for any $\star(\vec{A})$ we have:

$$
\begin{array}{ccccc}
 & & \star^r_{\mathbf{P}}(\overrightarrow{f_0(A)}) & & \\
 & & \cup\!\!\mid & & \\
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 & & \cup\!\!\mid & & \\
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Coherence recovers One Okada

- Lemma: If all the logical connectives in $\mathcal{L}$ are coherent, then there is a valuation f_0 such that for any $\star(\vec{A})$ we have:

$$\star^r(\overrightarrow{f_0(A)})$$

$$\cap|$$

$$\star(\vec{A}) \quad \in \quad f_0(\star(\vec{A})) \quad \subseteq \quad [\![\star(\vec{A})]\!]$$

$$\stackrel{def}{=} \star^l(\overrightarrow{f_0(A)}) \cap [\![\star(\vec{A})]\!]$$

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- To show reductive cut-elimination, more delicate argument is needed...

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- Split Okada's Lemma still holds. Together with rigidity,

$$\star(\vec{\alpha}) \overset{\text{Okada}^l}{\in} \star^l_{\mathbf{P}}(\overrightarrow{[\![X]\!]}) \overset{\text{Rigidity}}{\subseteq} \star^r_{\mathbf{P}}(\overrightarrow{[\![X]\!]}) \overset{\text{Okada}^r}{\subseteq} [\![\star(\vec{X})]\!].$$

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- I.e., $\star(\vec{X}) \Rightarrow \star(\vec{X})$ is cut-free derivable from atomic axioms.

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  If $\mathcal{L}_1$ and $\mathcal{L}_2$ admit cut-elimination, then so does $\mathcal{L}_1 \cup \mathcal{L}_2$.

Difficulty in classical case

- Everything so far works fine (assuming cyclicity). But LK does not satisfy reductive cut-elimination!

$$\dfrac{\dfrac{\Rightarrow A, B, B}{\Rightarrow A, B}\ (Contr) \qquad \dfrac{\Rightarrow B^{\perp}, B^{\perp}, C}{\Rightarrow B^{\perp}, C}\ (Contr)}{\Rightarrow A, C}\ (Cut)$$

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- Then what's the point of characterizing reductive cut-elimination?