Decomposition of Computation via Linear Logic

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Motivation (1)

- Linear Logic decomposes Intuitionistic Logic into

  Multiplicatives: $\otimes$, $\multimap$

  Additives: $\&$, $\oplus$

  Exponentials: $!$

- Eg,

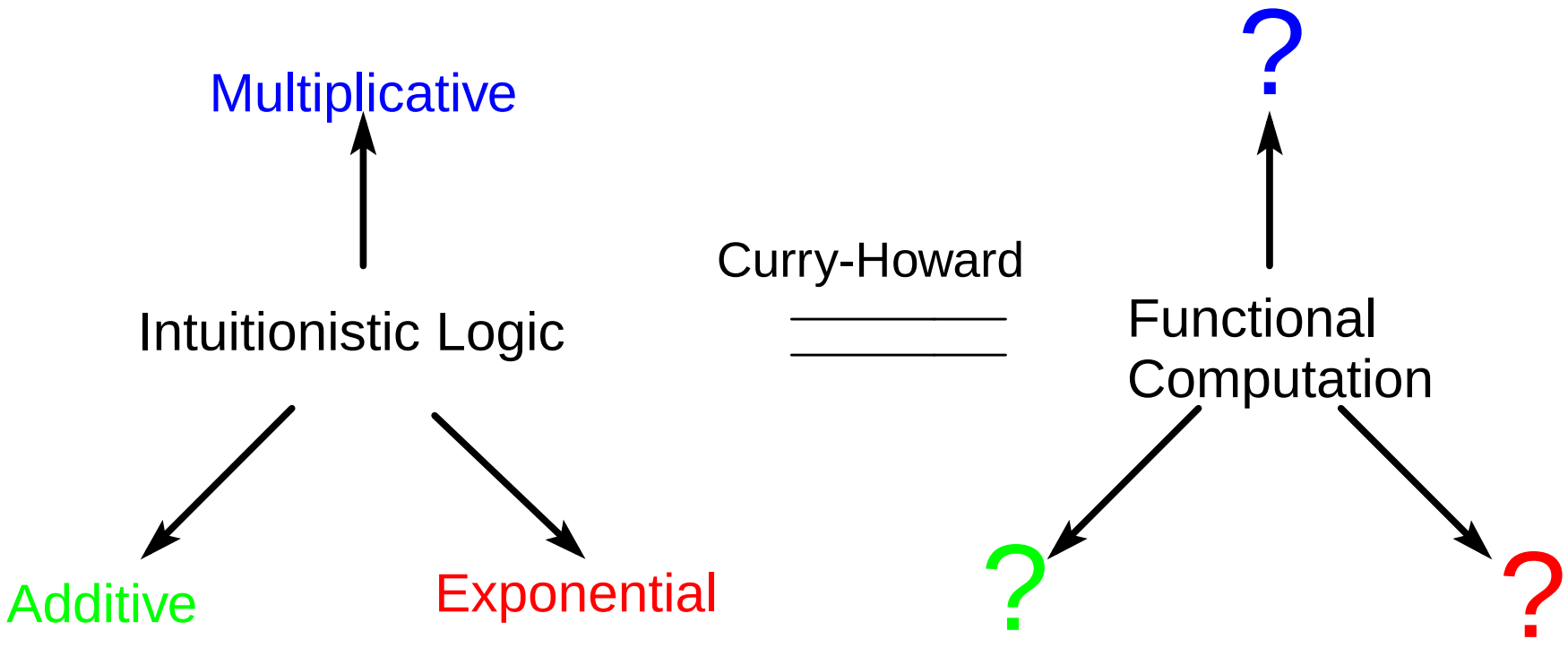
$$A \to B \equiv\ !A \multimap B$$

$$A \vee B \equiv\ !A \oplus !B$$

- Curry-Howard isomorphism relates Intuitionistic Logic to Functional Computation.

Motivation (2)

- How does Linear Logic decompose functional computation?



- Study from the viewpoint of computational complexity.

Summary

- MLL: complete for P
  — Size-efficient coding of boolean circuits.
- MALL: coNP-complete
  — Nondeterministic cut-elimination procedure.
- Generic MLL: captures Uniform P
  — A logical notion of uniformity.

Cut-Elimination as a Problem

- Cut-Elimination Problem (CEP):

  Given 2 proofs, do they reduce to the same normal form?

- Subsumes:

  Given a proof, does it reduce to “true”?

- CEPfor Linear Logic is non-elementary (Statman 1979).

- CEPfor **MLL** is in P.

Syntax of MLL

- We only consider the intuitionistic fragment **IMLL**.
- Identify **IMLL** proofs = untyped linear lambda terms.
- Justified by Hindley's theorem: any linear lambda term has a simple (propositional) type.
- Types ($\multimap, \forall$) are used neither for restriction nor for enrichment, but for classification.

$$\frac{}{x:A \vdash x:A} \qquad \frac{\Gamma \vdash u:A \quad x:A, \Delta \vdash t:C}{\Gamma, \Delta \vdash t[u/x]:C}$$

$$\frac{x:A, \Gamma \vdash t:B}{\Gamma \vdash \lambda x.t : A \multimap B} \qquad \frac{\Gamma \vdash u:A \quad x:B, \Delta \vdash t:C}{\Gamma, y:A \multimap B, \Delta \vdash t[yu/x]:C}$$

$$\frac{\Gamma \vdash t:A}{\Gamma \vdash t:\forall \alpha.A} \ \alpha \notin FV(\Gamma) \qquad \frac{x:A[B/\alpha], \Gamma \vdash t:C}{x:\forall \alpha.A, \Gamma \vdash t:C}$$

Defined Connectives

$$\mathbf{1} \equiv \forall\alpha.\alpha \multimap \alpha \qquad A \otimes B \equiv \forall\alpha.(A \multimap B \multimap \alpha) \multimap \alpha$$

$$\mathsf{I} \equiv \lambda x.x \qquad t \otimes u \equiv \lambda x.xtu$$

$$\text{let } t \text{ be I in } u \equiv tu \qquad \text{let } t \text{ be } x \otimes y \text{ in } u \equiv t(\lambda xy.u).$$

The above definitions are sound w.r.t.

$$\text{let I be I in } t \longrightarrow t \qquad \text{let } t \otimes u \text{ be } x \otimes y \text{ in } v \longrightarrow v[t/x, u/y]$$

(but *not* w.r.t. the commuting reduction rules)

$$\frac{}{\vdash \mathsf{I} : \mathbf{1}} \qquad \frac{\Gamma \vdash t : C}{x : \mathbf{1}, \Gamma \vdash \text{let } x \text{ be I in } t : C}$$

$$\frac{\Gamma \vdash t : A \quad \Delta \vdash u : B}{\Gamma, \Delta \vdash t \otimes u : A \otimes B} \qquad \frac{x : A, y : B, \Gamma \vdash t : C}{z : A \otimes B, \Gamma \vdash \text{let } z \text{ be } x \otimes y \text{ in } t : C}$$

Π_1 and $e\Pi_1$ types

- Π_1: constructed by $\multimap$, $\otimes$, **1** (viewed as primitives) and positive $\forall$.
- Example:

$$\mathbf{B} \equiv \forall\alpha.\alpha \multimap \alpha \multimap \alpha \otimes \alpha \quad \text{(multiplicative boolean type)}$$

$$\mathbf{W}\langle n\rangle \equiv \forall\alpha.(\mathbf{B} \multimap \alpha \multimap \alpha)^n \multimap \alpha \multimap \alpha \quad (\{0,1\}^n)$$

- Π_1 includes finite data types.
- $e\Pi_1$: like Π_1, but may contain negative inhabited types.
- Example: **B** is Π_1 inhabited. Hence **B** $\multimap$ **B** is $e\Pi_1$.
- $e\Pi_1$ includes functionals over finite data types.

Elimination of $\otimes$ and 1

- Proposition: Any Π_1 type is "isomorphic" to another Π_1 type not containing $\otimes$ nor $\mathbf{1}$. Similarly for $e\Pi_1$.
- Proof: Positive $\otimes$ and $\mathbf{1}$ are removed by their Π_1 definitions, while negative ones are removed by

$$((A \otimes B) \multimap C) \quad \circ\!\!-\!\!\circ \quad (A \multimap B \multimap C)$$

$$(\mathbf{1} \multimap C) \quad \circ\!\!-\!\!\circ \quad C$$

Weakening in MLL

- Theorem ($e\Pi_1$-Weakening): For any closed $e\Pi_1$ type A, there is a term w_A of type $A \multimap \mathbf{1}$.
- Examples:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}}{\mathbf{1} \vdash \mathbf{1}}}{\mathbf{1}, \mathbf{1} \vdash \mathbf{1}}}{\mathbf{1} \otimes \mathbf{1} \vdash \mathbf{1}}}{\mathbf{1} \multimap \mathbf{1} \otimes \mathbf{1} \vdash \mathbf{1}} \quad \vdash \mathbf{1}}{\mathbf{1} \multimap \mathbf{1} \multimap \mathbf{1} \otimes \mathbf{1} \vdash \mathbf{1}}}{\mathbf{B} \vdash \mathbf{1}} \qquad \dfrac{\dfrac{}{\vdash \mathbf{B}} \quad \dfrac{}{\mathbf{B} \vdash \mathbf{1}}}{\mathbf{B} \multimap \mathbf{B} \vdash \mathbf{1}}$$

- The left proof yields:

$$\mathsf{w}_{\mathbf{B}} \equiv \lambda z.\text{let } z\text{II be } x \otimes y \text{ in } \left(\text{let } y \text{ be I in } x\right) : \mathbf{B} \multimap \mathbf{1}.$$

Contraction in MLL

- Theorem(Π_1-Contraction): Let A be a closed inhabited Π_1 type (i.e. data type). Then there is a contraction map $\mathsf{cntr}_A : A \multimap A \otimes A$ such that for any closed term $t : A$,

$$\mathsf{cntr}_A(t) \longrightarrow^* t' \otimes t',$$

  where t' is $\beta\eta$-equivalent to t.

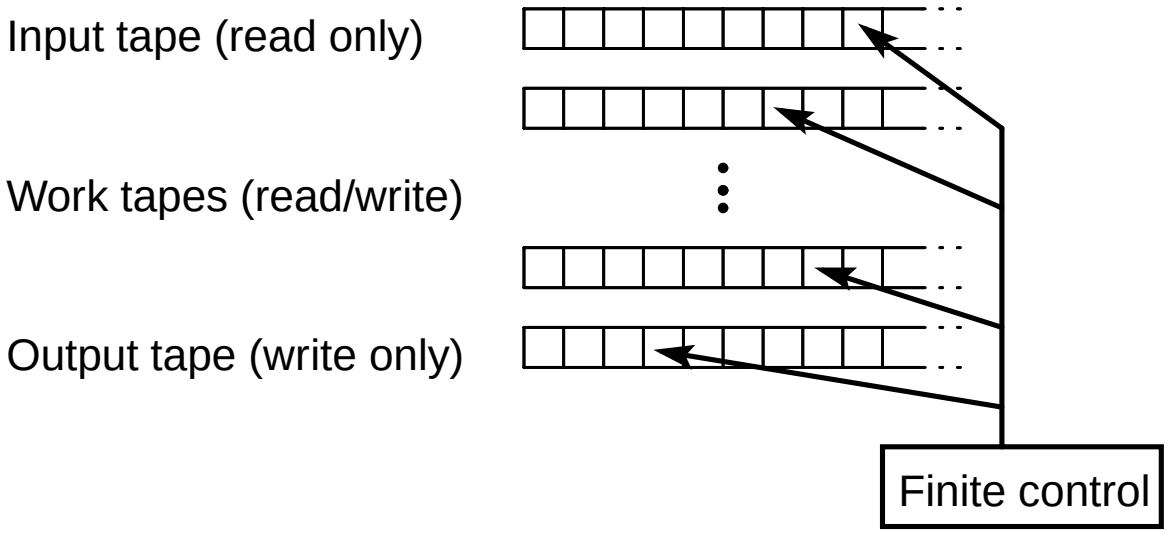
- Idea: For **B**, we have

$$\lambda z.\mathsf{if}\ z\ \mathsf{then}\ (\mathsf{true} \otimes \mathsf{true})\ \mathsf{else}\ (\mathsf{false} \otimes \mathsf{false}) : \mathbf{B} \multimap \mathbf{B} \otimes \mathbf{B}$$

- Data duplication is for free!

Turing Machines and Logspace Functions

- Multi-Tape Turing Machines

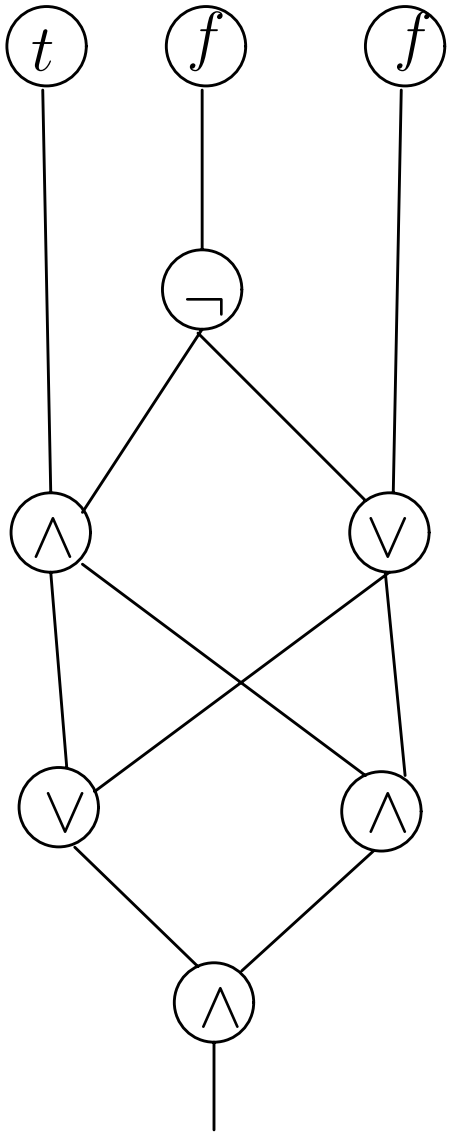


- $f : \{0,1\}^* \longrightarrow \{0,1\}^*$ is logspace if $f(w)$ can be computed within $O(\log n)$ workspace where $n = |w|$.
- Output may be polynomially large.
- The num of all possible config $= O(2^{k \log n}) = O(n^k)$ for some k. In particular, $L \subseteq P$.

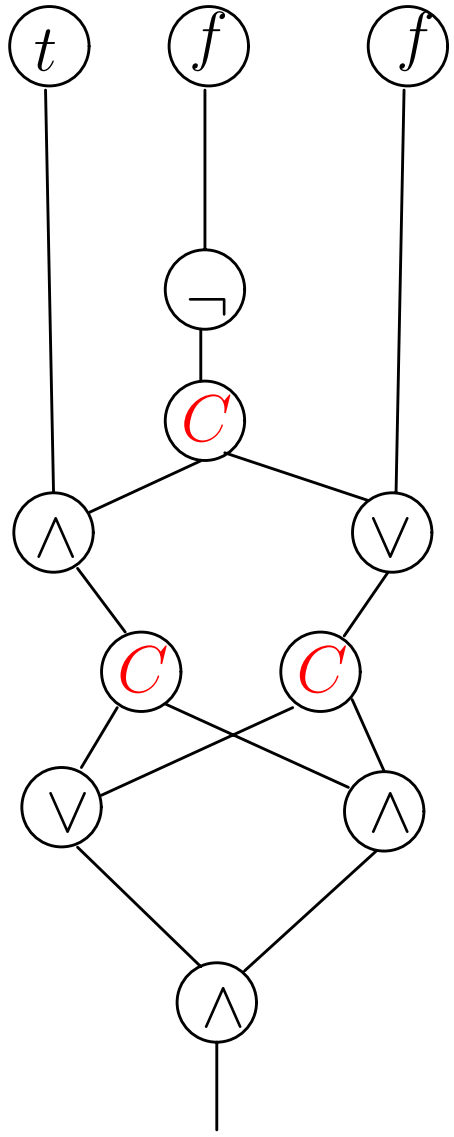
P-completeness

- A language $X \subseteq \{0,1\}^*$ is logspace reducible to $Y \subseteq \{0,1\}^*$ if there exists a logspace function $f : \{0,1\}^* \longrightarrow \{0,1\}^*$ such that $w \in X \Leftrightarrow f(w) \in Y$.
- X is P-complete if $X \in$ P and each $Y \in$ P is logspace reducible to X.
- The hardest problems in P.
- If X is P-complete, then $X \notin L$ unless $L =$ P.
- Circuit Value Problem (P-complete, Ladner 1975): Given a boolean circuit C with n inputs and 1 output, and n truth values $\vec{x} = x_1, \ldots, x_n$, is $\vec{x}$ accepted by C?

Boolean Circuits: implicit vs. explicit sharing



logspace
$\Longrightarrow$



Boolean Circuits in MLL

- Projection: for any $e\Pi_1$ type C,

$$\mathsf{fst}_C \equiv \lambda x.\text{let } x \text{ be } y \otimes z \text{ in } \big(\text{let } \mathsf{w}_C(z) \text{ be I in } y\big)$$

- For any closed term $t \otimes u : A \otimes C$, $\mathsf{fst}_C(t \otimes u) \longrightarrow^* t$.
- Boolean values and connectives:

true	$\equiv$	$\lambda xy.x \otimes y$	: **B**
false	$\equiv$	$\lambda xy.y \otimes x$	: **B**
not	$\equiv$	$\lambda Pxy.Pyx$	: **B** $\multimap$ **B**
or	$\equiv$	$\lambda PQ.\mathsf{fst}_{\mathbf{B}}(P\,\mathsf{true}\,Q)$	: **B** $\multimap$ **B** $\multimap$ **B**
cntr	$\equiv$	$\lambda P.\mathsf{fst}_{\mathbf{B}\otimes\mathbf{B}}\big(P(\mathsf{true} \otimes \mathsf{true})(\mathsf{false} \otimes \mathsf{false})\big)$	: **B** $\multimap$ **B** $\otimes$ **B**

Conditional

- Lemma ($e\Pi_1$-Conditional): Let

$$x_1 : C_1, \ldots, x_n : C_n \vdash t_1, t_2 : D$$

and the type $A \equiv C_1 \multimap \cdots C_n \multimap D$ is $e\Pi_1$. Then there is a conditional

$$b : \mathbf{B}, x_1 : C_1, \ldots, x_n : C_n \vdash \text{if } b \text{ then } t_1 \text{ else } t_2 : D,$$

such that $(\text{if true then } t_1 \text{ else } t_2) \longrightarrow t_1$ and $(\text{if false then } t_1 \text{ else } t_2) \longrightarrow t_2$.

- Proof: Let

$$\text{if } b \text{ then } t \text{ else } u \equiv \text{fst}_{\forall \vec{\alpha}.A}(b(\lambda \vec{x}.t)(\lambda \vec{x}.u))\vec{x}$$

P-completeness of MLL

- Theorem (Mairson2003, Mairson-Terui2004): There is a logspace algorithm which transforms a boolean circuit C with n inputs and m outputs into a term t_C of type $\mathbf{B}^n \multimap \mathbf{B}^m$, where the size of t_C is $O(|C|)$:

$$C \overset{\text{logspace}}{\Longrightarrow} t_C : \mathbf{B}^n \multimap \mathbf{B}^m$$

  As a consequence, the cut-elimination problem for **IMLL** is complete for P.

- Binary words $\{0,1\}^n$ represented by $\mathbf{B}^n$
- Any $f : \{0,1\}^n \longrightarrow \{0,1\}^m$ represented by a term $t_f : \mathbf{B}^n \multimap \mathbf{B}^m$.
- **MLL** captures all finite functions.

Remark

- **IMLL** proofs represent finite functions as size-efficient as boolean circuits.
- What about depth-efficiency?
- APN^i: the class of languages for which there are polynomial-size $\log^i$-depth proof nets in unbounded fan-in **MLL** (here depth = the logical depth of cut-formulas).
- $stconn^i$: St-connectivity gates for graphs of degree i.
- Theorem (Terui 2004): $APN^i = AC^i(stconn^2)$.
- Circuit depth corresponds to the depth of cut-formulas.

Syntax of MALL

- We only consider the intuitionistic $(\forall, \multimap, \&)$ fragment **IMALL**.
- Terms of **IMALL**: linear lambda terms plus the following;

  (i) if t and u are terms and $FV(t) = FV(u)$, then so is $\langle t, u \rangle$;

  (ii) if t is a term, then so are $\pi_1(t)$ and $\pi_2(t)$.
- Type assignment rules:

$$\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash \langle t_1, t_2 \rangle : A_1 \,\&\, A_2} \qquad \frac{x : A_i, \Gamma \vdash t : C}{y : A_1 \,\&\, A_2, \Gamma \vdash t[\pi_i(y)/x] : C} \; i = 1, 2.$$

- Reduction rule:

$$\pi_i \langle t_1, t_2 \rangle \longrightarrow t_i,$$

  for $i = 1, 2$.

Normalization in IMALL

- Normalization is exponential as it stands; let

$$\begin{aligned} t_0 &\equiv \lambda x.\langle x, x\rangle \\ t_{i+1} &\equiv \lambda x.t_i\langle x, x\rangle \end{aligned}$$

- The size of $nf(t_i)$ is exponential in i; e.g.

$$t_3 \equiv \lambda w.(\lambda x.(\lambda y.(\lambda z.\langle z, z\rangle)\langle y, y\rangle)\langle x, x\rangle)\langle w, w\rangle$$
$$\downarrow$$
$$\lambda w.(\lambda x.(\lambda y.\langle\langle y, y\rangle, \langle y, y\rangle\rangle)\langle x, x\rangle)\langle w, w\rangle$$
$$\downarrow$$
$$\lambda w.(\lambda x.\langle\langle\langle x, x\rangle, \langle x, x\rangle\rangle, \langle\langle x, x\rangle, \langle x, x\rangle\rangle\rangle)\langle w, w\rangle$$
$$\downarrow$$
$$\lambda w.\langle\langle\langle\langle w, w\rangle, \langle w, w\rangle\rangle, \langle\langle w, w\rangle, \langle w, w\rangle\rangle\rangle, \langle\langle\langle w, w\rangle, \langle w, w\rangle\rangle, \langle\langle w, w\rangle, \langle w, w\rangle\rangle\rangle$$

Slices

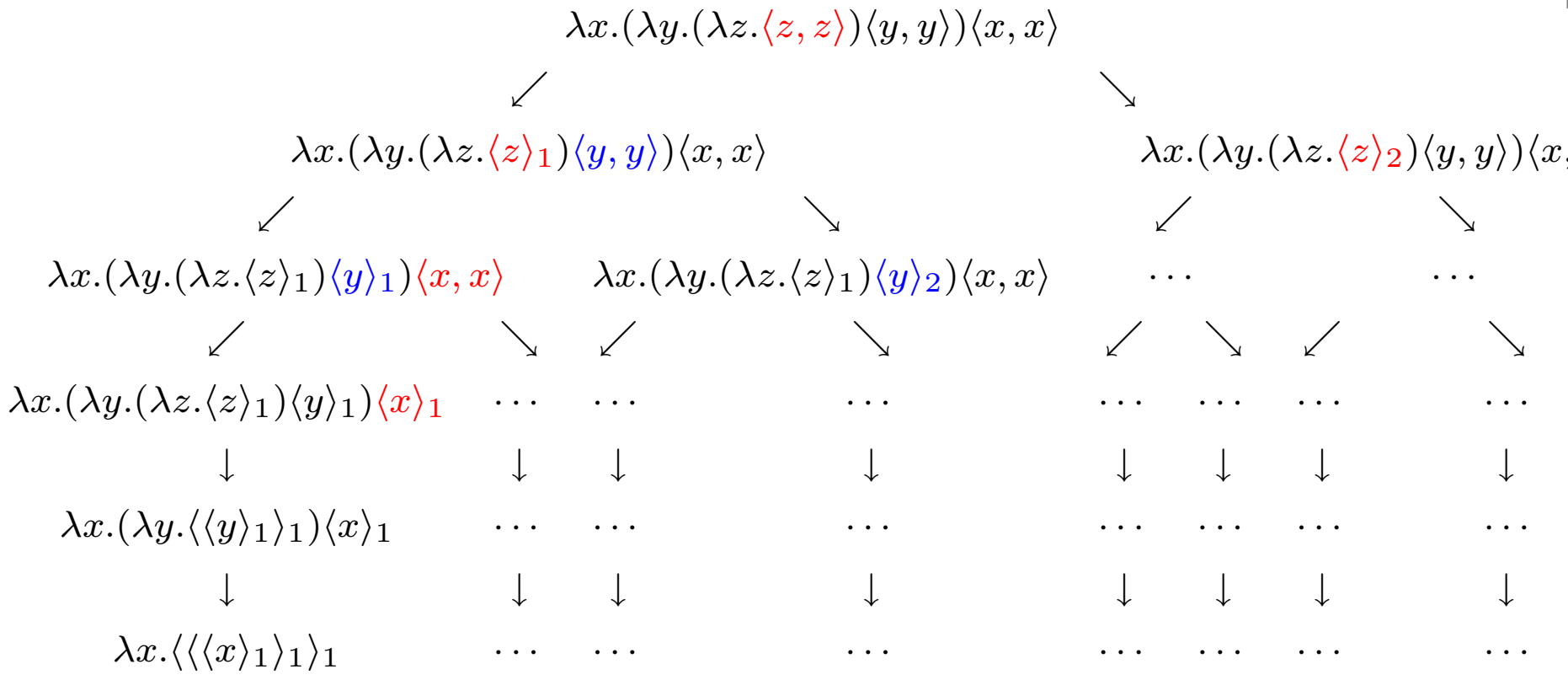
- How to avoid exponential explosion?
  - *Either* restrict to lazy additives (with no positive $\&$ in the conclusion type)
  - *Or* adopt nondeterministic cut-elimination with slices.
- A slice of a term t is obtained by applying the slicing operation:

  $\langle u, v\rangle \mapsto \langle u\rangle_1$, or $\langle u, v\rangle \mapsto \langle v\rangle_2$

  as many times as possible.
- Reduction rules for slices:

  $$(\lambda x.t)u \xrightarrow{sl} t[u/x], \quad \pi_i\langle t\rangle_i \xrightarrow{sl} t, \quad \pi_i\langle t\rangle_j \xrightarrow{sl} \text{fail}, \quad \text{if } i \neq j.$$

Nondeterministic Cut-Elimination with Slices



- Results in 8 normal forms:

$$\lambda x.\langle\langle\langle x\rangle_1\rangle_1\rangle_1 \quad \lambda x.\langle\langle\langle x\rangle_2\rangle_1\rangle_1 \quad \lambda x.\langle\langle\langle x\rangle_1\rangle_2\rangle_1 \quad \lambda x.\langle\langle\langle x\rangle_2\rangle_2\rangle_1$$
$$\lambda x.\langle\langle\langle x\rangle_1\rangle_1\rangle_2 \quad \lambda x.\langle\langle\langle x\rangle_2\rangle_1\rangle_2 \quad \lambda x.\langle\langle\langle x\rangle_1\rangle_2\rangle_2 \quad \lambda x.\langle\langle\langle x\rangle_2\rangle_2\rangle_2$$

Slicewise Checking

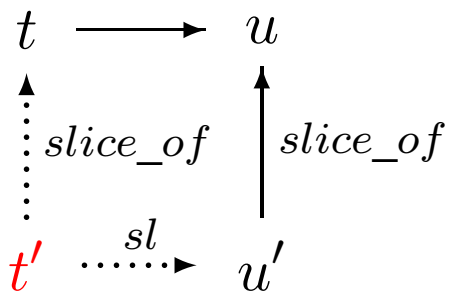
- Two slices t and u (of possibly different terms) are comparable if there is no context (i.e. a term with a hole) Φ such that $t \equiv \Phi[\langle t' \rangle_i]$, $u \equiv \Phi[\langle u' \rangle_j]$, and $i \neq j$.
- Example:

$\langle\langle\langle(\lambda x.x)\rangle_1\rangle_2\rangle_1$	$\langle\langle\langle yz\rangle_1\rangle_2\rangle_1$	: comparable;
$\langle\langle\langle(\lambda x.x)\rangle_1\rangle_2\rangle_1$	$\langle\langle\langle(\lambda x.x)\rangle_1\rangle_1\rangle_1$	: incomparable;
$\lambda x.x$	yz	: comparable.

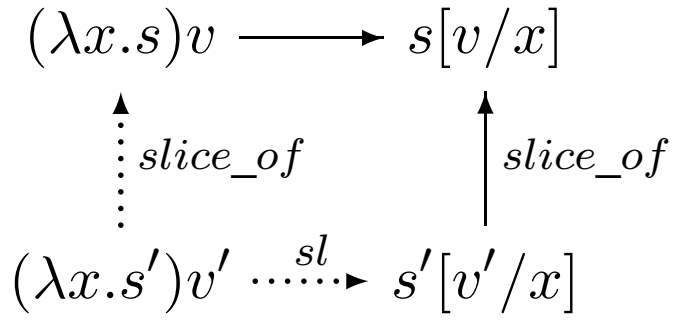
- Lemma (Slicewise Checking): Two terms t and u are equivalent iff for every comparable pair (t', u') of slices of t and u, $t' \equiv u'$.

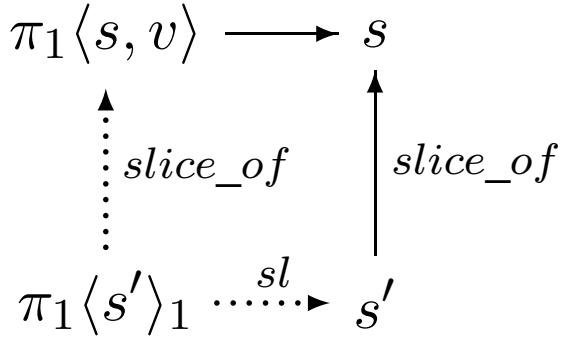
Pullback

- Lemma (Pullback): Let $t \longrightarrow^* u$ and u' be a slice of u. Then there is a unique slice t' of t such that $t' \xrightarrow{sl}{}^* u'$:



- Proof:





CEP for IMALL is in coNP

- Given t, u, suppose we want to show $nf(t) \not\equiv nf(u)$.
- By Slicewise Checking Lemma, it is sufficient to show that there is a comparabale pair of slices (t', u') of $nf(t)$ and $nf(u)$ such that $t' \not\equiv u'$.
- By Pullback Lemma, any slice of the normal terms $nf(t)$ and $nf(u)$. comes from a slice of the source terms t and u.
- Slices t' and u' can be obtained by the nondeterministic cut-elimination procedure with a suitable guess of slices, that works in quadratic time.
- Hence showing $nf(t) \not\equiv nf(u)$ belongs to NP.
- I.e., CEP for **IMALL** belongs to coNP.

Encoding a coNP-complete Problem (1)

- Logical Equivalence Problem (coNP-complete): Given two boolean formulas, are they logically equivalent?
- For any boolean formula C with n variables,

$$C \overset{\text{logspace}}{\Longrightarrow} t_C : \mathbf{B}^n \multimap \mathbf{B}^m.$$

- For each $1 \leq k \leq n$, let

$$\mathsf{ta}_k \equiv \lambda f.\lambda x_1 \cdots x_{k-1}.\langle f \text{ true } x_1 \cdots x_{k-1}, f \text{ false } x_1 \cdots x_{k-1} \rangle,$$

which is of type $\forall\alpha.(\mathbf{B}^{(k)} \multimap \alpha) \multimap (\mathbf{B}^{(k-1)} \multimap \alpha \,\&\, \alpha)$, and define

$$\mathsf{ta}(t_C) \equiv \mathsf{ta}_1(\cdots(\mathsf{ta}_n t_C)\cdots) : \underbrace{\mathbf{B} \,\&\, \cdots \,\&\, \mathbf{B}}_{2^n \ times}.$$

- $\mathsf{ta}(t_C)$ can be built from C in logspace.

Encoding a coNP-complete Problem (2)

- The normal form of $\mathsf{ta}(t_C)$ consists of 2^n boolean values, each of which corresponds to a 'truth assignment' to the formula C.
- Example: $\mathsf{ta}(\mathsf{or})$

$$
\begin{array}{lcl}
\mathsf{ta}_1(\mathsf{ta}_2\mathsf{or}) & \equiv & (\lambda f.\langle f\mathsf{true}, f\mathsf{false}\rangle)((\lambda g x.\langle g\mathsf{true}x, g\mathsf{false}x\rangle)\mathsf{or}) \\
& \longrightarrow & (\lambda f.\langle f\mathsf{true}, f\mathsf{false}\rangle)(\lambda x.\langle \mathsf{or\ true}\ x, \mathsf{or\ false}\ x\rangle) \\
& \longrightarrow^* & \langle\langle \mathsf{or\ true\ true}, \mathsf{or\ true\ false}\rangle, \langle \mathsf{or\ false\ true}, \mathsf{or\ false\ false}\rangle\rangle \\
& \longrightarrow^* & \langle\langle \mathsf{true}, \mathsf{true}\rangle, \langle \mathsf{true}, \mathsf{false}\rangle\rangle .
\end{array}
$$

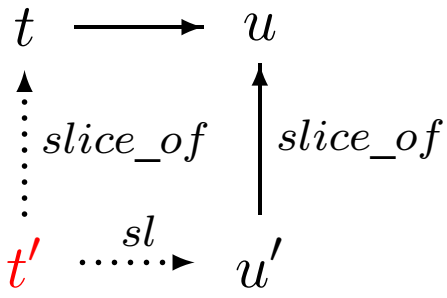
- Two formulas C and D with n variables are logically equivalent if and only if $\mathsf{ta}(t_C)$ and $\mathsf{ta}(t_D)$ reduce to the same normal form.
- Theorem (coNP-completeness of **IMALL**): The cut-elimination problem for **IMALL** is coNP-complete.

Remark (1)

- We do not claim that *the* complexity of **MALL** is coNP (If we considered the complement of CEP, the result would be NP-completeness).
- We do claim that additives have something to do with nondeterminism.

Remark (2)

- Pullback Lemma:



  is a syntactic counterpart of linearity in denotational semantics:

$$\bigcup a_i \sqsubset X \Longrightarrow F(\bigcup a_i) = \bigcup F(a_i)$$

- Slicewise computation is only available in linear logic framework.

Remark (3)

- In reality, functional computation is never nondeterministic.
- Nondeterministic computation can be simulated by deterministic one with an exponential overhead:
  $\Longrightarrow$ the complexity theoretic meaning of exponential isomorphism

$$!(A \,\&\, B) \circ\!\!-\!\!\circ\, !A \otimes !B.$$

Towards Infinite

- An **MLL** proof represents a finite function. What represents an infinite one?
- Analogy: A circuit C represents a finite predicate on $\{0,1\}^n$.
- A family $\{C_n\}_{n\in N}$ of boolean circuits (C_n has n inputs) represents an infinite predicate on $\{0,1\}^*$.
- Given an input w of length n, pick up C_n and evaluate $C_n(w)$.
- Such a family may represent a nonrecursive predicate.
- A family $\{C_n\}_{n\in N}$ is logspace uniform if

$$n \quad \overset{O(\log n)\text{ space}}{\Longrightarrow} \quad C_n.$$

- Theorem: $X \in \mathsf{P} \Longleftrightarrow$ there is a logspace uniform family $\{C_n\}_{n\in N}$ representing X.

Towards Logical Uniformity

- We could consider logspace uniform families of **MLL** proofs to capture P.
- But logspace uniformity is not a logical concept!
- Is there a purely logical notion of uniformity?
  $\Longrightarrow$ Generic exponentials (Lafont 2001)

Generic MLL(1)

- Types of **MGLL**:

$$A, B ::= \alpha \mid A \multimap B \mid \forall \alpha . A \mid !A$$

- Type assignment rules: **MLL** with generic promotion

$$\frac{x_1 : A_1, \ldots, x_n : A_n \vdash t : B}{x_1 :! A_1, \ldots, x_n :! A_n \vdash t :! B}$$

- Notation: $A^n \equiv \underbrace{A \otimes \cdots \otimes A}_{n \ times}$, $A^0 \equiv \mathbf{1}$.

Interpretation: MGLL $\longrightarrow$ MLL

- For each $n \in N$, define a "functor" $\langle n \rangle$ by

$$\begin{array}{lccc} \langle n \rangle : & \textbf{MGLL} & \longrightarrow & \textbf{MLL} \\ & !A & \mapsto & A^n \\ & \dfrac{\Gamma \vdash A}{!\Gamma \vdash !A} & \mapsto & \dfrac{\Gamma \vdash A}{\Gamma^n \vdash A^n} \end{array}$$

- Theorem: Let an **MGLL** proof $t : A$ be given. Then

$$n \overset{O(\log n)\text{ space}}{\Longrightarrow} t\langle n \rangle : A\langle n \rangle.$$

- Every **MGLL** proof t describes a logspace uniform family of infinitely many **MLL** proofs.

$$t\langle 1 \rangle, t\langle 2 \rangle, t\langle 3 \rangle, \ldots$$

Example

Generic Proof t	x :!**B**, y :!**B** $\vdash$	or(x, y) :!**B**
$t\langle 1\rangle$	x : **B**, y : **B** $\vdash$	or(x, y) : **B**
$t\langle 2\rangle$	x : **B** $\otimes$ **B**, y : **B** $\otimes$ **B** $\vdash$	let x be $x_1 \otimes x_2$ in let y be $y_1 \otimes y_2$ in $(\text{or}(x_1, y_1) \otimes \text{or}(x_2, y_2))$: **B** $\otimes$ **B**
$t\langle n\rangle$	x : $\mathbf{B}^n$, y : $\mathbf{B}^n$ $\vdash$	let x be $x_1 \otimes \cdots \otimes x_n$ in let y be $y_1 \otimes \cdots \otimes y_n$ in $(\text{or}(x_1, y_1) \otimes \cdots \otimes \text{or}(x_n, y_n))$: $\mathbf{B}^n$

Representing words in MGLL

- $\mathbf{W} \equiv \forall\alpha.!(\mathbf{B} \multimap \alpha \multimap \alpha) \multimap \alpha \multimap \alpha$.
- **W** has no proof in **MGLL**.
- $\mathbf{W}\langle n\rangle \equiv \forall\alpha.(\mathbf{B} \multimap \alpha \multimap \alpha)^n \multimap \alpha \multimap \alpha$ for each $n \in N$.
- $\mathbf{W}\langle n\rangle$ has proofs $\underline{w}$ representing $w \in \{0,1\}^n$.
- $\underline{010} \equiv \lambda f_1 \otimes f_2 \otimes f_3.\lambda x.(f_1\mathsf{false})(f_2\mathsf{true})(f_3\mathsf{false})x)) : \mathbf{W}\langle 3\rangle$

Representing predicates in MGLL(1)

- An **MGLL** proof $t : \mathbf{W}^l \multimap \mathbf{B}$ represents a predicate $X \subseteq \{0,1\}^*$ $\Longleftrightarrow$ for each word w of length n,

$$w \in X \iff t\langle n \rangle (\underbrace{\underline{w} \cdots \underline{w}}_{l \ times}) \to^* \text{true}$$

- Theorem: Every proof $t : \mathbf{W}^l \multimap \mathbf{B}$ in **MGLL** represents a P predicate.
- Proof: Given input w of length n, build $t\langle n \rangle$ in logspace, thus in polynomial time, and normalize $t\langle n \rangle (\underline{w} \cdots \underline{w})$ in quadratic time.

Representing predicates in MGLL(2)

- What about the converse?
- Theorem (Lafont2001): Every P predicate is representable in **MGLL** with additives.
- Theorem (Mairson-Terui2004): Every P predicate is representable in **MGLL**.
- Every P predicate can be programmed with a (generic) linear λ-term.
- Duplication-free program execution:
  1. Given an input w of length n, unfold t into **MLL** proofnet $t\langle n\rangle$.
  2. Normalize $t\langle n\rangle(\underline{w}\cdots\underline{w})$ (no sharing, no duplication, efficiently parallelizable)

Simulation of P Turing Machines (1)

- Polynomial clock n^k : $\mathbf{N} \multimap \mathbf{N}\langle X^k \rangle$
  — Already multiplicative in (Lafont 2001)
- One-step transition : $Conf \multimap Conf$
  — (Lafont 2001) uses additives
- Iteration: $A \multimap !(A \multimap A) \multimap \mathbf{N} \multimap A$
- Initialization, Acceptance-checking
  — OK.
- It suffices to give a multiplicative encoding of one-step transition.

Simulation of P Turing Machines (2)

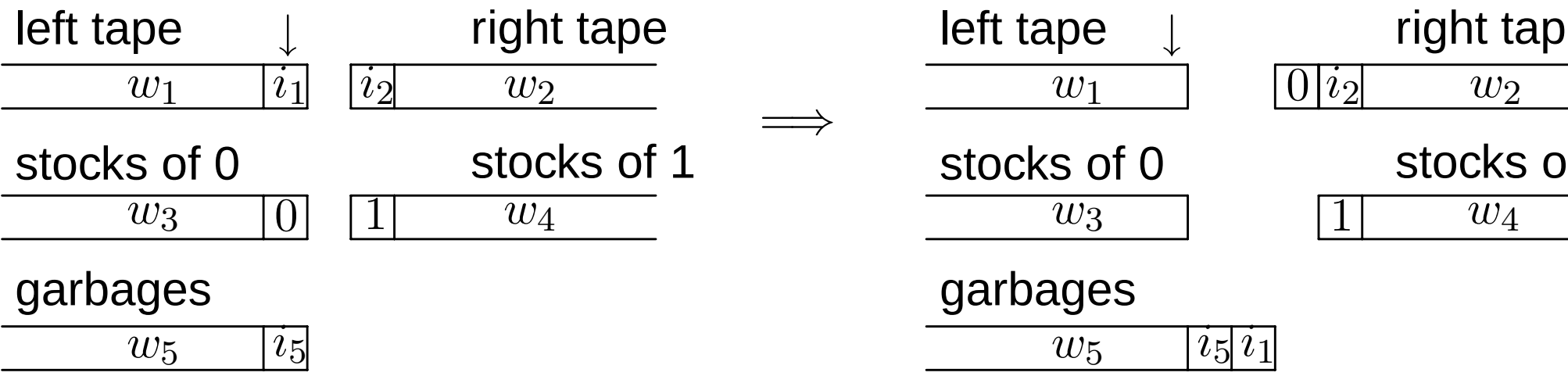
- Consider a TM with 2 symbols and 2^n states. Then,

$$Conf \equiv \forall\alpha.!(\mathbf{B} \multimap \alpha \multimap \alpha) \multimap ((\alpha \multimap \alpha)^5 \otimes \mathbf{B}^n)$$

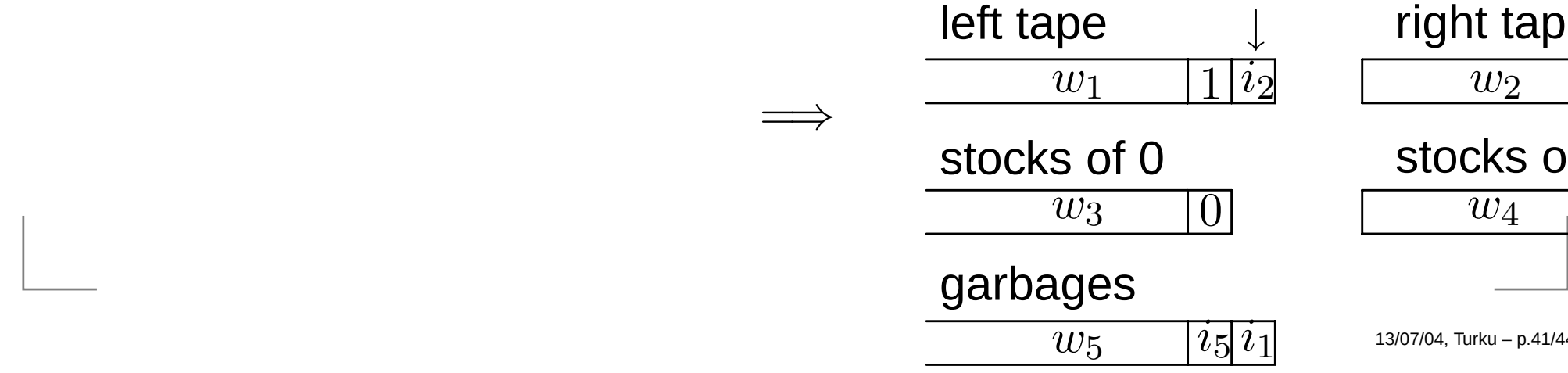
- $\mathbf{B}^n$ corresponds to the 2^n states.
- Usually one needs just 2 stacks $(\alpha \multimap \alpha)^2$, left-tape and right-tape, but we need 5.
- Difficulties:
  1. We cannot create a new tape cell.
  2. We cannot remove a redundant tape cell (as Weakening is available only for closed types).
- Solution: Use 5 stacks to represent each configuration: left-tape, right-tape, stocks of 0, stocks of 1, garbages.

Simulation of P Turing Machines (3)

(1) “Write 0 and move left”



(2) “Write 1 and move right”



Simulation of P Turing machines (4)

- One-step transition is obtained from:
  1. Lafont's ψ function to decompose a stack into the head and the tail
  2. Multiplicative conditional to branch according to the current state
  3. Combinatorial operations to rearrange 5 stacks

Multiplicative Soft Linear Logic

- **MSLL**: **MGLL** with multiplexing (generalization of dereliction, weakening and contraction)

$$!X \multimap X^n, \text{for each } n \in N$$

- Internalization of $\langle n \rangle$: **MGLL** $\longrightarrow$ **MSLL**
- **W** itself has inhabitants.
- Satisfies polynomial time strong normalization:
  Any proof t of depth d strongly normalizes in time $O(|t|^{d+2})$
  (depth d counts nesting of ! promotions)
- A self-contained logical system of polynomial time (like LLL).

Conclusion

- Multiplicatives: all finite computations (including booleans, conditionals)
- Additives: nondeterminism
- Generic Exponentials: uniformity