Densification of FL Chains via Residuated Frames

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ABSTRACT. We introduce a systematic method for densification, i.e., embedding a given chain into a dense one preserving certain identities, in the framework of FL algebras (pointed residuated lattices). Our method, based on residuated frames, offers a uniform proof to many of the known densification and standard completeness results in the literature.

   We propose a syntactic criterion for densification, called semi-anchoredness. We then prove that the semilinear varieties of integral FL algebras defined by semi-anchored equations admit densification, so that the corresponding fuzzy logics are standard complete. Our method also applies to (possibly non-integral) commutative FL chains. We prove that the semilinear varieties of commutative FL algebras defined by knotted axioms $x^m \leq x^n$ (with $m, n > 1$) admit densification. It provides a purely algebraic proof to the standard completeness of uninorm logic as well as its extensions by knotted axioms.

1. Introduction

Given a class K of ordered algebras and a chain $\mathbf{A}$ (i.e. a totally ordered algebra) in it, we would like to embed $\mathbf{A}$ into a dense chain in the same class K. This construction, referred to by *densification*, is important both in its own and as a key step towards *standard completeness* of various fuzzy logics, that is a completeness theorem with respect to the valuations of propositional variables into the real unit interval $[0, 1]$. See [18] for the background.

Many fuzzy logics fall into the class of *substructural logics*, whose algebraic semantics is given by *FL algebras* (or *pointed residuated lattices*) [17]. We thus consider densification of FL chains and standard completeness of associated fuzzy logics. More specifically, our aim is not to prove a new result, but rather to provide a *uniform, algebraic* account to densification. It is important since standard completeness is often proved by a proof theoretic argument, known as *density elimination* in the hypersequent calculus [2, 24, 10, 11, 4]. While density elimination is no doubt an interesting application of proof theory, it obscures the algebraic essence of densification. Even when densification is proved algebraically (as in [21, 22, 18]), it is not clear to what extent the employed technique generalizes.

To fulfil our goal, we employ *residuated frames* [16], that are effective devices to construct (complete) FL algebras with various properties. They are also a key to connect proof theory with algebraic studies; for instance one

can naturally define a residuated frame $\mathbf{W}$ from the sequent calculus $\mathbf{FL}$, so that validity in the dual algebra $\mathbf{W}^+$ directly implies cut-free derivability in $\mathbf{FL}$. This strong connection allowed us to prove that for a certain class of substructural logics, a strong form of cut-admissibility (*proof theory*) is equivalent to closure under completions (*algebra*), thus promoting a new approach to substructural logics, dubbed *algebraic proof theory for substructural logics* [8, 9].

In this paper, we use residuated frames to densify a given FL chain, preserving certain identities. Although our argument was originally inspired by density elimination along the spirit of algebraic proof theory, the resulting construction can be understood without any reference to proof theory: it provides a *purely algebraic* account.

The rest of this paper is organized as follows. Section 2 discusses densification in a general setting, and Section 3 specializes it to the FL algebras.

Section 4 reviews residuated frames, and Section 5 applies them to densify integral FL chains. This encompasses the standard completeness of monoidal t-norm logic and its noncommutative variant [21, 22]. Section 6 then addresses a more involved case: (possibly non-integral) commutative FL chains. It provides a purely algebraic proof to the standard completeness of uninorm logic, for which only proof theoretic arguments have been known before [24].

Section 7 recalls the concept of *substructural hierarchy* [6, 7, 8, 9], that is useful to classify equations in the language of FL. Based on the hierarchical classification, we introduce the class of *semi-anchored* $\mathcal{P}_3$ equations in Section 8, and prove a general result that every nontrivial semilinear variety of integral FL algebras defined by semi-anchored $\mathcal{P}_3$ equations admits densification and standard completeness. We then turn our attention to varieties of commutative FL algebras in Section 9. We prove that every semilinear variety of commutative FL algebras defined by knotted axioms $x^m \leq x^n$ with $m, n > 1$ admits densification and standard completeness. These results are again inspired by the proof theoretic arguments in [4, 3]. We conclude the paper with some remarks and open problems.

2. Densifiability

We begin with a general consideration on densification and standard completeness. In the sequel, we assume that every algebra $\mathbf{A}$ comes equipped with an order $\leq_{\mathbf{A}}$ defined by equations (e.g. $x \leq_{\mathbf{A}} y \Leftrightarrow x = x \wedge y$ if $\mathbf{A}$ has a lattice reduct).

Definition 2.1. Let $\mathbf{A}$ be a *chain*, i.e., $\leq_{\mathbf{A}}$ is a total order, of cardinality $\kappa > 1$. $\mathbf{A}$ is *dense* if $g < h$ $(g, h \in A)$ implies $g < p < h$ for some $p \in A$. Otherwise $\mathbf{A}$ contains a *gap*, that is a pair of elements $g, h \in A$ such that $g < h$ and there is no element $p \in A$ with $g < p < h$.

A chain $\mathbf{B}$ *fills* a gap (g, h) of $\mathbf{A}$ if there is an embedding $e : \mathbf{A} \longrightarrow \mathbf{B}$ and an element $p \in B$ such that $e(g) < p < e(h)$.

A nontrivial variety V is said to be *densifiable* if every gap of a chain in V can be filled by another chain in V.

Notice that by filling a gap one may introduce some undesirable elements that have nothing to do with the gap. Nevertheless, densifiability is indeed a sufficient condition for densification.

Proposition 2.2. *Let L be a language of algebras and* V *a densifiable variety of type L. Then every chain* $\mathbf{A}$ *of cardinality* $\kappa > 1$ *in* V *is embeddable into a dense chain of cardinality* $\kappa + \aleph_0 + |L|$ *in* V.

Proof. For simplicity let us assume $\kappa, |L| \leq \aleph_0$; it is clear that the argument below works for an arbitrary $\kappa > 1$ and L.

Let X be a countable set of variables and $T = T(X)$ be the set of terms in the language L over X. Let $(t_0, u_0), (t_1, u_1), \ldots$ be a countable sequence of elements of T^2 such that each $(t, u) \in T^2$ occurs infinitely many times in it.

For each $n \in \mathbb{N}$, we define a chain $\mathbf{B}_n$ in V as well as a partial valuation $f_n : X \rightharpoonup B_n$. Let $\mathbf{B}_0 := \mathbf{A}$ and f_0 be any surjective partial function onto A such that $X \backslash dom(f_0)$ is infinite.

For $n \geq 0$, if one of $f_n(t_n)$, $f_n(u_n)$ is undefined or $f_n(t_n) \not< f_n(u_n)$, then let $\mathbf{B}_{n+1} := \mathbf{B}_n$ and $f_{n+1} := f_n$.

Otherwise, let x be a variable taken from $X \backslash dom(f_n)$. If there is $p \in B_n$ such that $f_n(t_n) < p < f_n(u_n)$, then let $\mathbf{B}_{n+1} := \mathbf{B}_n$. If not, let $\mathbf{B}_{n+1}$ be a chain in V that fills the gap $(f_n(t_n), f_n(u_n))$ by $p \in B_{n+1}$. We assume $B_n \subseteq B_{n+1}$ and define $f_{n+1} : X \rightharpoonup B_{n+1}$ by extending f_n with $f_{n+1}(x) := p$.

Let $\mathbf{B} := \bigcup \mathbf{B}_n$, $f := \bigcup f_n$ and $\mathbf{C}$ be the subalgebra of $\mathbf{B}$ generated by $f[X]$ (so that $C = f[T]$). Clearly $\mathbf{C}$ is a countable chain in V that has $\mathbf{A}$ as subalgebra since $A \subseteq f[X]$. Moreover $\mathbf{C}$ is dense, since for every pair $(g, h) \in C^2$ with $g < h$, there is $n \in \mathbb{N}$ such that $g = f_n(t_n)$ and $h = f_n(u_n)$ so that we have $g < f_{n+1}(x) < h$. $\qquad \square$

Given a class K of algebras of the same type, the semantic consequence relation $\models_\mathsf{K}$ is defined as usual. Namely, given a set $E \cup \{s = t\}$ of equations in the language of K, $E \models_\mathsf{K} s = t$ holds if E entails $s = t$ in every algebra $\mathbf{A} \in \mathsf{K}$.

A variety V is *semilinear* if $\models_\mathsf{V} = \models_{\mathsf{V}_C}$, where V_C consists of all chains in V. This is equivalent to say that every subdirectly irreducible algebra in V is a chain (cf. [18]). V is said to be *(strongly) standard complete* if $\models_\mathsf{V} = \models_{\mathsf{V}_{[0,1]}}$, where $\mathsf{V}_{[0,1]}$ consists of all *standard* chains in V, namely those over the real unit interval $([0,1], \leq)$. This conforms to the terminology in fuzzy logics under the identification of an (algebraizable) logic with the corresponding variety. There is actually a weaker notion of standard completeness: $\emptyset \models_\mathsf{V} s = t$ iff $\emptyset \models_{\mathsf{V}_{[0,1]}} s = t$. In the rest of this paper we mean by standard completeness the stronger version.

We need a few concepts concerning completions. Given an algebra $\mathbf{A}$, a *completion* of $\mathbf{A}$ consists of a complete algebra $\mathbf{B}$ together with an embedding $e : \mathbf{A} \longrightarrow \mathbf{B}$. A completion $(\mathbf{B}, e)$ is *join-dense* if $x = \bigvee\{a \in e[A] : a \leq_{\mathbf{B}} x\}$, and *meet-dense* if $x = \bigwedge\{a \in e[A] : x \leq_{\mathbf{B}} a\}$ for every $x \in B$. A join-dense and meet-dense completion is called a *MacNeille completion*. It is known that the lattice reduct of a MacNeille completion is uniquely determined (up to isomorphism that fixes A) by join and meet density [1, 27]. For instance, the MacNeille completion of the rational unit interval $([0,1]_\mathbb{Q}, \leq)$ is just $([0,1], \leq)$.

Proposition 2.3. *Let L be a finite or countable language of algebras and V a variety of type L. If V is semilinear, densifiable and every chain in V has a MacNeille completion in it, V is standard complete.*

Proof. Let $E \cup \{s = t\}$ be a set of equations, and suppose that $E \not\models_\mathsf{V} s = t$. By semilinearity, there is a finite or countable chain $\mathbf{A}$ such that $E \not\models_\mathbf{A} s = t$. By Proposition 2.2, $\mathbf{A}$ is embeddable into a countable dense chain $\mathbf{B}$ in V. It is well known that $(B, \leq_\mathbf{B})$ is isomophic to one of $(0,1)_\mathbb{Q}$, $(0,1]_\mathbb{Q}$, $[0,1)_\mathbb{Q}$ and $[0,1]_\mathbb{Q}$, whose MacNeille completion is $[0,1]$. Hence $\mathbf{B}$ is embeddable into a standard chain $\mathbf{C}$ in $\mathsf{V}_{[0,1]}$, and we have $E \not\models_{\mathsf{V}_{[0,1]}} s = t$. $\square$

Hence there are two key factors for standard completeness: densifiability and closure under MacNeille completions.

3. FL Algebras

In this section, we recall the concept of FL algebra. A standard reference on this topic is [17].

Definition 3.1. A *residuated lattice* is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$, such that $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid and for all $x, y, z \in A$,

$$x \cdot y \leq z \iff y \leq x \backslash z \iff x \leq z/y.$$

An *FL algebra* is a residuated lattice $\mathbf{A}$ with a distinguished element $0 \in A$. The constant 0 is used to define negations: $\sim x := x \backslash 0$, $-x := 0/x$. $\mathbf{A}$ is

(e) *commutative* if $x \cdot y = y \cdot x$ for every $x, y \in A$,
(c) *contractive* if $x \leq x \cdot x$ for every $x \in A$,
(i) *integral* if 1 is the greatest element,
(o) *0-bounded* if 0 is the least element.

We remark that $\mathbf{A}$ is a *chain* if and only if the following *communication property* holds for every $x, y, z, w \in A$:

$$x \leq z \ \text{and} \ y \leq w \implies x \leq w \ \text{or} \ y \leq z. \qquad (com)$$

Also important is the fact that if (g, h) is a gap in a FL chain, then we have $h \backslash g < 1$. Indeed, $1 \leq h \backslash g$ would imply $h \leq g$.

The two divisions $\backslash$ and $/$ coincide in any commutative FL algebra. So we write $x \to y := x \backslash y = y/x$ in that case.

We write FL for the variety of FL algebras and use subscripts $\mathsf{e},\mathsf{c},\mathsf{i},\mathsf{o}$ to indicate the properties (e), (c), (i), (o) above. For instance, $\mathsf{FL}_{\mathsf{ei}}$ denotes the variety of commutative integral FL algebras. It is known (cf. [17, 18]) that a variety of FL algebras is semilinear if and only if it satisfies the four-variable equation

$$\lambda_a(x \vee y\backslash y) \vee \rho_b(x \vee y\backslash x) = 1,$$

where λ_a and ρ_b are conjugate operators defined by:

$$\lambda_a(x) := (a\backslash xa) \wedge 1, \qquad \rho_b(x) := (bx/b) \wedge 1.$$

Given a variety V of FL algebras, we denote by V^ℓ the subvariety obtained by imposing the above equation. Notice that it is equivalent to the familiar *prelinearity axiom* $(x \to y) \vee (y \to x) = 1$ in the $\mathsf{FL}_{\mathsf{ei}}$ algebras.

Unfortunately FL^ℓ is not standard complete (cf. [18]). On the other hand, it is not hard to see that every subvariety of $\mathsf{FL}_{\mathsf{i}}^\ell$ defined by a combination of (e), (c), (o) is densifiable and closed under MacNeille completions, so is standard complete [21, 22].

A short proof of the densifiability of $\mathsf{FL}_{\mathsf{i}}^\ell$ is as follows. Let $\mathbf{A}$ be an integral FL chain with a gap (g, h). We insert a new element p between g and h:

$$A^p := A \cup \{p\}, \qquad g < p < h.$$

The meet and join operations are naturally extended to A^p. To extend multiplication $\cdot$ and divisions $\backslash, /$, note that for every $a \in A$, either $ah = h$ or $ah \leq g$ holds. For every $a \in A$, we define:

$$
\begin{array}{llll}
p \cdot p & := \; p & (h^2 = h) & \qquad p\backslash p \;\; := \;\; 1 \\
 & := \; h^2 & (h^2 \leq g) & \qquad p\backslash a \;\; := \;\; h\backslash a \\
a \cdot p & := \; p & (ah = h) & \qquad a\backslash p \;\; := \;\; p \quad\; (ah = h) \\
 & = \; ah & (ah \leq g) & \qquad\quad\;\; = \;\; a\backslash g \;\; (ah \leq g)
\end{array}
$$

Other cases $p \cdot a$, p/p, a/p and p/a are defined analogously.

This gives rises to a new algebra $\mathbf{A}^p$ in $\mathsf{FL}_{\mathsf{i}}^\ell$ that fills the gap (g, h) of $\mathbf{A}$. While it is possible to check the correctness manually, our approach is rather to derive $\mathbf{A}^p$ by a general residuated frame construction (Section 5). Our approach will explain the rationale behind $\mathbf{A}^p$, and provide a generic recipe for proving further standard completeness results. It also leads to an algebraic proof of the standard completeness of $\mathsf{FL}_{\mathsf{e}}^\ell$ (uninorm logic [24]) in Section 6.

4. Residuated Frames and MacNeille Completions

Just as Kripke frames are useful devices to build various Heyting and modal algebras, residuated frames are useful devices to build various FL algebras. In this section, we introduce residuated frames and recall some relevant facts from [16, 8].

Definition 4.1. A *frame* $\mathbf{W}$ (for FL algebras) is a tuple $(W, W', N, \circ, \varepsilon, \epsilon)$ where $(W, \circ, \varepsilon)$ is a monoid, $N \subseteq W \times W'$ and $\epsilon \in W'$. It is *residuated* if there are functions $\backslash\!\backslash : W \times W' \longrightarrow W'$ and $/\!\!/ : W' \times W \longrightarrow W'$ such that

$$x \circ y \; N \; z \quad \Longleftrightarrow \quad y \; N \; x \backslash\!\backslash z \quad \Longleftrightarrow \quad x \; N \; z /\!\!/ y.$$

We often omit $\circ$ and write xy for $x \circ y$.

Given a frame $\mathbf{W} = (W, W', N, \circ, \varepsilon, \epsilon)$, there is a canonical way to make it residuated: let $\tilde{W}' := W \times W' \times W$ and define $\tilde{N} \subseteq W \times \tilde{W}'$ by

$$x \; \tilde{N} \; (v_1, z, v_2) \quad \Longleftrightarrow \quad v_1 x v_2 \; N \; z.$$

Then $\tilde{\mathbf{W}} := (W, \tilde{W}', \tilde{N}, \circ, \varepsilon, (\varepsilon, \epsilon, \varepsilon))$ is a residuated frame, since

$$
\begin{aligned}
x \circ y \; \tilde{N} \; (v_1, z, v_2) \quad &\Longleftrightarrow \quad v_1 x y v_2 \; N \; z \\
&\Longleftrightarrow \quad y \; \tilde{N} \; (v_1 x, z, v_2) \\
&\Longleftrightarrow \quad x \; \tilde{N} \; (v_1, z, y v_2).
\end{aligned}
$$

As we have said at the beginning, the primary purpose of residuated frames is to build residuated lattices. Let us now describe the construction.

Let $\mathbf{W} = (W, W', N, \circ, \varepsilon, \epsilon)$ be a residuated frame. Given $X, Y \subseteq W$ and $Z \subseteq W'$, let:

$$
\begin{aligned}
X \circ Y \quad &:= \quad \{x \circ y : x \in X, y \in Y\}, \\
X^{\triangleright} \quad &:= \quad \{z \in W' : X \; N \; z\}, \\
Z^{\triangleleft} \quad &:= \quad \{x \in W : x \; N \; Z\},
\end{aligned}
$$

where $X \; N \; z$ holds iff $x \; N \; z$ for every $x \in X$, and $x \; N \; Z$ iff $x \; N \; z$ for every $z \in Z$.

We write $x^{\triangleright}$ and $z^{\triangleleft}$ instead of $\{x\}^{\triangleright}$ and $\{z\}^{\triangleleft}$. The pair $(\triangleright, \triangleleft)$ forms a *Galois connection*:

$$X \subseteq Z^{\triangleleft} \quad \Longleftrightarrow \quad X^{\triangleright} \supseteq Z,$$

so that $\gamma(X) := X^{\triangleright\triangleleft}$ defines a closure operator on $\mathcal{P}(W)$ (the powerset of W):

(1) $X \subseteq \gamma(X)$,
(2) $X \subseteq Y \Longrightarrow \gamma(X) \subseteq \gamma(Y)$,
(3) $\gamma(\gamma(X)) = \gamma(X)$.

Furthermore, γ is a *nucleus*, namely it satisfies

(4) $\gamma(X) \circ \gamma(Y) \subseteq \gamma(X \circ Y)$.

It is for this property that a frame has to be residuated.

Let $\mathcal{P}(W)$ be the powerset of W and $\gamma[\mathcal{P}(W)] \subseteq \mathcal{P}(W)$ be its image under γ. Then a set X belongs to $\gamma[\mathcal{P}(W)]$ iff it is *Galois-closed*, namely $X = \gamma(X)$, iff $X = Z^{\triangleleft}$ for some $Z \subseteq W'$. For $X, Y \in \mathcal{P}(W)$, let

$$
\begin{aligned}
X \circ_{\gamma} Y \quad &:= \quad \gamma(X \circ Y), \\
X \cup_{\gamma} Y \quad &:= \quad \gamma(X \cup Y), \\
X \backslash Y \quad &:= \quad \{y : X \circ \{y\} \subseteq Y\}, \\
Y / X \quad &:= \quad \{y : \{y\} \circ X \subseteq Y\}.
\end{aligned}
$$

Proposition 4.2 ([16]). *Let* $\mathbf{W} = (W, W', N, \circ, \varepsilon, \epsilon)$ *be a residuated frame. Then the dual algebra defined by*

$$\mathbf{W}^+ := (\gamma[\mathcal{P}(W)], \cap, \cup_\gamma, \circ_\gamma, \backslash, /, \gamma(\{\varepsilon\}), \epsilon^{\triangleleft})$$

is a complete FL algebra.

As an example, let $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1, 0)$ be an FL algebra. Then we may define a frame by $\mathbf{W_A} := (A, A, N, \cdot, 1, 0)$, where N is the lattice ordering $\leq$ of $\mathbf{A}$. $\mathbf{W_A}$ is residuated precisely because $\mathbf{A}$ is residuated:

$$a \cdot b \ N \ c \iff b \ N \ a\backslash c \iff a \ N \ c/b.$$

Hence by the previous proposition, $\mathbf{W_A}^+$ is a complete FL algebra. We want $\mathbf{W_A}^+$ to be commutative (resp. contractive, integral, 0-bounded, totally ordered) whenever $\mathbf{A}$ is. The following rules ensure that.

$$\frac{xy \ N \ z}{yx \ N \ z} \ (e^N) \qquad \frac{xx \ N \ z}{x \ N \ z} \ (c^N) \qquad \frac{\varepsilon \ N \ z}{x \ N \ z} \ (i^N)$$

$$\frac{x \ N \ \epsilon}{x \ N \ z} \ (o^N) \qquad \frac{x \ N \ z \ \mathsf{and} \ y \ N \ w}{x \ N \ w \ \mathsf{or} \ y \ N \ z} \ (com^N)$$

It is clear that $\mathbf{W_A}$ satisfies (e^N) (resp. (c^N), (i^N), (o^N), (com^N)), whenever $\mathbf{A}$ is commutative (resp. contractive, integral, 0-bounded, totally ordered). These properties are in turn propagated to the dual algebra $\mathbf{W_A}^+$. This holds for any residuated frame.

Lemma 4.3. *Let* $\mathbf{W}$ *be a residuated frame. If* $\mathbf{W}$ *satisfies* (e^N) *(resp.* (c^N), (i^N), (o^N), (com^N)*), then* $\mathbf{W}^+$ *is commutative (resp. contractive, integral, 0-bounded, totally ordered).*

Proof. Let us only prove that (com^N) implies $\mathbf{W}^+$ being totally ordered. Suppose that there are $X, Y \in \gamma[\mathcal{P}(W)]$ for which $X \nsubseteq Y$ and $Y \nsubseteq X$. The former means that there are $x \in X$ and $w \in Y^{\triangleright}$ such that $x \ N \ w$ does not hold (since $Y = Y^{\triangleright\triangleleft}$). Similarly, the latter means that there are $y \in Y$ and $z \in X^{\triangleright}$ such that $y \ N \ z$ does not hold. On the other hand, we have $x \ N \ z$ and $y \ N \ w$ by definition of $X^{\triangleright}, Y^{\triangleright}$. Hence the rule (com^N) implies that at least one of $x \ N \ w$ and $y \ N \ z$ should hold, a contradiction. $\square$

Finally, we would like to have an embedding of $\mathbf{A}$ into $\mathbf{W_A}^+$. More generally, let $\mathbf{A}$ be an FL algebra and $\mathbf{W} = (W, W', N, \circ, \varepsilon, \epsilon)$ a residuated frame. Suppose that there are injections $i : A \longrightarrow W$ and $i' : A \longrightarrow W'$ by means of which we identify A with a subset of W and of W'. In such a situation the rules in Figure 1, called *Gentzen rules*, ensure the existence of a homomorphism.

Lemma 4.4 ([16]).

(1) *If* $(\mathbf{W}, \mathbf{A})$ *satisfies the Gentzen rules for every* $x \in W$, $z \in W'$ *and* $a, b \in A$, *then* $e(a) := \gamma(\{a\})$ *defines a homomorphism* $e : \mathbf{A} \longrightarrow \mathbf{W}^+$.

(2) *Furthermore, if* $a \ N \ b$ *implies* $a \leq_{\mathbf{A}} b$ *for every* $a, b \in A$, *then* e *is an embedding.*

$$\frac{x \ N \ a \quad a \ N \ z}{x \ N \ z} \ (\text{Cut}) \qquad \frac{}{a \ N \ a} \ (\text{Id})$$

$$\frac{x \ N \ a \quad b \ N \ z}{a\backslash b \ N \ x\backslash\!\backslash z} \ (\backslash\text{L}) \qquad \frac{x \ N \ a\backslash\!\backslash b}{x \ N \ a\backslash b} \ (\backslash\text{R})$$

$$\frac{x \ N \ a \quad b \ N \ z}{b/a \ N \ z/\!\!/x} \ (/\text{L}) \qquad \frac{x \ N \ b/\!\!/a}{x \ N \ b/a} \ (/\text{R})$$

$$\frac{a \circ b \ N \ z}{a \cdot b \ N \ z} \ (\cdot\text{L}) \qquad \frac{x \ N \ a \quad y \ N \ b}{x \circ y \ N \ a \cdot b} \ (\cdot\text{R})$$

$$\frac{a \ N \ z}{a \wedge b \ N \ z} \ (\wedge\text{L}\ell) \qquad \frac{b \ N \ z}{a \wedge b \ N \ z} \ (\wedge\text{L}r) \qquad \frac{x \ N \ a \quad x \ N \ b}{x \ N \ a \wedge b} \ (\wedge\text{R})$$

$$\frac{a \ N \ z \quad b \ N \ z}{a \vee b \ N \ z} \ (\vee\text{L}) \qquad \frac{x \ N \ a}{x \ N \ a \vee b} \ (\vee\text{R}\ell) \qquad \frac{x \ N \ b}{x \ N \ a \vee b} \ (\vee\text{R}r)$$

$$\frac{\varepsilon \ N \ z}{1 \ N \ z} \ (1\text{L}) \qquad \frac{}{\varepsilon \ N \ 1} \ (1\text{R}) \qquad \frac{}{0 \ N \ \epsilon} \ (0\text{L}) \qquad \frac{x \ N \ \epsilon}{x \ N \ 0} \ (0\text{R})$$

FIGURE 1. Gentzen rules

Remark 4.5. Actually Lemma 4.4 holds for much more general situations. For instance, $\mathbf{A}$ can be an arbitrary, even partial, algebra in the language of FL, and i, i' need not be injections as far as the Gentzen rules are satisfied.

Since $(\mathbf{W_A}, \mathbf{A})$ trivially satisfies the Gentzen rules, we see that $(\mathbf{W_A^+}, e)$ is a completion of $\mathbf{A}$. Moreover, it is join-dense and meet-dense since

$$\begin{aligned} X &= \textstyle\bigcup_\gamma \{\gamma(a) : a \in X\} &=& \textstyle\bigcup_\gamma \{e(a) : e(a) \subseteq X\} \\ &= \textstyle\bigcap \{a^\triangleleft : X \ N \ a\} &=& \textstyle\bigcap \{e(a) : X \subseteq e(a)\} \end{aligned} \qquad (*)$$

holds for every Galois-closed set X. The last equality holds because $e(a) = a^{\triangleright\triangleleft} = a^\triangleleft$ and $X \ N \ a$ iff $X \subseteq a^\triangleleft$.

Corollary 4.6. $(\mathbf{W_A^+}, e)$ *is a MacNeille completion of* $\mathbf{A}$. *Hence for every* $\mathsf{x} \subseteq \{\mathsf{e}, \mathsf{c}, \mathsf{i}, \mathsf{o}\}$, *every chain* $\mathbf{A} \in \mathsf{FL}_\mathsf{x}^\ell$ *has a MacNeille completion in* $\mathsf{FL}_\mathsf{x}^\ell$.

The varieties $\mathsf{FL}_\mathsf{x}^\ell$ with $\mathsf{x} \subseteq \{\mathsf{e}, \mathsf{c}, \mathsf{i}, \mathsf{o}\}$ are just a few examples. We will see in Section 8 that the same holds for many more subvarieties of FL.

5. Densification of Integral FL Chains

Residuated frames are useful not just for completion, but also for densification. In this section, we prove the densifiability of $\mathsf{FL}_\mathsf{x}^\ell$ with $\{\mathsf{i}\} \subseteq \mathsf{x} \subseteq \{\mathsf{e}, \mathsf{c}, \mathsf{i}, \mathsf{o}\}$ by using residuated frames. Our proof gives a rationale behind the concrete definition of $\mathbf{A}^p$ in Section 3, and moreover serves as a warm-up before the more involved case of (nonintegral) commutative FL chains in the next section.

Let us fix an integral FL chain $\mathbf{A}$, a gap (g, h) in it and a new element p. Our purpose is to define a residuated frame whose dual algebra fills the gap (g, h) by p.

We define a frame $\mathbf{W}_{\mathbf{A}}^p = (W, W', N, \circ, \varepsilon, \epsilon)$ as follows.

- $(W, \circ, \varepsilon)$ is the free monoid generated by $A \cup \{p\}$.
- $W' := A \cup \{p\}$, $\epsilon := 0 \in A$. N is defined below.

Thus each element $x \in W$ is a finite sequence of elements from $A \cup \{p\}$. We denote by A^* the subset of W that consists of finite sequences of elements from A (without any occurrence of p). Also, given $x \in W$ we denote by $\overline{x}$ the product (in $\mathbf{A}$) of all elements of x *where p is replaced by h*. For instance, if $x = papb \in W$ with $a, b \in A$, then $\overline{x} = hahb \in A$.

Let us now define N. Under the intuition that $g < p < h$ should hold and N should be an extension of $\leq_{\mathbf{A}}$, it is natural to require that $a \, N \, p$ iff $a \leq g$, and $p \, N \, a$ iff $h \leq a$ for every $a \in A$. We also require that $p \, N \, p$. The definition below embodies these requirements. For every $x \in W$ and $a \in A$:

$$
\begin{array}{lll}
x \, N \, a & \Longleftrightarrow & \overline{x} \leq_{\mathbf{A}} a \\
x \, N \, p & \Longleftrightarrow & \overline{x} \leq_{\mathbf{A}} g \quad (\text{if } x \in A^*) \\
x \, N \, p & \text{always holds} & \text{(otherwise)}
\end{array}
$$

As explained in Section 4, the frame $\mathbf{W}_{\mathbf{A}}^p$ induces a residuated frame $\tilde{\mathbf{W}}_{\mathbf{A}}^p$.

Notice that, being $\mathbf{A}$ an integral chain, $\mathbf{W}_{\mathbf{A}}^p$ satisfies

$$
\frac{uv \, N \, z}{uxv \, N \, z}
$$

for every $u, x, v \in W$ and $z \in W'$. Hence $\tilde{\mathbf{W}}_{\mathbf{A}}^p$ satisfies the rule (i^N), so the dual algebra $\tilde{\mathbf{W}}_{\mathbf{A}}^{p+}$ is integral by Lemma 4.3.

To have a closer look at the residuated frame $\tilde{\mathbf{W}}_{\mathbf{A}}^p$, it is convenient to partition the set $\tilde{W}' = W \times W' \times W$ into three, in accordance with the case distinctions in the definition of N:

$$
\begin{array}{lll}
\tilde{W}_1' & := & \{(u, a, v) \in \tilde{W}' : a \in A\}, \\
\tilde{W}_2' & := & \{(u, p, v) \in \tilde{W}' : u, v \in A^*), \\
\tilde{W}_3' & := & \{(u, p, v) \in \tilde{W}' : u \notin A^* \text{ or } v \notin A^*\}.
\end{array}
$$

Just as we associated an element $\overline{x} \in A$ to each $x \in W$, we associate an element $\overline{z} \in A$ to each $z \in \tilde{W}'$ as follows:

$$
\begin{array}{llll}
\overline{z} & := & \overline{u} \backslash a / \overline{v} & (z = (u, a, v) \in \tilde{W}_1') \\
& := & \overline{u} \backslash g / \overline{v} & (z = (u, p, v) \in \tilde{W}_2') \\
& := & 1 & (z = (u, p, v) \in \tilde{W}_3')
\end{array}
$$

We finally define $A^\circ := \tilde{W}_1' \cup \tilde{W}_3'$. A pair $(x, z) \in W \times \tilde{W}'$ is said to be *stable* if either $x \in A^*$ or $z \in A^\circ$. We also say that a statement $x \, \tilde{N} \, z$ is *stable* if (x, z) is.

The following lemma explains why we have defined the sets A^*, A° and the concept of stability.

Lemma 5.1.

(1) *If (x, z) is stable, then $x \tilde{N} z$ iff $\overline{x} \leq_{\mathbf{A}} \overline{z}$. If not, $x \tilde{N} z$ always holds.*

(2) *If $x \notin A^*$, then $\overline{x} \leq_{\mathbf{A}} h$.*

(3) *If $z \notin A^\circ$, then $g \leq_{\mathbf{A}} \overline{z}$.*

Proof. (1) When $z = (u, a, v) \in \tilde{W}'_1$, we have $x \tilde{N} z$ iff $uxv N a$ iff $\overline{uxv} \leq a$ iff $\overline{x} \leq \overline{z}$. When $z = (u, p, v) \in \tilde{W}'_3$, both $uxv N p$ and $\overline{x} \leq 1 = \overline{z}$ hold. When $z \in \tilde{W}'_2$ and $x \in A^*$, $x \tilde{N} z$ iff $uxv N p$ iff $\overline{uxv} \leq g$ iff $\overline{x} \leq \overline{z}$. When $z \in \tilde{W}'_2$ and $x \notin A^*$, $x \tilde{N} z$ always holds.

(2) $x \notin A^*$ means that the sequence x contains an occurrence of p, that is interpreted by h. Hence the claim holds by integrality. (3) is proved in a similar way. $\qquad\square$

Lemma 5.2. *$\tilde{\mathbf{W}}^p_{\mathbf{A}}$ satisfies the rule (com^N). Hence $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$ is a chain.*

Proof. We verify:

$$\frac{x \tilde{N} z \;\; \mathsf{and} \;\; y \tilde{N} w}{x \tilde{N} w \;\; \mathsf{or} \;\; y \tilde{N} z} \; (com^N)$$

In case at least one of the conclusions is not stable, (com^N) immediately holds by Lemma 5.1(1). Notice that this is always the case when both premises $x \tilde{N} z$ and $y \tilde{N} w$ are not stable. Hence we only need to consider the cases when both conclusions are stable and either both or only one of the premises is stable.

(i) If both premises $x \tilde{N} z$ and $y \tilde{N} w$ are stable, (com^N) boils down to

$$\overline{x} \leq \overline{z} \;\; \mathsf{and} \;\; \overline{y} \leq \overline{w} \;\; \Longrightarrow \;\; \overline{x} \leq \overline{w} \;\; \mathsf{or} \;\; \overline{y} \leq \overline{z},$$

that holds by the communication property in $\mathbf{A}$.

(ii) Assume only one premise is stable. For instance, let $y \tilde{N} w$ be stable and $x \tilde{N} z$ not stable. We have $x \notin A^*$, $z \notin A^\circ$. Moreover, as both conclusions $x \tilde{N} w$ and $y \tilde{N} z$ are assumed to be stable, we have $w \in A^\circ$, $y \in A^*$. We have either $\overline{y} \leq g$ or $h \leq \overline{y}$ since (g, h) is a gap. If $\overline{y} \leq g$, then $\overline{y} \leq g \leq \overline{z}$ by Lemma 5.1(3), so the right conclusion holds. If $h \leq \overline{y}$, Lemma 5.1(2) and the right premise imply $\overline{x} \leq h \leq \overline{y} \leq \overline{w}$, so the left conclusion holds. The case where $y \tilde{N} w$ is not stable and $x \tilde{N} z$ is stable is symmetrical. $\qquad\square$

For the next lemma, we consider injections i, i' from A to W and W' given by $i(a) := a \in W$ and $i'(a) := (\varepsilon, a, \varepsilon) \in \tilde{W}'$, and identify a with $i(a)$ and $i'(a)$.

Lemma 5.3. *$(\tilde{\mathbf{W}}^p_{\mathbf{A}}, \mathbf{A})$ satisfies all Gentzen rules. Moreover $a \tilde{N} b$ implies $a \leq_{\mathbf{A}} b$ for every $a, b \in A$. Hence $e(a) := \gamma(a)$ is an embedding of $\mathbf{A}$ into $\tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$.*

Proof. Observe that all Gentzen rules (Figure 1, where N is replaced by $\tilde{N}$) have stable premises. If the conclusion is also stable, then it follows from the premises by Lemma 5.1(1). Otherwise (as it may happen for the rule (Cut)), the conclusion holds automatically. $\qquad\square$

Lemma 5.4. *Let γ be the embedding of $\mathbf{A}$ into $\tilde{\mathbf{W}}_{\mathbf{A}}^{p+}$ as in Lemma 5.3. The following hold.*

(1) *For every $z \in A \cup \{p\}$, $\gamma(z) = z^{\triangleright\triangleleft} = z^{\triangleleft}$.*

(2) *$\gamma(g) \subsetneq \gamma(p) \subsetneq \gamma(h)$.*

Proof. (1) Suppose that $z = a \in A$. We have $a \in a^{\triangleleft}$ by (Id). Hence $a^{\triangleright\triangleleft} \subseteq a^{\triangleleft}$. To show the other inclusion, let $x \in a^{\triangleleft}$ and $z \in a^{\triangleright}$. Then $x \tilde{N} a$ and $a \tilde{N} z$, so $x \tilde{N} z$ by (Cut). This shows that $a^{\triangleleft} \subseteq a^{\triangleright\triangleleft}$.

For $z = p$, the above reasoning suggests that it is sufficient to verify (Id) and (Cut) for p too:

$$\frac{}{p \; \tilde{N} \; p} \; \text{(Id)} \qquad \frac{x \; \tilde{N} \; p \quad p \; \tilde{N} \; z}{x \; \tilde{N} \; z} \; \text{(Cut)}$$

(Here we identify p on the right hand side with $(\varepsilon, p, \varepsilon) \in \tilde{W}'$. (Id) is obvious. For (Cut), if the conclusion is unstable, it holds automatically. Otherwise, we distinguish three cases. If $x \in A^*$ and $z \notin A^{\circ}$, Lemma 5.1(3) and the left premise (which is stable) imply $\overline{x} \leq g \leq \overline{z}$. If $x \notin A^*$ and $z \in A^{\circ}$, Lemma 5.1(2) and the right premise (which is stable) imply $\overline{x} \leq h \leq \overline{z}$. If $x \in A^*$ and $z \in A^{\circ}$, we have $\overline{x} \leq g < h \leq \overline{z}$.

(2) We have $g \; \tilde{N} \; p$ and $p \; \tilde{N} \; h$, so $g \in p^{\triangleleft}$ and $p \in h^{\triangleleft}$, that imply $\gamma(g) \subseteq \gamma(p) \subseteq \gamma(h)$ by (1). On the other hand, we have neither $p \; \tilde{N} \; g$ nor $h \; \tilde{N} \; p$ (that would mean $h \leq g$). Hence the two inclusions are strict. $\square$

We have proved that the chain $\tilde{\mathbf{W}}_{\mathbf{A}}^{p+}$ fills the gap (g, h) of $\mathbf{A}$. It is easy to see that $\tilde{\mathbf{W}}_{\mathbf{A}}^{p}$ satisfies (e^N), (c^N), (o^N) whenever $\mathbf{A}$ satisfies (e), (c), (o). Hence together with Lemma 4.3, Corollary 4.6 and Proposition 2.3, we conclude:

Theorem 5.5. $\mathsf{FL}_{\times}^{\ell}$ *with $\{i\} \subseteq \times \subseteq \{e, c, i, o\}$ is densifiable and standard complete.*

Thus we have given a uniform proof to the standard completeness of Gödel logic ($\mathsf{FL}_{\mathsf{ecio}}^{\ell}$), monoidal t-norm logic ($\mathsf{FL}_{\mathsf{eio}}^{\ell}$) and its noncommutative counterpart ($\mathsf{FL}_{\mathsf{io}}^{\ell}$) [21, 22].

Structure of $\tilde{\mathbf{W}}_{\mathbf{A}}^{p+}$. We have obtained a chain $\tilde{\mathbf{W}}_{\mathbf{A}}^{p+}$ filling a gap of $\mathbf{A}$, but we have not yet seen what kind of chain it is. By looking into its structure, it turns out that it is just a MacNeille completion of the chain $\mathbf{A}^{p}$ presented in Section 3.

We will show that the restriction of $\tilde{\mathbf{W}}_{\mathbf{A}}^{p+}$ to $e[A] \cup \{\gamma(p)\}$ forms a subalgebra by giving a concrete description. To simplify the notation, we write

$$\hat{x} := \gamma(x) = x^{\triangleright\triangleleft} = x^{\triangleleft}$$

for every $x \in A \cup \{p\}$ (cf. Lemma 5.4 (1)), and $\hat{x} \cdot \hat{y} := \hat{x} \circ_{\gamma} \hat{y}$. The lattice structure of $e[A] \cup \{\hat{p}\}$ is already clear (cf. Lemma 5.4 (2)). Moreover, since $e(a) = \hat{a}$ is an embedding, we have $\hat{a} \star \hat{b} = \widehat{a \star b}$ for every $a, b \in A$ and $\star \in \{\cdot, \backslash, /\}$. Hence it is sufficient to determine the operations $\cdot, \backslash, /$ applied to $\hat{a}$ and $\hat{p}$.

Proposition 5.6. *For every $a \in A$, we have:*

$$
\begin{array}{llll}
\hat{p} \cdot \hat{p} &=& \hat{p} \quad (h^2 = h) & \hat{p} \backslash \hat{p} = \hat{1} \\
&=& \hat{h}^2 \quad (h^2 \leq g) & \hat{p} \backslash \hat{a} = \widehat{h \backslash a} \\
\hat{a} \cdot \hat{p} &=& \hat{p} \quad (ah = h) & \hat{a} \backslash \hat{p} = \hat{p} \quad (ah = h) \\
&=& \widehat{ah} \quad (ah \leq g) & \qquad\quad = \widehat{a \backslash g} \quad (ah \leq g)
\end{array}
$$

Similar equalities hold for $\hat{p} \cdot \hat{a}$, $\hat{p}/\hat{p}$, $\hat{a}/\hat{p}$ and $\hat{p}/\hat{a}$. Hence the restriction of $\tilde{\mathbf{W}}_{\mathbf{A}}^{p+}$ to $e[A] \cup \{\hat{p}\}$ forms a subalgebra that is isomorphic to $\mathbf{A}^p$, and $\tilde{\mathbf{W}}_{\mathbf{A}}^{p+}$ is its MacNeille completion.

Proof. Notice that $\hat{x} \cdot \hat{y} = \gamma(\gamma(x) \circ \gamma(y)) = (xy)^{\rhd\lhd}$. Hence to see the equivalence between $\hat{x} \cdot \hat{y}$ and $\hat{u}$, it is sufficient to check $(xy)^{\rhd} = u^{\rhd}$, which holds exactly when $xy \, \tilde{N} \, z$ iff $u \, \tilde{N} \, z$ for every $z \in \tilde{W}'$.

If $z \in A^\circ$, stability implies:

- $pp \, \tilde{N} \, z$ iff $h^2 \leq \overline{z}$ iff $h \leq \overline{z}$ iff $p \, \tilde{N} \, z$ (when $h^2 = h$).
- $pp \, \tilde{N} \, z$ iff $h^2 \leq \overline{z}$ iff $h^2 \, \tilde{N} \, z$ (when $h^2 \leq g$).
- $ap \, \tilde{N} \, z$ iff $ah \leq \overline{z}$ iff $h \leq \overline{z}$ iff $p \, \tilde{N} \, z$ (when $ah = h$).
- $ap \, \tilde{N} \, z$ iff $ah \leq \overline{z}$ iff $ah \, \tilde{N} \, z$ (when $ah \leq g$).

If $z \notin A^\circ$, both sides of the above four hold by Lemma 5.1(1) and (3).

To prove the equalities for $\backslash$, notice that $\hat{w} = w^\lhd$ and $\hat{x} \backslash \hat{z} = x^{\rhd\lhd} \backslash z^\lhd = \{x\} \backslash z^\lhd$ for every $w, x, z \in A \cup \{p\}$. Hence to see $\hat{x} \backslash \hat{z} = \hat{w}$, it is sufficient to check that $xy \, \tilde{N} \, z$ iff $y \, \tilde{N} \, w$ for every $y \in W$.

We have $\hat{p} \backslash \hat{p} = \hat{1}$ and $\hat{p} \backslash \hat{a} = \widehat{h \backslash a}$ since:

- both $py \, \tilde{N} \, p$ and $y \, \tilde{N} \, 1$ hold,
- $py \, \tilde{N} \, a$ iff $h\overline{y} \leq a$ iff $\overline{y} \leq h \backslash a$ iff $y \, \tilde{N} \, h \backslash a$.

For the equality for $\hat{a} \backslash \hat{p}$, we distinguish two cases. If $y \in A^*$, stability implies:

- $ay \, \tilde{N} \, p$ iff $a\overline{y} \leq g$ iff $\overline{y} \leq g$ iff $y \, \tilde{N} \, p$ (when $ah = h$). To see the second equivalence, $\overline{y} \leq g$ obviously implies $a\overline{y} \leq g$. Conversely, suppose that $\overline{y} \leq g$ does not hold. Then $h \leq \overline{y}$, so $h = ah \leq a\overline{y}$. Hence $a\overline{y} \leq g$ does not hold.
- $ay \, \tilde{N} \, p$ iff $a\overline{y} \leq g$ iff $\overline{y} \leq a \backslash g$ iff $y \, \tilde{N} \, a \backslash g$ (when $ah \leq g$).

If $y \notin A^*$, both sides of the above two hold. In particular, $y \, \tilde{N} \, a \backslash g$ holds since $a\overline{y} \leq ah \leq g$ by Lemma 5.1(2), so $\overline{y} \leq a \backslash g$.

Finally $\tilde{\mathbf{W}}_{\mathbf{A}}^{p+}$ is a MacNeille completion of $\mathbf{A}^p$, since $e[A] \cup \{\hat{p}\} = \{\hat{x} : x \in W\} = \{z^\lhd : z \in W'\}$, and any Galois-closed set X is both a join of elements from the second set and a meet of elements from the third set (cf. (*) in Section 4). $\qquad\square$

One might call into question the significance of our construction based on residuated frames, since the resulting algebra admits a much simpler presentation as given in Section 3. Our justifications are as follows.

- Our method has a heuristic value, as it provides a general recipe how to find a chain that fills a gap. In essence, it amounts to a combinatorial task of finding a residuated frame satisfying (com^N) and Gentzen rules.

Once such a frame has been found, we are done. See the next section for another application of our method.

- To prove standard completeness, we usually need to show both densifiability and closure under MacNeille completions (Proposition 2.3). Since residuated frames unify the two tasks to a large extent, it is an economical way after all.

- Residuated frames are intimately connected to the sequent calculus in proof theory, as one can see in the Gentzen rules of Figure 1. This allows us to translate various proof theoretic arguments into algebraic ones. Indeed, our construction was inspired by a proof theoretic argument for standard completeness: first introduce the *density rule* in the hypersequent calculus, that enforces the intended algebraic models to be dense chains, and then show that it can be eliminated from a given proof, thus relating dense chains with non-dense ones [2, 24, 10, 11, 4]. Further references on the subject can be found in [25, 23]. Our frame construction precisely mirrors the way how the density rule is eliminated in [11, 4]. It is amazing that such a proof theoretic argument, devised independently of algebraic considerations, translates into an algebraic one quite smoothly. It suggests a deep connection between proof theory and algebra, perhaps much deeper than usually believed. Finding such a connection is the main goal of our long span project: *algebraic proof theory for substructural logics* [8, 9].

6. Densification of Commutative FL Chains

We now turn to another class of chains: commutative FL chains. It is known that the logic corresponding to this class, called *uninorm logic* [24], is standard complete. However, all the known proofs of this fact are proof theoretic, based on elimination of the density rule in a hypersequent calculus. In this section, we translate the proof theoretic argument into an algebraic one based on residuated frames. This gives rise to a first algebraic proof of standard completeness for uninorm logic.

Let $\mathbf{A}$ be a commutative FL chain with a gap (g, h) and p a new element. We again build a residuated frame whose dual algebra fills the gap (g, h). Although we could define W as before, we can exploit commutativity to simplify the construction.

We define a frame $\mathbf{W}_{\mathbf{A}}^p := (W, W', N, \circ, \varepsilon, \epsilon)$ as follows:

- $W := A \times \mathbb{N}$. Each element $(a, m) \in W$ is denoted by ap^m as if it were a polynomial in the variable p. We identify A with the subset $\{ap^0 : a \in A\}$ of W.
- $ap^m \circ bp^n := (ab)p^{m+n}$, $\varepsilon := 1 = 1p^0$.
- $W' := A \cup \{p\}$, $\epsilon := 0 \in A$.

- There are three types of elements in $W \times W'$: (ap^n, b), (a, p) and (ap^{n+1}, p) with $a, b \in A$ and $n \in \mathbb{N}$. N is defined accordingly:

$$
\begin{aligned}
ap^n \ N \ b &\iff ah^n \leq_{\mathbf{A}} b, \\
a \ N \ p &\iff a \leq_{\mathbf{A}} g, \\
ap^{n+1} \ N \ p &\iff ah^n \leq_{\mathbf{A}} 1.
\end{aligned}
$$

Notice that this is compatible with the previous definition. In particular, $ap^{n+1} \ N \ p$ always holds if $\mathbf{A}$ is integral. As before, the frame $\mathbf{W}_{\mathbf{A}}^p$ induces a residuated frame $\tilde{\mathbf{W}}_{\mathbf{A}}^p := (W, \tilde{W}', \tilde{N}, \circ, \varepsilon, (\varepsilon, \epsilon))$. Because of commutativity, the definitions of $\tilde{W}'$ and $\tilde{N}$ are slightly simplified:

$$
\tilde{W}' := W \times W', \qquad x \ \tilde{N} \ (y, z) \text{ iff } x \circ y \ N \ z.
$$

As in the integral case the set $\tilde{W}'$ can be partitioned into three:

$$
\begin{aligned}
\tilde{W}'_1 &:= \{(ap^n, b) : a, b \in A, n \geq 0\}, \\
\tilde{W}'_2 &:= \{(a, p) : a \in A\}, \\
\tilde{W}'_3 &:= \{(ap^{n+1}, p) : a \in A, n \geq 0\}.
\end{aligned}
$$

Elements of $W, \tilde{W}'$ are again interpreted by elements of A. For $x = ap^n \in W$, let $\overline{x} := ah^n \in A$. For $z \in \tilde{W}'$, we define:

$$
\begin{aligned}
\overline{z} &:= ah^n \to b &(z = (ap^n, b) \in \tilde{W}'_1), \\
&:= a \to g &(z = (a, p) \in \tilde{W}'_2), \\
&:= ah^n \to 1 &(z = (ap^{n+1}, p) \in \tilde{W}'_3).
\end{aligned}
$$

As before, $A^{\circ} := \tilde{W}'_1 \cup \tilde{W}'_3$. A pair $(x, z) \in W \times \tilde{W}'$ is *stable* if either $x \in A$ or $z \in A^{\circ}$. Similarly to Lemma 5.1(1), we have:

Lemma 6.1. *If (x, z) is stable, then $x \ \tilde{N} \ z$ iff $\overline{x} \leq_{\mathbf{A}} \overline{z}$.*

Proof. When $x \in A$ and $z = (a, p) \in \tilde{W}'_2$, we have $x \ \tilde{N} \ z$ iff $xa \ N \ p$ iff $xa \leq_{\mathbf{A}} g$ iff $x \leq a \to g$ iff $\overline{x} \leq_{\mathbf{A}} \overline{z}$.

When $x = x'p^m$ and $z = (ap^n, b) \in \tilde{W}'_1$, we have $x \ \tilde{N} \ z$ iff $x'ap^{m+n} \ N \ b$ iff $x'ah^{m+n} \leq_{\mathbf{A}} b$ iff $x'h^m \leq_{\mathbf{A}} ah^n \to b$ iff $\overline{x} \leq_{\mathbf{A}} \overline{z}$.

When $x = x'p^m$ and $z = (ap^{n+1}, p) \in \tilde{W}'_3$, we have $x \ \tilde{N} \ z$ iff $x'ap^{m+n+1} \ N \ p$ iff $x'ah^{m+n} \leq_{\mathbf{A}} 1$ iff $x'h^m \leq_{\mathbf{A}} ah^n \to 1$ iff $\overline{x} \leq_{\mathbf{A}} \overline{z}$. $\square$

Lemma 6.2. $\tilde{\mathbf{W}}_{\mathbf{A}}^p$ *satisfies the rule* (com^N).

Proof. We verify:

$$
\frac{x \ \tilde{N} \ z \ \text{ and } \ y \ \tilde{N} \ w}{x \ \tilde{N} \ w \ \text{ or } \ y \ \tilde{N} \ z} \ (com^N)
$$

(i) If $x, y \in A$ or $w, z \in A^{\circ}$, all the pairs $\{x, y\} \times \{z, w\}$ are stable. By Lemma 6.1, the rule boils down to

$$
\overline{x} \leq \overline{z} \ \text{ and } \ \overline{y} \leq \overline{w} \implies \overline{x} \leq \overline{w} \ \text{ or } \ \overline{y} \leq \overline{z},
$$

that holds by the communication property.

(ii) Suppose that $w \notin A^{\circ}$ and $z \notin A^{\circ}$. Then $w = (a, p)$ and $z = (b, p)$ so that (com^N) becomes:

$$\frac{xb \ N \ p \ \text{ and } \ ya \ N \ p}{xa \ N \ p \ \text{ or } \ yb \ N \ p} \qquad \Longleftrightarrow \qquad \frac{b \ \tilde{N} \ (x, p) \ \text{ and } \ a \ \tilde{N} \ (y, p)}{a \ \tilde{N} \ (x, p) \ \text{ or } \ b \ \tilde{N} \ (y, p)}$$

Since $a, b \in A$, it reduces to the case (i).

(iii) Suppose that $w \in A^{\circ}$ and $z \notin A^{\circ}$. We write $w = (w_1, w_2)$ and $z = (a, p)$. There are three subcases.

First, suppose that $x, y \notin A$. Then we may write $x = x'p$ and $y = y'p$ so that (com^N) becomes:

$$\frac{x' \ \tilde{N} \ (pa, p) \ \text{ and } \ y' \ \tilde{N} \ (pw_1, w_2)}{x' \ \tilde{N} \ (pw_1, w_2) \ \text{ or } \ y' \ \tilde{N} \ (pa, p)}$$

Since $(pa, p), (pw_1, w_2) \in A^{\circ}$, it reduces to the case (i).

Second, suppose that $x \in A$ and $y \notin A$, so that we may write $y = y'p$. Notice that $x \ \tilde{N} \ z$ iff $xa \ N \ p$ iff $xa \leq g$. Also, $y \ \tilde{N} \ z$ iff $y'pa \ N \ p$ iff $\overline{y}'a \leq 1$. Thus what we have to check is

$$xa \leq g \ \text{ and } \ \overline{y} \leq \overline{w} \ \implies \ x \leq \overline{w} \ \text{ or } \ \overline{y}'a \leq 1.$$

By the communication property, the premises imply either $x \leq \overline{w}$ or $\overline{y}a \leq g$. On the other hand, the latter means $\overline{y}'ha \leq g$, so $\overline{y}'a \leq h \to g < 1$ (since $g < h$). So we are done.

Finally, suppose that $x \notin A$ and $y \in A$, so that we may write $x = x'p$. Note that $x \ \tilde{N} \ z$ iff $x'pa \ N \ p$ iff $\overline{x}'a \leq 1$. Also, $y \ \tilde{N} \ z$ iff $ya \ N \ p$ iff $ya \leq g$. Thus what we have to check is

$$\overline{x}'a \leq 1 \ \text{ and } \ y \leq \overline{w} \ \implies \ \overline{x} \leq \overline{w} \ \text{ or } \ ya \leq g.$$

If $ya \leq g$, we are done. Otherwise $h \leq ya$. Hence together with the premises we obtain $\overline{x} = \overline{x}'h \leq \overline{x}'ya \leq y \leq \overline{w}$. $\qquad \square$

Lemma 6.3. $(\tilde{\mathbf{W}}^p_{\mathbf{A}}, \mathbf{A})$ *satisfies all the Gentzen rules, and moreover $a \ N \ b$ implies $a \leq_{\mathbf{A}} b$ for every $a, b \in A$.*

Proof. As in the proof of Lemma 5.3, notice that all Gentzen rules except (Cut) have stable premises and conclusion. Hence we only have to check the (Cut) rule

$$\frac{x \ \tilde{N} \ a \quad a \ \tilde{N} \ z}{x \ \tilde{N} \ z} \ (\text{Cut})$$

where (x, z) is unstable. We may write $x = x'p$ and $z = (b, p)$. By noting that $x \ \tilde{N} \ z$ iff $x'pb \ N \ p$ iff $\overline{x}'b \leq 1$, it amounts to

$$\overline{x}'h \leq a \ \text{ and } \ ab \leq g \ \implies \ \overline{x}'b \leq 1.$$

Now the premises imply $\overline{x}'hb \leq g$, so $\overline{x}'b \leq h \to g < 1$.

$\qquad \square$

Lemma 6.4. *Let γ be the embedding of $\mathbf{A}$ into $\tilde{\mathbf{W}}_{\mathbf{A}}^{p+}$ in Lemma 6.3. The following hold*

(1) *For every $z \in W'$, $\gamma(z) = z^{\triangleright\triangleleft} = z^{\triangleleft}$.*
(2) *$\gamma(g) \subsetneq \gamma(p) \subsetneq \gamma(h)$.*

Proof. In view of the proof of Lemma 5.4, it is sufficient to show that (Id) and (Cut) hold for p. For (Cut):

$$\frac{x \; \tilde{N} \; p \quad p \; \tilde{N} \; z}{x \; \tilde{N} \; z} \; \text{(Cut)}$$

If $x \in A$ and $z \in A^\circ$, then all of (x, z), (x, p) and (p, z) are stable. Hence the premises imply $\overline{x} \leq g < h \leq \overline{z}$.

If $x \notin A$ and $z \in A^\circ$, we may write $x = x'p$. The premises amount to $\overline{x}' \leq 1$ and $h \leq \overline{z}$, so we obtain $\overline{x} = \overline{x}'h \leq \overline{z}$.

If $x \notin A$ and $z \notin A^\circ$, we may write $x = x'p$ and $z = (a, p)$. The premises amount to $\overline{x}' \leq 1$ and $a \leq 1$, so we obtain $\overline{x}'a \leq 1$.

Finally if $x \in A$ and $z \notin A^\circ$, we may write $z = (a, p)$. The premises amount to $x \leq g$ and $a \leq 1$, so we obtain $xa \leq g$.

In any case we obtain the conclusion $x \; \tilde{N} \; z$. $\qquad\qquad \square$

We have proved that the chain $\tilde{\mathbf{W}}_{\mathbf{A}}^{p+}$ fills the gap (g, h) of $\mathbf{A}$. As for the integral case, it is easy to see that $\tilde{\mathbf{W}}_{\mathbf{A}}^{p}$ satisfies (e^N), (c^N), (o^N) whenever $\mathbf{A}$ satisfies (e), (c), (o). Thus we have the following.

Theorem 6.5. *Every variety $\mathsf{FL}_{\mathsf{x}}^{\ell}$ with $\{\mathsf{e}\} \subseteq \mathsf{x} \subseteq \{\mathsf{e}, \mathsf{c}, \mathsf{i}, \mathsf{o}\}$ is densifiable, hence is standard complete.*

Since uninorm logic is complete with respect to $\mathsf{FL}_{\mathsf{e}}^{\ell}$, our argument provides a purely algebraic proof to the standard completeness of uninorm logic [24].

It is known that $\mathsf{FL}_{\mathsf{x}}^{\ell}$ does not admit densification for $\mathsf{x} \subseteq \{\mathsf{c}, \mathsf{o}\}$. Hence the situation, already known in the literature (cf. [18])), is as follows.

Corollary 6.6. *Let $\mathsf{x} \subseteq \{\mathsf{e}, \mathsf{c}, \mathsf{i}, \mathsf{o}\}$. The variety $\mathsf{FL}_{\mathsf{x}}^{\ell}$ is densifiable and standard complete if and only if $\mathsf{e} \in \mathsf{x}$ or $\mathsf{i} \in \mathsf{x}$.*

7. Substructural Hierarchy and MacNeille Completions

The rest of this paper is devoted to densification of subvarieties of $\mathsf{FL}_{\mathsf{i}}^{\ell}$ and $\mathsf{FL}_{\mathsf{e}}^{\ell}$. The concept of substructural hierarchy [6, 7, 8, 9] is useful to deal with those subvarieties systematically.

Definition 7.1. For each $n \geq 0$, the sets $\mathcal{P}_n, \mathcal{N}_n$ of terms are defined as follows:

(0) $\mathcal{P}_0 = \mathcal{N}_0 = $ the set of variables.
(P1) 1 and all terms $t \in \mathcal{N}_n$ belong to $\mathcal{P}_{n+1}$.
(P2) If $t, u \in \mathcal{P}_{n+1}$, then $t \vee u, t \cdot u \in \mathcal{P}_{n+1}$.
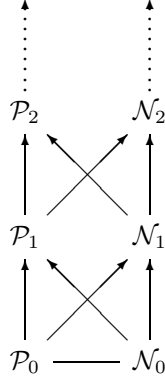(N1) 0 and all terms $t \in \mathcal{P}_n$ belong to $\mathcal{N}_{n+1}$.

FIGURE 2. The Substructural Hierarchy

(N2) If $t, u \in \mathcal{N}_{n+1}$, then $t \wedge u \in \mathcal{N}_{n+1}$.

(N3) If $t \in \mathcal{P}_{n+1}$ and $u \in \mathcal{N}_{n+1}$, then $t \backslash u, u / t \in \mathcal{N}_{n+1}$.

In other words, $\mathcal{P}_n$ and $\mathcal{N}_n$ $(n \geq 1)$ are generated by the following BNF grammar:

$$\begin{aligned} \mathcal{P}_n &::= \mathcal{N}_{n-1} \mid 1 \mid \mathcal{P}_n \vee \mathcal{P}_n \mid \mathcal{P}_n \cdot \mathcal{P}_n, \\ \mathcal{N}_n &::= \mathcal{P}_{n-1} \mid 0 \mid \mathcal{N}_n \wedge \mathcal{N}_n \mid \mathcal{P}_n \backslash \mathcal{N}_n \mid \mathcal{N}_n / \mathcal{P}_n. \end{aligned}$$

By residuation, any equation $u = v$ can be written as $1 \leq t$. We say that $u = v$ belongs to $\mathcal{P}_n$ ($\mathcal{N}_n$, resp.) if t does.

The classes $(\mathcal{P}_n, \mathcal{N}_n)$ constitute the *substructural hierarchy* (Figure 2). Among those classes, relevant to subsequent arguments are $\mathcal{N}_2$ and $\mathcal{P}_3$. The former includes:

$xy \leq yx$	(e)
$x \leq xx$	(c)
$x \leq 1$	(i)
$0 \leq x$	(o)
$x^m \leq x^n$	(knotted axioms, $m, n \geq 0$)
$1 \leq \sim (x \wedge \sim x)$	(no-contradiction)

$\mathcal{P}_3$ includes:

$1 \leq x \vee \sim x$	(excluded middle)
$1 \leq \sim x \vee \sim\sim x$	(weak excluded middle)
$1 \leq \sim (x \cdot y) \vee (x \wedge y \backslash x \cdot y)$	(weak nilpotent minimum)
$1 \leq \sim (x \cdot y)^n \vee ((x \wedge y)^{n-1} \backslash (x \cdot y)^n)$	(wnm^n)
$1 \leq \bigvee_{i=0}^{k} (p_0 \wedge \cdots \wedge p_{i-1} \backslash p_i)$	(bounded width k)
$1 \leq p_0 \vee (p_0 \backslash p_1) \vee \cdots \vee (p_0 \wedge \cdots \wedge p_{k-1} \backslash p_k)$	(bounded size k)

The classes $\mathcal{N}_2$ and $\mathcal{P}_3$ are intimately related to the classes of structural quasiequations and structural clauses defined below.

Definition 7.2. By a *clause*, we mean a classical first-order formula of the form:

$$t_1 \leq u_1 \text{ and } \cdots \text{ and } t_m \leq u_m \implies t_{m+1} \leq u_{m+1} \text{ or } \cdots \text{ or } t_n \leq u_n, \quad (q)$$

where t_i, u_i are terms of FL and all variables are assumed to be universally quantified. Each $t_i \leq u_i$ $(1 \leq i \leq m)$ is called a *premise*, while each $t_j \leq u_j$ $(m+1 \leq j \leq n)$ is a *conclusion*. (q) is a *quasiequation* if $n = m + 1$. It is *structural* if $t_1, \ldots, t_n$ are products of variables (including the empty product 1) and $u_1, \ldots, u_n$ are either a variable or 0. Given a structural clause (q), let $L(q)$ be the set of variables occurring in $t_{m+1}, \ldots, t_n$, and $R(q)$ the set of variables occurring in $u_{m+1}, \ldots, u_n$. (q) is said to be *analytic* if the following conditions are satisfied:

> *Separation:* $L(q)$ and $R(q)$ are disjoint.
> *Linearity:* Each variable in $L(q) \cup R(q)$ occurs exactly once in the conclusions $t_{m+1} \leq u_{m+1}, \ldots, t_n \leq u_n$.
> *Inclusion:* Each of $t_1, \ldots, t_m$ is a product of variables in $L(q)$, while each of $u_1, \ldots, u_m$ is either a variable in $R(q)$ or 0.

Theorem 7.3.

(1) *Every equation in $\mathcal{N}_2$ is equivalent in the integral FL algebras to a set of analytic quasiequations.*
(2) *Every equation in $\mathcal{P}_3$ is equivalent in the integral FL chains to a set of analytic clauses.*

Proof. (1) is proved in [8]. For (2), we have

$$\mathbf{A} \models t \vee u = 1 \iff \mathbf{A} \models (t = 1 \text{ or } u = 1)$$
$$\mathbf{A} \models t \cdot u = 1 \iff \mathbf{A} \models (t = 1 \text{ and } u = 1)$$

for every integral FL chain $\mathbf{A}$. Thus each $\mathcal{P}_3$ equation is equivalent to a set of disjunctions of the form $(t_1 = 1 \text{ or } \cdots \text{ or } t_n = 1)$. The rest of the proof proceeds as in [7, 9]. $\square$

Example 7.4. Our running example is the weak nilpotent minimum axiom $1 \leq \sim(xy) \vee (x \wedge y \backslash xy)$ that belongs to $\mathcal{P}_3$. It is equivalent in the integral FL chains to:

$$xy \leq z, \quad xv \leq z, \quad vy \leq z, \quad vv \leq z \implies xy \leq 0 \text{ or } v \leq z. \quad (wnm)$$

Structural clauses are useful because they can be expressed as rules for residuated frames. Moreover, analytic ones are preserved under the dual algebra construction. To make it more precise, let

$$t_1 \leq u_1 \text{ and } \cdots \text{ and } t_m \leq u_m \implies t_{m+1} \leq u_{m+1} \text{ or } \cdots \text{ or } t_n \leq u_n. \quad (q)$$

be a structural clause and $\mathbf{W} = (W, W', N, \circ, \varepsilon, \epsilon)$ a residuated frame. We can naturally translate each t_i into a term over $(\circ, \varepsilon)$, and each u_i into either a

variable or ϵ. The resulting terms are still denoted by t_i, u_i. Corresponding to the clause (q), we have:

$$t_1 \ N \ u_1 \ \text{ and } \ \cdots \ \text{ and } \ t_m \ N \ u_m \ \implies \ t_{m+1} \ N \ u_{m+1} \ \text{ or } \ \cdots \ \text{ or } \ t_n \ N \ u_n. \tag{q^N}$$

Example 7.5. The clause (wnm) corresponds to the following rule for residuated frames:

$$xy \ N \ z, \quad xv \ N \ z, \quad vy \ N \ z, \quad vv \ N \ z \ \implies \ xy \ N \ \epsilon \ \text{ or } \ v \ N \ z. \tag{wnm^N}$$

It holds by definition that if an FL algebra **A** satisfies (q), then the residuated frame $\mathbf{W_A}$ satisfies (q^N). Moreover, Lemma 4.3 generalizes to all analytic clauses.

Theorem 7.6. *Let (q) be an analytic clause. If a residuated frame* **W** *satisfies (q^N), then the dual algebra* $\mathbf{W}^+$ *satisfies (q).*

The correctness of the above theorem should be obvious from the example below as well as the case of (com) handled by Lemma 4.3. The case of quasiequations is detailed in [8] and the case of clauses is implicit in [7]; [9] contains a more general result.

Example 7.7. Suppose that **W** satisfies (wnm^N). Our goal is to show that $\mathbf{W}^+$ satisfies (wnm), namely

$$XY \subseteq Z, \quad XV \subseteq Z, \quad VY \subseteq Z, \quad VV \subseteq Z \ \implies \ XY \subseteq \epsilon^\triangleleft \ \text{ or } \ V \subseteq Z$$

holds for every Galois-closed sets X, Y, V, Z. Suppose that neither of the conclusions holds. Then there are $x \in X, y \in Y, v \in V$ and $z \in Z^\triangleright$ such that neither $xy \ N \ \epsilon$ nor $v \ N \ z$ holds (since $Z = Z^{\triangleright\triangleleft}$).

On the other hand, the premises yield $xy \ N \ z$, $xv \ N \ z$, $vy \ N \ z$ and $vv \ N \ z$, that contradict the assumption that **W** satisfies the rule (wnm^N).

Corollary 4.6 and Theorem 7.6 lead to the following general result on MacNeille completions.

Theorem 7.8. *Let* V *be a variety of FL algebras.*

(1) *If* V *is defined by equations equivalent to analytic quasiequations and* $\mathbf{A} \in \mathsf{V}$, *then its MacNeille completion belongs to* V.

(2) *If* V *is defined by equations equivalent to analytic clauses (over chains) and* $\mathbf{A}$ *is a chain in* V, *then its MacNeille completion belongs to* V.

8. Densification of Subvarieties of FL_i^ℓ

We now focus on subvarieties of FL_i^ℓ defined by $\mathcal{P}_3$ equations. By Theorems 7.3 and 7.8, such varieties are always closed under MacNeille completions (applied to chains). However, there are such varieties that do not admit densification. A typical counterexample is the variety BA of Boolean algebras, whose

only nontrivial chain is the two element one. Notice that BA is defined by excluded middle $x \vee \neg x = 1$ in $\mathcal{P}_2$, which is equivalent to

$$xy \leq z \quad \Longrightarrow \quad x \leq 0 \quad \text{or} \quad y \leq z. \qquad (em)$$

This type of analytic clauses causes a problem. We thus need some criteria for densifiability.

Before we proceed further, let us make it precise what it means that the specific residuated frame $\tilde{\mathbf{W}}^p_{\mathbf{A}}$ defined in Section 5 satisfies (q^N). Recall that an analytic clause (q) is of the form

$$t_1 \leq z_1 \quad \text{and} \quad \cdots \quad \text{and} \quad t_m \leq z_m \quad \Longrightarrow \quad t_{m+1} \leq z_{m+1} \quad \text{or} \quad \cdots \quad \text{or} \quad t_n \leq z_n.$$

For the purpose of this section, it is convenient to write (q) as $P \Longrightarrow C$, where

$$\begin{aligned} P &:= \{t_1 \leq z_1, \ldots, t_m \leq z_m\}, \\ C &:= \{t_{m+1} \leq z_{m+1}, \ldots, t_n \leq z_n\}. \end{aligned}$$

Recall that each equation in P and C consists of variables $L(q)$ and $R(q)$. To each $x \in L(q)$ we associate an element $x^\bullet \in W = (A \cup \{p\})^*$, so that each term t is interpreted by $t^\bullet \in W$. Likewise, to each $z \in R(q)$ we associate a triple $z^\bullet \in \tilde{W}' = W \times W' \times W$, where $W' = A \cup \{p\}$. The interpretations of constants $1, 0$ are already fixed: $1^\bullet := \varepsilon \in W$ and $0^\bullet := (\varepsilon, \epsilon, \varepsilon) = (\varepsilon, 0, \varepsilon) \in \tilde{W}'$. It is now clear when $\tilde{\mathbf{W}}^p_{\mathbf{A}}$ satisfies (q^N). It is true just in case the following holds for each such interpretation $\bullet$:

$$\{t^\bullet \, \tilde{N} \, z^\bullet : t \leq z \in P\} \quad \Longrightarrow \quad \{t^\bullet \, \tilde{N} \, z^\bullet : t \leq z \in C\}. \qquad (*)$$

Let us now come back to criteria for densifiability. We begin with a naive one.

Definition 8.1. Let (q) be an analytic clause. A variable $x \in L(q)$ is said to be *anchored* if for every $z \in R(q)$, whenever there is a premise of the form $uxv \leq z$, there is a conclusion of the form $u'xv' \leq z$. We say that (q) is *anchored* if every $x \in L(q)$ is anchored.

Clearly (em) does not satisfy this condition as x is not anchored, while any analytic quasiequation satisfies it due to the inclusion condition.

Lemma 8.2. *Let $\mathbf{A}$ be an integral FL chain with a gap (g, h) and (q) an anchored analytic clause. If $\mathbf{A}$ satisfies (q), then the residuated frame $\tilde{\mathbf{W}}^p_{\mathbf{A}}$ satisfies (q^N).*

In particular, if $\mathbf{A}$ satisfies an analytic quasiequation (q), $\tilde{\mathbf{W}}^p_{\mathbf{A}}$ satisfies (q^N).

Proof. Assume that $\mathbf{A}$ satisfies an anchored clause (q). Our goal is to verify $(*)$ above. If there is a conclusion $t \leq z \in C$ such that $(t^\bullet, z^\bullet)$ is not stable, then we have $t^\bullet \, \tilde{N} \, z^\bullet$ by Lemma 5.1(1), so $(*)$ holds.

Otherwise, $(t^\bullet, z^\bullet)$ is stable for every $t \leq z \in C$, so that $t^\bullet \, \tilde{N} \, z^\bullet$ iff $\overline{t}^\bullet \leq \overline{z}^\bullet$, by Lemma 5.1(1).

We claim that the same holds for each premise $t \leq z \in P$. Suppose for contradiction that $t^\bullet \notin A^*$ and $z^\bullet \notin A^\circ$. The former means that t contains a variable x such that $x^\bullet \notin A^*$, i.e., the sequence $x^\bullet$ contains p. Since x is anchored, there must be a conclusion $u \leq z \in C$ (with x occurring in u), so that $(u^\bullet, z^\bullet)$ is not stable. But that has been already ruled out.

As a consequence, $(*)$ amounts to

$$\{\overline{t}^\bullet \leq_{\mathbf{A}} \overline{z}^\bullet : t \leq z \in P\} \quad \Longrightarrow \quad \{\overline{t}^\bullet \leq_{\mathbf{A}} \overline{z}^\bullet : t \leq z \in C\},$$

that holds since $\mathbf{A}$ satisfies (q). $\square$

The previous lemma does not apply to many clauses. For instance, it does not apply to (wnm):

$$xy \leq z, \quad xv \leq z, \quad vy \leq z, \quad vv \leq z \quad \Longrightarrow \quad xy \leq 0 \text{ or } v \leq z, \quad (wnm)$$

since x and y are not anchored. To deal with this and more involved clauses, we extend the definition of anchoredness slightly.

Definition 8.3. Let $(q) : P \Longrightarrow C$ be an analytic clause. We say that (q) is *semi-anchored* if for every $\{y_1, \ldots, y_n\} \subseteq L(q)$ there is a substitution $\sigma : \{y_1, \ldots, y_n\} \longrightarrow L(q)$ such that every $\sigma(y_i)$ is anchored in (q) and $\sigma[P] \subseteq P$ (where $\sigma[P] := \{\sigma(t) \leq z : t \leq z \in P\}$).

Note that the new definition generalizes the previous one (just take the identity substitutions).

Example 8.4. (wnm) is semi-anchored because the substitution $\sigma(x) = \sigma(y) = \sigma(v) = v$ as well as its restrictions will work: v is anchored in (wnm) and for every restriction σ_0 of σ, we have $\sigma_0[P] \subseteq P$.

Likewise, we can show that (wnm^n) is equivalent to a semi-anchored clause for every n. This conforms to the standard completeness of monoidal t-norm logic with (wnm^n) proved in [4].

Lemma 8.5. *Let $\mathbf{A}$ be an integral FL chain with a gap (g, h) and $(q) : P \Longrightarrow C$ a semi-anchored analytic clause. If $\mathbf{A}$ satisfies (q), then $\tilde{\mathbf{W}}^p_{\mathbf{A}}$ satisfies (q^N).*

Proof. Our purpose is again to show that

$$\{t^\bullet \tilde{N} z^\bullet : t \leq z \in P\} \quad \Longrightarrow \quad \{t^\bullet \tilde{N} z^\bullet : t \leq z \in C\} \qquad (*)$$

holds in $\tilde{\mathbf{W}}^p_{\mathbf{A}}$ for every interpretation $\bullet$. As in the previous proof, we may assume that $(t^\bullet, z^\bullet)$ is stable for every conclusion $t \leq z$ in C. But this time we cannot assume that it holds for all premises. So let $t_1 \leq z_1, \ldots, t_m \leq z_m$ be the premises that violate stability. Namely for every i, $z_i^\bullet \notin A^\circ$ and $t_i^\bullet \notin A^*$. Let $y_1, \ldots, y_n$ be the variables occurring in $t_1, \ldots, t_m$ such that $y_j^\bullet \notin A^*$. By Lemma 5.1 (2) and (3), we have:

$$g \leq z_i^\bullet \ (1 \leq i \leq m), \qquad y_j^\bullet \leq h \ (1 \leq j \leq n). \qquad (!)$$

By semi-anchoredness, there is a substitution $\sigma : \{y_1, \ldots, y_n\} \longrightarrow L(q)$ such that every $x_j := \sigma(y_j)$ is anchored and $\sigma[P] \subseteq P$. We claim that

($\sharp$) for every non stable premise $t_i^\bullet \le z_i^\bullet$, the premise $\sigma(t_i)^\bullet \le z_i^\bullet$ is stable.

To check that $\sigma(t_i)^\bullet \le z_i^\bullet$ is stable, we only need to show that for any $x_j = \sigma(y_j)$, we have $x_j^\bullet \in A^*$. Recall that each x_j is anchored, hence there is always a conclusion $t \le z$ such that $z = z_i$ and t contains x_j. As we have assumed that all conclusions are stable and $z_i^\bullet \notin A^\circ$, we have $x_j^\bullet \in A^*$, for every $1 \le j \le n$.

Since (g, h) is a gap, we have either $x_j^\bullet \le g$ or $h \le x_j^\bullet$ for each $1 \le j \le n$. We distinguish two cases.

(i) There is some x_j such that $x_j^\bullet \le g$. Assume x_j belongs to a premise $\sigma(t_i) \le z_i$. As x_j is anchored, there is a conclusion $t \le z$ in C such that t contains x_j and $z = z_i$. We have $t^\bullet \le x_j^\bullet \le g \le \overline{z_i}^\bullet$ by integrality and (!). So we obtain a true conclusion $t^\bullet \tilde{N} z^\bullet$.

(ii) For every x_j we have $h \le x_j^\bullet$. First notice that clauses are preserved under substitution, so $\mathbf{A}$ satisfies

$$\sigma[P] \Longrightarrow \sigma[C]. \qquad (\sigma(q))$$

We now assume that all the premises of (*) hold. This includes the premises $\{t^\bullet \; \tilde{N} \; z^\bullet : t \le z \in \sigma[P]\}$. Moreover, since the substitution makes all premises stable by ($\sharp$), the above premises are equivalent to $\{\overline{t}^\bullet \le \overline{z}^\bullet : t \le z \in \sigma[P]\}$, which implies $\{\overline{t}^\bullet \le \overline{z}^\bullet : t \le z \in \sigma[C]\}$ by ($\sigma(q)$). Therefore, there is some conclusion $t \le z$ in $\sigma[C]$ such that $\overline{t}^\bullet \le \overline{z}^\bullet$ holds.

Since $t \le z \in \sigma[C]$, we may write $t = t(x_1, \ldots, x_n)$ so that if we set $s := t(y_1, \ldots, y_n)$ we obtain an original conclusion $s \le z \in C$. Because $y_j^\bullet \le h \le x_j^\bullet$ by (!), we have $\overline{s}^\bullet \le \overline{t}^\bullet \le \overline{z}^\bullet$. This shows $s^\bullet \; \tilde{N} \; z^\bullet$ as required. $\qquad \square$

To state our main theorem, let us call an equation *semi-anchored* if it is equivalent to a set of semi-anchored analytic clauses in the integral FL chains.

Theorem 8.6. *Let* V *be a nontrivial subvariety of* FL_i^ℓ *defined by a set of semi-anchored equations. Then* V *is densifiable, so is standard complete.*

Proof. Let Q be the set of semi-anchored analytic clauses equivalent to the defining equations of V. Let $\mathbf{A} \in \mathsf{V}$ be a chain with a gap (g, h). Then $\mathbf{A}$ satisfies all the clauses in Q, so by the previous lemma $\tilde{\mathbf{W}}_{\mathbf{A}}^p$ satisfies (q^N) for all $(q) \in Q$. Hence $\tilde{\mathbf{W}}_{\mathbf{A}}^{p+}$, filling the gap (g, h) of $\mathbf{A}$, satisfies Q by Theorem 7.6, i.e., $\tilde{\mathbf{W}}_{\mathbf{A}}^{p+} \in \mathsf{V}$. $\qquad \square$

9. Densification of Subvarieties of FL_e^ℓ

Now we turn our attention to subvarieties of FL_e^ℓ algebras. The situation here is considerably more complicated than for FL_i^ℓ, and we have no idea how to deal with $\mathcal{P}_3$, or even $\mathcal{N}_2$, equations uniformly. We thus limit ourselves to the subvarieties of FL_e^ℓ defined by *knotted axioms* $x^m \le x^n$, with distinct $m, n > 1$.

To begin with, any knotted axiom implies a curious fact, that $yx^k \leq 1$ holds if and only if $yx^l \leq 1$, as far as $k, l \geq 1$.

Lemma 9.1. *Let* **A** *be a commutative FL chain satisfying* $x^m \leq x^n$ *for some distinct* $m, n > 0$. *Then* **A** *satisfies the following quasiequations.*

$$xxy \leq 1 \implies xy \leq 1, \tag{c_1}$$

$$xy \leq 1 \implies xxy \leq 1. \tag{w_1}$$

Proof. Notice that (c_1) and (w_1) are mutually derivable in the commutative FL chains. Therefore we will only show that (c_1) holds in case $m < n$. In case $n < m$, one can prove in a symmetric way that (w_1) holds. Given $a, b \in A$, assume that $aab \leq 1$ holds in **A**. It implies (1) $a^{2n}b^n \leq 1$. We have either $1 \leq a$ or $a \leq 1$. In the former case, we immediately obtain $ab \leq aab \leq 1$. In the latter case, we have (2) $a^n \leq a^l$ for every $l \leq n$. Now choose $k, l \in \mathbb{N}$ such that $2n = k(n - m) + l$ and $m \leq l < n$. Notice that, being $m \leq l$, we have by the knotted axiom $a^l = a^{(l-m)}a^m \leq a^{(l-m)}a^n = a^l a^{n-m}$. Hence we get (3) $a^l \leq a^l a^{k(n-m)} = a^{2n}$. By combining (1) - (3), we obtain $a^n b^n \leq a^l b^n \leq a^{2n}b^n \leq 1$. Since **A** is a chain, one can easily show that $ab \leq 1$ follows from the latter. $\square$

Another important fact is that any knotted axiom is equivalent to a very simple form of analytic quasiequation in the commutative FL chains.[1]

Lemma 9.2. *Let* **A** *be a commutative FL chain .* **A** *satisfies* $x^m \leq x^n$, *for distinct* $m, n > 0$, *if and only if it satisfies the following quasiequation:*

$$x_1^n \leq z \quad \text{and} \quad \ldots \quad \text{and} \quad x_m^n \leq z \implies x_1 \cdots x_m \leq z. \tag{$knot_m^n$}$$

Proof. Assume $(knot_m^n)$ holds in **A**. Given $a \in A$, we interpret all $x_1, \ldots, x_m$ by a and z by a^n. Then all the premises hold, and the conclusion is $a^m \leq a^n$.

Conversely, assume $x^m \leq x^n$ holds in **A**. Suppose that $a_1^n \leq b, \cdots, a_m^n \leq b$ hold for $a_1, \ldots, a_m, b \in A$. Our goal is to show $a_1 \cdots a_m \leq b$. Since **A** is a chain, there is a maximum among $a_1, \ldots, a_m$, say a_k. Then we have $a_1 \ldots a_m \leq a_k^m \leq a_k^n \leq b$. $\square$

Lemma 9.3. *Let* **A** *be a commutative FL chain with a gap* (g, h), *satisfying* $(knot_m^n)$ *for some distinct* $m, n > 1$. *The residuated frame* $\tilde{\mathbf{W}}_{\mathbf{A}}^p$ *defined in Section 6 satisfies* $(knot_m^{nN})$.

Proof. We need to show that $\tilde{\mathbf{W}}_{\mathbf{A}}^p$ satisfies

$$x_1^n \,\tilde{N}\, z \quad \text{and} \quad \ldots \quad \text{and} \quad x_m^n \,\tilde{N}\, z \implies x_1 \cdots x_m \,\tilde{N}\, z, \tag{$knot_m^{nN}$}$$

for every $x_1, \ldots, x_m \in W = A \times \mathbb{N}$ and $z \in \tilde{W}' = W \times W' = W \times (A \cup \{p\})$.

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[1] Notice that, being the knotted axioms in the class $\mathcal{N}_2$, we could have just used the procedure in [7, 8] to obtain equivalent analytic quasiequations. The quasiequations we consider here are a simplified version, which are equivalent to the axioms only when we restrict to chains.

The conclusion is stable if and only if all the premises are. If it is the case, the claim easily follows from Lemma 6.1 and from the fact that $\mathbf{A}$ satisfies $(knot_m^n)$.

So assume that some of the premises violate stability, for instance, w.l.o.g., $x_1^n \, \tilde{N} \, z, \ldots, x_k^n \, \tilde{N} \, z$ with $1 \le k \le m$. This means that there are $a_1, \ldots, a_m, b \in A$ and natural numbers $e_1, \ldots, e_k \ge 1$ such that

$$z = (b, p), \quad x_i = a_i p^{e_i} \ (1 \le i \le k), \quad x_j = a_j \ (k+1 \le j \le m).$$

Then $(knot_m^{nN})$ amounts to:

$$\left. \begin{array}{ccc} a_1^n b h^{n e_1 - 1} \le 1, & \ldots, & a_k^n b h^{n e_k - 1} \le 1, \\ a_{k+1}^n b \le g, & \ldots, & a_m^n b \le g \end{array} \right\} \implies a_1 \cdots a_m b h^{e_1 + \cdots + e_k - 1} \le 1.$$

By combining all the premises on the first line and by applying Lemma 9.1 (noting that $n > 1$), we obtain

$$a_1^n \cdots a_k^n b^k h^l \le 1 \qquad\qquad (*)$$

for any $l \ge 1$. By combining all those on the second line, we obtain

$$a_{k+1}^n \cdots a_m^n b^{m-k} \le g^{m-k} \le h^{m-k}. \qquad\qquad (**)$$

If $e_1 + \cdots + e_k - 1 \ge 1$, the two inequalities $(*)$ and $(**)$ with $l := m - k + 1$ implies $a_1^n \cdots a_m^n b^m h \le 1$, which leads to the conclusion by Lemma 9.1.

Otherwise $k = 1$ and $e_1 = 1$. Since $m > 1$, we have $m - k \ge 1$. Hence $(*)$ and $(**)$ with $l := m - k$ implies $a_1^n \cdots a_m^n b^m \le 1$, which leads to the conclusion. $\qquad\qquad\square$

Finally, we obtain the main theorem of this section.

Theorem 9.4. *Let V be a subvariety of $\mathsf{FL}_\mathsf{e}^\ell$ defined by $x^m \le x^n$ with distinct $m, n > 1$. Then V is densifiable, so is standard complete.*

Proof. Let $\mathbf{A} \in \mathsf{V}$ be a chain with a gap (g, h). By Lemma 9.2 and 9.3, $\tilde{\mathbf{W}}_\mathbf{A}^p$ satisfies $(knot_m^{nN})$. Hence $\tilde{\mathbf{W}}_\mathbf{A}^{p+}$, filling the gap (g, h) of $\mathbf{A}$, satisfies $(knot_m^{nN})$ by Theorem 7.6, i.e., $\tilde{\mathbf{W}}_\mathbf{A}^{p+} \in \mathsf{V}$. $\qquad\qquad\square$

Notice that $x^m \le x^0$ with $m > 0$ is equivalent to the integrality $x \le 1$, that has been already dealt with; indeed the former implies $x \le (1 \vee x)^m \le (1 \vee x)^0 = 1$. Likewise, $x^0 \le x^n$ with $n > 0$ is equivalent to $1 \le x$, which defines the trivial variety.

The only remaining cases are $x^m \le x^1$ and $x^1 \le x^n$ with $m, n > 1$, which are respectively equivalent to $x^2 \le x$ and $x \le x^2$ in FL^ℓ. Unfortunately, our result in this section, as well as its proof theoretic origin [3], does not cover these cases. It would require a construction of a residuated frame different from the one given in Section 6.

10. Final remarks and open problems

The results presented here subsume most of the results on *strong* standard completeness of fuzzy logics in the literature [4, 5, 13, 14, 15, 18, 21, 22, 24, 28].

Unfortunately, $\mathcal{N}_3$-subvarieties (i.e., subvarieties defined by $\mathcal{N}_3$ equations) of FL^ℓ, or even $\mathsf{FL}^\ell_{\mathsf{eio}}$, cannot be dealt with by our method. This is the case, for instance, of the varieties corresponding to *basic logic*, *Łukasiewicz logic*, *product logic*, **WCMTL** and Π**MTL** (see e.g. [12, 13, 14, 26, 19]). It is certainly a limitation of our approach, but notice that these varieties only enjoy the *weak* form of standard completeness ($\emptyset \models_{\mathsf{V}} s = t \Leftrightarrow \emptyset \models_{\mathsf{V}_{[0,1]}} s = t$), and it is actually proved that none of them admits the strong form studied in this paper.

It is an open problem to what extent the $\mathcal{N}_2$-subvarieties of $\mathsf{FL}^\ell_{\mathsf{e}}$ admit densification and standard completeness. Section 9 only gives a partial solution (for knotted axioms $x^m \leq x^n$ with $m, n > 1$). Another possible research direction would be to generalize the concept of semi-anchoredness, that is a sufficient condition for densification of $\mathcal{P}_3$-subvarieties of $\mathsf{FL}^\ell_{\mathsf{i}}$. While it is a fairly general condition, we are aware of a counterexample which is not amenable to semi-anchoredness. That is the variety $\Omega(S_n MTL)$, corresponding to the totally decomposable n-contractive MTL chains [20]. Although the defining equations are $\mathcal{P}_3$, they do not seem to be equivalent to semi-anchored clauses.

In this paper we have not considered *involutive* subvarieties of FL, i.e., those defined by $- \sim x = \sim - x = x$. For involutive $\mathsf{FL}^\ell_{\mathsf{ei}}$, that corresponds to *involutive monoidal t-norm logic*, strong standard completeness has been proved algebraically in [14] and proof-theoretically in [24]. We believe that this result can be reproved by employing involutive residuated frames of [16]. On the other hand, an important open problem in this direction is the standard completeness of involutive $\mathsf{FL}^\ell_{\mathsf{e}}$, that correspond to *involutive uninorm logic*, for which we are not sure whether our method applies or not.

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