Disjunction Property and Complexity of Substructural Logics

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Abstract

We systematically identify a large class of substructural logics that satisfy the disjunction property (DP), and show that every consistent substructural logic with the DP is PSPACE-hard. Our results are obtained by using algebraic techniques. PSPACE-completeness for many of these logics is furthermore established by proof theoretic arguments.

Keywords: substructural logics, disjunction property, computational complexity

1. Introduction

Logics that may lack some of the structural rules (exchange, weakening and contraction) are generally called *substructural logics* [22, 10]. They include various nonclassical logics (such as relevance, superintuitionistic and fuzzy logics [8, 4, 14]), and arise from various algebraic structures (such as ordered groups, relation algebras, and ideal lattices of rings). Remarkably, some logics popular in computer science can also be thought of as extensions of substructural logics (such as linear logic [12], the logic of bunched implications [20], and separation logic [23]). Since substructural logics are abundant in various fields, it is important to establish basic logical properties and complexity results not just for each logic independently, but also for a wide class of logics uniformly. In fact, such a systematic study was undertaken in 90's, and the field has been rapidly growing since then. See [10] for the current state of the art.

In the spirit of the latter systematic approach, this paper aims to establish a uniform complexity result on the decision problem for a wide class of substructural logics. It is well known that there are several substructural logics which are PSPACE-complete, for instance intuitionistic logic [25] and the multiplicative, additive fragment of linear logic **MALL** [19]. The same holds for *full Lambek calculus* **FL** (the multiplicative-additive

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fragment of intuitionistic noncommutative linear logic) [17] and some of its extensions. See [18, 15] for surveys.

Such results often rely on proof theoretic methods, and presuppose that the logic under consideration possesses a good sequent calculus for which the cut elimination theorem holds. In contrast, we consider arbitrary extensions of the base logic **FL** by axioms and inference rules. Instead of relying on the existence of cut-free sequent calculi, we extensively use algebraic techniques, as is common in the study of substructural logics (cf. [8, 10]).

More specifically, we focus on the *disjunction property* (DP), which provides a sufficient condition for PSPACE-hardness. We define the class of *ℓ-monoidal inference rules*, which basically consists of rules in the language of lattice conjunction, disjunction and monoid multiplication. We also define the class of $\mathcal{M}_2$ *axioms*, which naturally correspond to the ℓ-monoidal inference rules. These classes are sufficiently large and contain many rules and axioms that often appear in the literature (see Figure 4 in Section 3.3). We then prove:

(i) Every extension of **FL** by ℓ-monoidal inference rules and $\mathcal{M}_2$ axioms satisfies the DP (Section 3).

(ii) Every consistent extension of **FL** with the DP is PSPACE-hard (Section 4).

These two results together establish that a wide range of substructural logics are PSPACE-hard.

In proving (i), we develop a way of constructing suitable well-connected algebras, which substantially generalizes the construction of [24]. Our algebraic methodology turns out to be far more applicable than the usual proof theoretic one based on cut-free proof analysis.

The statement (ii) is a generalization of the same result for superintuitionistic logics [4, Theorem 18.30]. To prove this, we modify the coding of quantified Boolean formulas by [15] along the idea of [26]. In passing, we also note that every consistent substructural logic is coNP-hard.

Finally in Section 5, we turn to the problem of membership in PSPACE. By a standard proof theoretic argument, we show that substructural logics defined by *analytic* and *shrinking* structural rules are PSPACE-complete.

2. Preliminaries

2.1. Substructural logics

Given a set S, we denote by S^* a set of all finite sequences of elements from S.

Our base logic is the *Full-Lambek calculus* **FL** (see [10]). The language of **FL** consists of propositional variables, constants 0, 1 and binary connectives $\wedge, \vee, \cdot, \backslash, /$. Constant 0 is primarily used to define negations:

$$\sim\alpha = \alpha\backslash 0\,, \qquad -\alpha = 0/\alpha.$$

When the distinction between $\alpha\backslash\beta$ and β/α (resp. $\sim\alpha$ and $-\alpha$) is irrelevant, we denote either of them by $\alpha \to \beta$ (resp. $\neg\alpha$). The set of all formulas in this language (**FL**-formulas) is denoted by Fm.

FL	1	0	$\top$	$\bot$	$\cdot$	$\backslash$ $(\to)$	$/$	$\wedge$	$\vee$
Linear Logic	$\mathbf{1}$	$\bot$	$\top$	$\mathbf{0}$	$\otimes$	$\multimap$	$\circ\!-$	$\&$	$\oplus$

Figure 1: Correspondence with linear logic connectives

Two constants $\top$ and $\bot$ are often added to the language of **FL**. While we do not officially include them, we stress that *all the results of this paper hold in their presence.*

There is a quite unfortunate conflict of notation between substructural logics and linear logic. Most problematically, 0 in **FL** corresponds to $\bot$ in linear logic and *vice versa*. Figure 1 clarifies the notational correspondence.

The provability relation is defined by a sequent calculus. A *sequent* is an expression of the form $\Gamma \Rightarrow \varphi$ where $\Gamma \in Fm^*$ and φ is a formula or the empty sequence. The sequent calculus consists of the following initial sequents and rules:

Initial sequents:

$$\alpha \Rightarrow \alpha \qquad\qquad \Rightarrow 1 \qquad\qquad 0 \Rightarrow$$

Rules:

$$\frac{\Gamma \Rightarrow \alpha \qquad \Sigma, \alpha, \Pi \Rightarrow \varphi}{\Sigma, \Gamma, \Pi \Rightarrow \varphi}\ (\mathrm{cut})$$

$$\frac{\Gamma \Rightarrow \alpha \qquad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\Rightarrow\wedge) \qquad \frac{\Gamma, \alpha, \Sigma \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta, \Sigma \Rightarrow \varphi}\ (\wedge\Rightarrow) \qquad \frac{\Gamma, \beta, \Sigma \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta, \Sigma \Rightarrow \varphi}\ (\wedge\Rightarrow)$$

$$\frac{\Gamma, \alpha, \Sigma \Rightarrow \varphi \qquad \Gamma, \beta, \Sigma \Rightarrow \varphi}{\Gamma, \alpha \vee \beta, \Sigma \Rightarrow \varphi}\ (\vee\Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow\vee) \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow\vee)$$

$$\frac{\Gamma, \alpha, \beta, \Sigma \Rightarrow \varphi}{\Gamma, \alpha \cdot \beta, \Sigma \Rightarrow \varphi}\ (\cdot\Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha \qquad \Sigma \Rightarrow \beta}{\Gamma, \Sigma \Rightarrow \alpha \cdot \beta}\ (\Rightarrow\cdot)$$

$$\frac{\Gamma \Rightarrow \alpha \qquad \Pi, \beta, \Sigma \Rightarrow \varphi}{\Pi, \Gamma, \alpha\backslash\beta, \Sigma \Rightarrow \varphi}\ (\backslash\Rightarrow) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\backslash\beta}\ (\Rightarrow\backslash)$$

$$\frac{\Gamma \Rightarrow \alpha \qquad \Pi, \beta, \Sigma \Rightarrow \varphi}{\Pi, \beta/\alpha, \Gamma, \Sigma \Rightarrow \varphi}\ (/\Rightarrow) \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta/\alpha}\ (\Rightarrow/)$$

$$\frac{\Gamma, \Sigma \Rightarrow \varphi}{\Gamma, 1, \Sigma \Rightarrow \varphi}\ (1\Rightarrow) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (\Rightarrow 0)$$

We say that a sequent $\Gamma \Rightarrow \varphi$ is *provable* in **FL** and write $\vdash_{\mathbf{FL}} \Gamma \Rightarrow \varphi$ if $\Gamma \Rightarrow \varphi$ can be obtained from the initial sequents by repeated applications of the rules of **FL**. More generally, given a set Ψ of formulas, we say that $\Gamma \Rightarrow \varphi$ is provable from Ψ and write $\Psi \vdash_{\mathbf{FL}} \Gamma \Rightarrow \varphi$ if the sequent $\Gamma \Rightarrow \varphi$ is derivable in the sequent calculus for **FL** extended by initial sequents $\Rightarrow \psi$ for each $\psi \in \Psi$. We write $\vdash_{\mathbf{FL}} \varphi$ (resp. $\Psi \vdash_{\mathbf{FL}} \varphi$) if $\vdash_{\mathbf{FL}} \Rightarrow \varphi$ (resp. $\Psi \vdash_{\mathbf{FL}} \Rightarrow \varphi$). It is easy to see that $\Psi \vdash_{\mathbf{FL}} \alpha_1, \ldots, \alpha_n \Rightarrow \beta$ is equivalent to $\Psi \vdash_{\mathbf{FL}} (\alpha_1 \cdots \alpha_n)\backslash\beta$. Notice also that $\Psi \vdash_{\mathbf{FL}} \alpha\backslash\beta$ iff $\Psi \vdash_{\mathbf{FL}} \beta/\alpha$, so we write $\Psi \vdash_{\mathbf{FL}} \alpha \to \beta$ in such a case.

Usually substructural logics are defined to be axiomatic extensions of **FL**. Let Φ be a set of formulas closed under substitutions. The *axiomatic extension* of **FL** by Φ is the calculus obtained from **FL** by adding new initial sequents $\Rightarrow \varphi$ for all formulas $\varphi \in \Phi$.

For the purpose of this paper, it is more convenient to consider substructural logics to be defined by inference rules. An *inference rule* is an expression of the form:

$$\frac{\Gamma_1 \Rightarrow \varphi_1 \quad \cdots \quad \Gamma_n \Rightarrow \varphi_n}{\Gamma_0 \Rightarrow \varphi_0}$$

The *rule extension* of **FL** is obtained from **FL** by adding a set Φ of inference rules closed under substitutions. In this paper, a *substructural logic* refers to a rule extension of **FL**.

The most prominent extensions of **FL** are extensions by combinations of the structural rules of exchange (e), contraction (c), left and right weakening (i), (o):

$$\frac{\Sigma_1, \Gamma, \Delta, \Sigma_2 \Rightarrow \varphi}{\Sigma_1, \Delta, \Gamma, \Sigma_2 \Rightarrow \varphi}\,(\mathrm{e}) \qquad \frac{\Sigma_1, \Gamma, \Gamma, \Sigma_2 \Rightarrow \varphi}{\Sigma_1, \Gamma, \Sigma_2 \Rightarrow \varphi}\,(\mathrm{c})$$

$$\frac{\Sigma_1, \Sigma_2 \Rightarrow \varphi}{\Sigma_1, \Gamma, \Sigma_2 \Rightarrow \varphi}\,(\mathrm{i}) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \varphi}\,(\mathrm{o})$$

Let S be a subset of $\{\mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$. Then $\mathbf{FL}_S$ denotes the extension of **FL** by adding the structural rules from S. The combination of (i) and (o) is abbreviated by (w); for instance $\mathbf{FL}_{\mathrm{ew}}$ is the extension of **FL** by (e), (i), and (o). Given $S \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$, it is a well-known fact that $\mathbf{FL}_S$ can be viewed as an axiomatic extension of **FL**. The following axiomatic schemata correspond respectively to (e),(c), (i) and (o):

$$\alpha \cdot \beta \to \beta \cdot \alpha\,, \qquad \alpha \to \alpha \cdot \alpha\,, \qquad \alpha \to 1\,, \qquad 0 \to \alpha\,. \tag{1}$$

We have:

- $\vdash_{\mathbf{FL}_{\mathrm{e}}} \alpha\backslash\beta \to \beta/\alpha, \quad \vdash_{\mathbf{FL}_{\mathrm{e}}} \beta/\alpha \to \alpha\backslash\beta$,
- $\vdash_{\mathbf{FL}_{\mathrm{w}}} \alpha \cdot \beta \to \alpha \wedge \beta$,
- $\vdash_{\mathbf{FL}_{\mathrm{c}}} \alpha \wedge \beta \to \alpha \cdot \beta$,

Hence $\mathbf{FL}_{\mathrm{ewc}}$ is nothing but intuitionistic logic.

Another important class of substructural logics is given by the law of double-negation elimination:

$$\sim\!-\alpha \to \alpha\,, \qquad -\!\sim\!\alpha \to \alpha.$$

In presence of (e), these two just amount to $\neg\neg\alpha \to \alpha$. The extension of any substructural logic $\mathbf{L}$ by the law of double-negation elimination is denoted by $\mathbf{InL}$. In terms of proof theory, this amounts to extending the sequent calculus to a *multi-conclusion* one. In particular, $\mathbf{InFL_e}$ is the multiplicative additive fragment of linear logic, $\mathbf{MALL}$.

Let $\mathbf{L}$ be a substructural logic. As before, the symbol $\vdash_{\mathbf{L}}$ denotes the provability relation in $\mathbf{L}$ and we will use it in all its forms like in $\mathbf{FL}$, i.e., $\Psi \vdash_{\mathbf{L}} \varphi$ means that $\Rightarrow \varphi$ is derivable in $\mathbf{L}$ from $\{ \Rightarrow \psi \mid \psi \in \Psi\}$ for any set of formulas $\Psi \cup \{\varphi\}$ and $\vdash_{\mathbf{L}} \Gamma \Rightarrow \varphi$ means that the sequent $\Gamma \Rightarrow \varphi$ is provable in $\mathbf{L}$. It is known that $\vdash_{\mathbf{L}}$ is a substitution invariant consequence relation, i.e., it satisfies the following properties for every $\Phi, \Psi \subseteq Fm$ and formulas φ, ψ:

- if $\varphi \in \Phi$, then $\Phi \vdash_{\mathbf{L}} \varphi$,
- if $\Phi \vdash_{\mathbf{L}} \Psi$ and $\Psi \vdash_{\mathbf{L}} \psi$, then $\Phi \vdash_{\mathbf{L}} \psi$ and
- if $\Phi \vdash_{\mathbf{L}} \varphi$, then $\sigma[\Phi] \vdash_{\mathbf{L}} \sigma(\varphi)$ for every substitution σ,

where $\Phi \vdash_{\mathbf{L}} \Psi$ stands for $\Phi \vdash_{\mathbf{L}} \psi$ for all $\psi \in \Psi$.

A substructural logic $\mathbf{L}$ is said to be *consistent* if there is a formula φ such that $\nvdash_{\mathbf{L}} \varphi$. This definition of consistency is suitable for our purposes. Note that one can define other reasonable non-equivalent notions of consistency. For instance, one can define $\mathbf{L}$ to be consistent if $\vdash_{\mathbf{L}} \varphi$ and $\vdash_{\mathbf{L}} \neg\varphi$ for no formula φ.

It is important to observe the distinction between the two symbols $\vdash$ and $\Rightarrow$ for entailment. Thanks to the (cut) rule, $\Phi \vdash_{\mathbf{L}} \Gamma, \psi, \Delta \Rightarrow \varphi$ implies $\Phi \cup \{\psi\} \vdash_{\mathbf{L}} \Gamma, \Delta \Rightarrow \varphi$ for arbitrary $\Phi, \Gamma, \Delta, \psi, \varphi$, whereas the converse direction, i.e., the *deduction theorem*, does not necessarily hold. Indeed, $\Phi \cup \{\psi\} \vdash_{\mathbf{L}} \Gamma, \Delta \Rightarrow \varphi$ implies $\Phi \vdash_{\mathbf{L}} \Gamma, \psi, \Delta \Rightarrow \varphi$ if and only if $\mathbf{L}$ validates the structural rules (e), (i), (c). If $\mathbf{L}$ further validates (o), (the $(\wedge, \vee, \backslash, 0)$-fragment of) $\mathbf{L}$ becomes a *superintuitionistic logic*.

Remark 2.1 In this paper, we do not consider nonassociative substructural logics. The latter often behave quite differently from the associative ones; for instance, the $(\cdot, \backslash, /, 1)$-fragment of the nonassociative $\mathbf{FL}$ is decidable in P [13, 2], in sharp contrast to the NP-completeness of the same fragment of associative $\mathbf{FL}$. Thus some special care is needed for nonassociative substructural logics. Indeed, our coding of existential quantifier in Section 4 does not work for them.

2.2. FL-algebras

Now we are going to define an algebraic semantics for substructural logics.

An *FL-algebra* is an algebraic structure $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1, 0\rangle$ where $\langle A, \wedge, \vee\rangle$ is a lattice, $\langle A, \cdot, 1\rangle$ is a monoid, and for all $x, y, z \in A$ we have the *residuation property*:

$$x \cdot y \leq z \text{ iff } y \leq x\backslash z \text{ iff } x \leq z/y.$$

The residuation property is equivalent to the existence of maximum solutions of the inequality $x \cdot y \leq z$ for x and y. These maximum solutions are z/y for x and $x\backslash z$ for y.

The element 0 can be arbitrary chosen in A. It is used to define negations: $\sim\alpha = \alpha\backslash 0$, $-\alpha = 0/\alpha$. The operations $\backslash$ and $/$ are called respectively left and right division. As before, $x \to y$ (resp. $\neg x$) denotes either of $x\backslash y$ and y/x (resp. $\sim x$ and $-x$) when the distinction is irrelevant. In the absence of parentheses we assume that $\cdot$ is performed first followed by $\backslash, /$ and then by $\wedge, \vee$. We often write xy for $x \cdot y$. For reader's convenience the following lemma lists basic properties of FL-algebras.

Lemma 2.2 *The following identities hold in any FL-algebra.*

1. $x(y \vee z) = xy \vee xz$ *and* $(y \vee z)x = yx \vee zx$,
2. $x\backslash(y \wedge z) = (x\backslash y) \wedge (x\backslash z)$ *and* $(y \wedge z)/x = (y/x) \wedge (z/x)$,
3. $(y \vee z)\backslash x = (y\backslash x) \wedge (z\backslash x)$ *and* $x/(y \vee z) = (x/y) \wedge (x/z)$,
4. $(x/y)y \leq x$ *and* $y(y\backslash x) \leq x$,
5. $(x/y)/z = x/(zy)$ *and* $z\backslash(y\backslash x) = (yz)\backslash x$,
6. $x\backslash(y/z) = (x\backslash y)/z$,
7. $x/1 = x = 1\backslash x$,
8. $1 \leq x\backslash x$ *and* $1 \leq x/x$,
9. $(z/y)(y/x) \leq z/x$ *and* $(x\backslash y)(y\backslash z) \leq x\backslash z$.

Terms in the language of FL-algebras are just formulas of **FL**. For naturality we often write $s, t, u, \ldots$ for elements of Fm in algebraic contexts. Let $E \cup \{t = u\}$ be a set of identities (equations) in the language of FL-algebras. Given an evaluation v into $\boldsymbol{A}$, we write $E \models_{\boldsymbol{A},v} t = u$ if $v(s_1) = v(s_2)$ for all $s_1 = s_2 \in E$ implies $v(t) = v(u)$. We write $E \models_{\boldsymbol{A}} t = u$ if $E \models_{\boldsymbol{A},v} t = u$ holds for every evaluation v into $\boldsymbol{A}$. Let K be a class of FL-algebras. Then we write $E \models_{\mathsf{K}} t = u$ if $E \models_{\boldsymbol{A}} t = u$ holds for every $\boldsymbol{A} \in \mathsf{K}$. When E is empty, we simply write $\models_{\boldsymbol{A}} t = u$ and $\models_{\mathsf{K}} t = u$. Since FL-algebras have a lattice reduct, we can express each inequality $t \leq u$ as the identity $t \vee u = u$. Thus we shortly write $\models_{\mathsf{K}} t \leq u$ instead of $\models_{\mathsf{K}} t \vee u = u$.

In addition to identities that correspond to axioms, we are also interested in quasi-identities that correspond to inference rules. A *quasi-identity* is an expression of the form

$$t_1 = u_1 \ \mathsf{and} \ \ldots \ \mathsf{and} \ t_n = u_n \Longrightarrow t_0 = u_0. \qquad (q)$$

We write $\models_{\boldsymbol{A}} (q)$ if $\{t_1 = u_1, \ldots, t_n = u_n\} \models_{\boldsymbol{A}} t_0 = u_0$. Note that identities are special cases of quasi-identities.

We say that a set Q of quasi-identities *defines* a class K of FL-algebras if $\boldsymbol{A} \in \mathsf{K} \Longleftrightarrow \models_{\boldsymbol{A}} (q)$ for every $(q) \in Q$. Analogously one can define a class of FL-algebras defined by a set of identities.

Let K be a class of algebras in the same language. The most fundamental in universal algebra is *Birkhoff's theorem* showing that K is defined by a set of identities if and only if K is a *variety*, that is a class of algebras closed under homomorphic images, subalgebras, and products. Its analogue for classes of algebras defined by quasi-identities is also well known (see [3]). Namely, K is defined by a set of quasi-identities if and only if K is a *quasivariety*, that is a class of algebras closed under isomorphic images, subalgebras, products and ultraproducts containing a trivial algebra.

It is known that the class FL of all FL-algebras is a variety (see [10]). By Birkhoff's theorem and its analogue for quasivarieties, any subclass of FL defined by equations is a variety, and any subclass defined by quasi-identities is a quasivariety.

The axiomatic schemata (1) correspond respectively to the following identities:

$$(e)\ xy \leq yx\,, \qquad (c)\ x \leq x^2\,, \qquad (i)\ x \leq 1\,, \qquad (o)\ 0 \leq x\,. \tag{2}$$

The corresponding FL-algebras and varieties of FL-algebras are denoted in the same way as logics, i.e., given $S \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$, the subvariety of FL defined by S is denoted by FL_S and its members are called FL_S-algebras. $\mathrm{FL_e}$-algebras and $\mathrm{FL_i}$-algebras are respectively called *commutative* and *integral*. FL-algebras satisfying $\sim\!-x \leq x$ and $-\!\sim x \leq x$ (algebraic counterpart of the law of double-negation elimination) are called *involutive*.

2.3. Correspondence between logic and algebra

It is known that the logic **FL** is algebraizable and its equivalent algebraic semantics is the variety FL [11]. In more detail, extending **FL** by an axiomatic schema φ is equivalent to restricting FL to the subvariety defined by $1 \leq \varphi$. This induces a dual-isomorphism V from the lattice of axiomatic extensions of **FL** to the subvariety lattice of FL. Further, we have the following completeness theorem.

Theorem 2.3 ([11]) *Let* $\mathbf{L}$ *be an axiomatic extension of* **FL** *and* $\mathsf{V}(\mathbf{L})$ *the corresponding variety of FL-algebras. Then there are translations* τ, ρ *such that for any* $\Phi \cup \{\varphi\} \subseteq Fm$ *and any set* $E \cup \{t = u\}$ *of identities we have:*

$$\Phi \vdash_{\mathbf{L}} \varphi \quad \textit{iff} \quad \tau(\Phi) \models_{\mathsf{V}(\mathbf{L})} \tau(\varphi)\,.$$

$$E \models_{\mathsf{V}(\mathbf{L})} t = u \quad \textit{iff} \quad \rho(E) \vdash_{\mathbf{L}} \rho(t = u)\,.$$

The translations τ, ρ *are defined as follows:* $\tau(\varphi)$ *is* $1 \leq \varphi$ *for* $\varphi \in Fm$ *and* $\rho(t = u)$ *is* $(u\backslash t) \wedge (t\backslash u)$ *for an identity* $t = u$.

This algebraization result can be generalized to a correspondence between rule extensions of **FL** and subquasivarieties of FL as follows.

To each sequent $\Gamma \Rightarrow \varphi$ we associate an identity $\Gamma^{\cdot} \leq \varphi^{\cdot}$, where $\Gamma^{\cdot}$ denotes the product of formulas in Γ ($\Gamma^{\cdot} = 1$ if Γ is the empty sequence), and $\varphi^{\cdot}$ denotes φ itself if φ is a formula ($\varphi^{\cdot} = 0$ if φ is the empty sequence). Given an inference rule

$$\frac{\Gamma_1 \Rightarrow \varphi_1 \quad \cdots \quad \Gamma_n \Rightarrow \varphi_n}{\Gamma_0 \Rightarrow \varphi_0}\ (r)$$

we associate to it the quasi-identity

$$\Gamma_1^{\cdot} \leq \varphi_1^{\cdot} \ \mathsf{and}\ \ldots\ \mathsf{and}\ \Gamma_n^{\cdot} \leq \varphi_n^{\cdot} \Longrightarrow \Gamma_0^{\cdot} \leq \varphi_0^{\cdot}. \tag{$r^{\cdot}$}$$

This induces a dual-isomorphism Q from the lattice of rule extensions of **FL** (substructural logics in our sense) to the lattice of quasivarieties of FL-algebras. We again have:

Logic	Algebra
logic **FL**	variety FL
axiom φ	identity $1 \leq \varphi$
inference rule (r)	quasi-identity $(r^{\cdot})$
axiomatic extension **L** of **FL**	subvariety $\mathsf{V}(\mathbf{L})$ of FL
rule extension **L** of **FL**	subquasivariety $\mathsf{Q}(\mathbf{L})$ of FL
consistent	nontrivial

Figure 2: Correspondence between logical and algebraic concepts

Theorem 2.4 *Let* $\mathbf{L}$ *be a substructural logic and* $\mathsf{Q}(\mathbf{L})$ *the corresponding quasivariety of FL-algebras. Then for any* $\Phi \cup \{\varphi\} \subseteq Fm$ *and any set* $E \cup \{t = u\}$ *of identities we have:*

$$\Phi \vdash_{\mathbf{L}} \varphi \quad \textit{iff} \quad \tau(\Phi) \models_{\mathsf{Q}(\mathbf{L})} \tau(\varphi)\,,$$

$$E \models_{\mathsf{Q}(\mathbf{L})} t = u \quad \textit{iff} \quad \rho(E) \vdash_{\mathbf{L}} \rho(t = u)\,,$$

where the translations τ, ρ *are defined as in Theorem 2.3.*

In view of this theorem, it is easy to see that a substructural logic $\mathbf{L}$ is consistent if and only if $\mathsf{Q}(\mathbf{L})$ is *nontrivial*, in the sense that it contains an algebra other than the trivial one-element FL-algebra $\{1\}$.

Let $\mathbf{L}$ be a substructural logic and $\alpha_1, \dots, \alpha_n, \varphi \in Fm$. Note that according to Theorem 2.4 and the residuation property we have the following chain of equivalent statements:

$$\vdash_{\mathbf{L}} \alpha_1, \dots, \alpha_n \Rightarrow \varphi \text{ iff } \vdash_{\mathbf{L}} (\alpha_1 \cdots \alpha_n)\backslash\varphi \text{ iff } \models_{\mathsf{Q}(\mathbf{L})} 1 \leq (\alpha_1 \cdots \alpha_n)\backslash\varphi \text{ iff } \models_{\mathsf{Q}(\mathbf{L})} \alpha_1 \cdots \alpha_n \leq \varphi.$$

We summarize the correspondence between logical and algebraic concepts in Figure 2.

3. Disjunction property

3.1. Disjunction property and its algebraic form

In this subsection we recall the definition of the disjunction property and introduce its algebraic counterpart.

Definition 3.1 (Disjunction Property) Let $\mathbf{L}$ be a substructural logic. $\mathbf{L}$ satisfies the *Disjunction Property* (DP) if for all formulas φ, ψ we have $\vdash_{\mathbf{L}} \varphi \vee \psi$ implies $\vdash_{\mathbf{L}} \varphi$ or $\vdash_{\mathbf{L}} \psi$.

Substructural logics satisfying the DP have the following property, which will be crucial to show the correctness of our coding of quantified Boolean formulas in Section 4.

Lemma 3.2 *Let* $\mathbf{L}$ *be a substructural logic satisfying the DP,* φ, ψ *formulas and* V *a set of propositional variables. Then* $V \vdash_{\mathbf{L}} \varphi \vee \psi$ *implies* $V \vdash_{\mathbf{L}} \varphi$ *or* $V \vdash_{\mathbf{L}} \psi$*.*

Proof: Let σ be the substitution such that $\sigma(x) = x \vee 1$ if $x \in V$ and $\sigma(x) = x$ otherwise. By Theorem 2.4 and noting that $1 \leq x$ means that $x = x \vee 1$, we have:

$$V \vdash_{\mathbf{L}} \varphi \quad \text{iff} \quad \{1 \leq x \mid x \in V\} \models_{\mathsf{Q}(\mathbf{L})} 1 \leq \varphi \quad \text{iff} \quad \models_{\mathsf{Q}(\mathbf{L})} 1 \leq \sigma(\varphi) \quad \text{iff} \quad \vdash_{\mathbf{L}} \sigma(\varphi)$$

for every formula φ. Hence the lemma reduces to the DP.

In more detail, the second statement implies the third because for any evaluation v we can define a new evaluation v' by $v'(x) = v(\sigma(x))$. We then have $1 \leq v'(x)$ for every $x \in V$, so $1 \leq v'(\varphi)$. We also have $v'(\varphi) = v(\sigma(\varphi))$, so $1 \leq v(\sigma(\varphi))$. On the other hand, the third implies the second because for any evaluation v such that $1 \leq v(x)$ for every $x \in V$, we have $v(x) = v(\sigma(x))$, and so $v(\varphi) = v(\sigma(\varphi)) \geq 1$. □

¿From the proof theoretic perspective, substructural logics with a single-conclusion cut-free sequent calculus usually have the DP. This class includes $\mathbf{FL}_S$ for any $S \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$. Other examples of substructural logics in this class are extensions of **FL** by $\neg(\alpha \wedge \neg\alpha)$ and/or axiomatic schemata $\alpha^n \to \alpha^m$ for $n, m \geq 0$ denoted by (knot^n_m). Furthermore, some substructural logics with a multi-conclusion cut-free sequent calculus without the right contraction also have the DP. This class includes involutive substructural logics $\mathbf{InFL}_S$ for any $S \subseteq \{\mathrm{e}, \mathrm{w}\}$ (rules (i) and (o) are derivable from each other in **InFL**).

There is also an algebraic way to prove the DP for a substructural logic. It involves the following algebraic characterization of the DP. Recall that an FL-algebra $\boldsymbol{A}$ is called *well-connected* if for all $x, y \in A$, $x \vee y \geq 1$ implies $x \geq 1$ or $y \geq 1$.

Theorem 3.3 ([10]) *Let* $\mathbf{L}$ *be an axiomatic extension of* $\mathbf{FL}$*. Then the following are equivalent:*

1. $\mathbf{L}$ *has the DP.*
2. *For all* $\boldsymbol{A}_1, \boldsymbol{A}_2 \in \mathsf{V}(\mathbf{L})$ *there is a well-connected FL-algebra* $\boldsymbol{C} \in \mathsf{V}(\mathbf{L})$ *such that* $\boldsymbol{A}_1 \times \boldsymbol{A}_2$ *is a homomorphic image of* $\boldsymbol{C}$*.*

Let $\mathbf{L}$ be any of the logics $\mathbf{FL}$, $\mathbf{FL_e}$, $\mathbf{FL} + (knot^n_m)$ and $\mathbf{FL_e} + (knot^n_m)$. Using Theorem 3.3 it is proved in [24] that the extension of $\mathbf{L}$ by the lattice distributivity axiom (dis) (i.e., $\alpha \wedge (\beta \vee \gamma) \to (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$) enjoys the DP. Further, [24] proves that $\mathbf{InFL}$, $\mathbf{InFL_e}$, $\mathbf{InFL} + (dis)$ and $\mathbf{InFL_e} + (dis)$ enjoy the DP. Thus the relevance logic **RW** satisfies the DP as well because **RW** is equivalent to the constant-free fragment of $\mathbf{InFL_e} + (dis)$ expanded by negation.

For the purpose of this paper, we need a generalization of Theorem 3.3 working for all substructural logics (not only axiomatic extensions of **FL**). We also a little bit simplify the characterizing algebraic condition so that it is easier to work with it.

Theorem 3.4 *Let* $\mathbf{L}$ *be a substructural logic (i.e. a rule extension of* $\mathbf{FL}$*). Then* $\mathbf{L}$ *has the DP iff the following condition holds:*

(*) *for every* $\boldsymbol{A} \in \mathsf{Q}(\mathbf{L})$ *there is a well-connected FL-algebra* $\boldsymbol{C} \in \mathsf{Q}(\mathbf{L})$ *such that* $\boldsymbol{A}$ *is a homomorphic image of* $\boldsymbol{C}$*.*

Proof: Let $\mathsf{K} = \mathsf{Q}(\mathbf{L})$. Assume first that $\mathbf{L}$ has the DP and $\boldsymbol{A} \in \mathsf{K}$. Then $\boldsymbol{A}$ is a homomorphic image of a K-free algebra $\boldsymbol{C}$. Since K is a quasi-variety, we have $\boldsymbol{C} \in \mathsf{K}$. Moreover, it is easy to show using Theorem 2.4 that every K-free algebra is well-connected because $\mathbf{L}$ has the DP.

Conversely, assume that (*) holds. In view of Theorem 2.4, it is sufficient to prove the following: if there are $\boldsymbol{A}_1, \boldsymbol{A}_2 \in \mathsf{K}$ such that $\not\models_{\boldsymbol{A}_1} 1 \leq t_1$ and $\not\models_{\boldsymbol{A}_2} 1 \leq t_2$, there is $\boldsymbol{C} \in \mathsf{K}$ such that $\not\models_{\boldsymbol{C}} 1 \leq t_1 \vee t_2$.

Let $\boldsymbol{A} = \boldsymbol{A}_1 \times \boldsymbol{A}_2$. It belongs to K since K is a quasivariety. Hence condition (*) gives us a well-connected algebra $\boldsymbol{C} \in \mathsf{K}$ together with a surjective homomorphism $f : \boldsymbol{C} \longrightarrow \boldsymbol{A}$. Given evaluations v_i into $\boldsymbol{A}_i$ $(i = 1, 2)$ such that $1 \not\leq v_i(t_i)$, we choose an evaluation v into $\boldsymbol{C}$ in such a way that $f(v(x)) = \langle v_1(x), v_2(x) \rangle$ holds for every variable x.

We claim that $1 \not\leq v(t_1 \vee t_2)$. Otherwise, the well-connectedness implies $1 \leq v(t_1)$ or $1 \leq v(t_2)$, say $1 \leq v(t_1)$. But then $\langle 1, 1 \rangle = f(1) \leq f(v(t_1)) = \langle v_1(t_1), v_2(t_1) \rangle$. Hence we have $1 \leq v_1(t_1)$, contradicting the assumption. □

Thus we say that a class K of FL-algebras has the DP if the condition (*) holds for K.

3.2. Lattice-monoidal quasi-identities

We will now generalize the construction from [24] and prove the DP also for other substructural logics. More specifically, we will prove that any quasivariety K of FL-algebras defined by the following type of quasi-identities satisfies the DP.

Definition 3.5 (ℓ-monoidal quasi-identity) A quasi-identity

$$t_1 \leq u_1 \text{ and } \dots \text{ and } t_n \leq u_n \Longrightarrow t_0 \leq u_0 \tag{q}$$

is said to be *ℓ-monoidal* if for every $0 \leq i \leq n$, t_i is in the language $\{\cdot, \wedge, \vee, 1\}$ and u_i is either 0 or in the language $\{\cdot, \wedge, \vee, 1\}$.

Accordingly, an inference rule schema

$$\frac{\Gamma_1 \Rightarrow \varphi_1 \quad \cdots \quad \Gamma_n \Rightarrow \varphi_n}{\Gamma_0 \Rightarrow \varphi_0}\ (r)$$

is said to be *ℓ-monoidal* if for every $0 \leq i \leq n$, Γ_i is a sequence of formulas in the language $\{\cdot, \wedge, \vee, 1\}$ and φ_i is either the empty sequence or a formula in the language $\{\cdot, \wedge, \vee, 1\}$.

In our construction, the key role will be played by $\mathrm{FL_i}$-algebras $\boldsymbol{B}$ with a *unique subcover* of 1, that is an element s such that $x < 1$ iff $x \leq s$ for every $x \in B$. Such an algebra $\boldsymbol{B}$ is well-connected, since $x < 1$ and $y < 1$ imply $x, y \leq s$, so $x \vee y \leq s < 1$.

Thus the first step is to find in the given quasivariety K an $\mathrm{FL_i}$-algebra $\boldsymbol{B}$ with a unique subcover of 1. In doing so, two types of FL-algebras have to be distinguished depending on the position of 0.We say that an FL-algebra $\boldsymbol{A}$ is *of type* $1 \leq 0$ if $\models_{\boldsymbol{A}} 1 \leq 0$ holds; $\boldsymbol{A}$ is *of type* $1 \not\leq 0$ otherwise.

Lemma 3.6 *Let $\boldsymbol{B}$ be a nontrivial FL-algebra. There is an element $a \in B$ such that $a < 1$.*

Proof: Since $\boldsymbol{B}$ is nontrivial, there is an element $b \in B$ such that $b \neq 1$. If $1 \not\leq b$ then $a = b \wedge 1 < 1$. If $1 < b$ then we take $a = b\backslash 1$. Clearly we have $a \leq 1\backslash 1 = 1$. Moreover, $a < 1$; otherwise $b = b \cdot a = b \cdot (b\backslash 1) \leq 1$. $\Box$

Lemma 3.7 *Let* K *be a quasivariety of FL-algebras defined by* ℓ*-monoidal quasi-identities. Then for any nontrivial algebra* $\boldsymbol{A} \in \mathsf{K}$*, there is an* FL_{ei}*-algebra* $\boldsymbol{B} \in \mathsf{K}$ *which is of the same type as* $\boldsymbol{A}$ *and has a unique subcover of* 1*.*

Proof: Let $\boldsymbol{A}$ be a nontrivial algebra from K. We distinguish two cases depending on the type of $\boldsymbol{A}$.

First suppose that $\boldsymbol{A}$ is of type $1 \leq 0$. By Lemma 3.6 there is $a \in A$ such that $a < 1$. Consider the submonoid B of $\boldsymbol{A}$ generated by a, namely $B = \{a^n : n \geq 0\}$. This submonoid inherits join, meet and product operations from $\boldsymbol{A}$, and is commutative and dually well ordered:

$$\cdots a^4 \leq a^3 \leq a^2 \leq a < 1.$$

Hence B gives rise to an $\mathrm{FL_{ei}}$-algebra $\boldsymbol{B}$ of type $1 \leq 0$ by setting

$$\begin{aligned} x \to y &= sup\{z \in B : xz \leq y\} \\ 0_{\boldsymbol{B}} &= 1. \end{aligned}$$

It is clear that a is the unique subcover of 1.

It remains to show that $\boldsymbol{B} \in \mathsf{K}$. Let v be an evaluation of variables into B, which can also be considered an evaluation into A. We claim that

(*) $\models_{\boldsymbol{A},v} t \leq u$ if and only if $\models_{\boldsymbol{B},v} t \leq u$ for every ℓ-monoidal identity $t \leq u$.

When both t and u are in the language $\{\cdot, \wedge, \vee, 1\}$, the claim is obvious since $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$ with respect to this language. When $u = 0$, the claim amounts to

$$\models_{\boldsymbol{A},v} t \leq 0 \text{ if and only if } \models_{\boldsymbol{B},v} t \leq 1$$

by our definition of $0_{\boldsymbol{B}}$. But both sides trivially hold because $v(t) \leq 1$ and $1 \leq 0_{\boldsymbol{A}}$. Since quasi-identities are just Horn implications over identities, it immediately follows that any ℓ-monoidal quasi-identity valid in $\boldsymbol{A}$ is also valid in $\boldsymbol{B}$. This ensures $\boldsymbol{B} \in \mathsf{K}$.

Next suppose that $\boldsymbol{A}$ is of type $1 \not\leq 0$. Then by the proof of Lemma 3.6, we may take $a = 0 \wedge 1 < 1$, and define an $\mathrm{FL_{ei}}$-algebra $\boldsymbol{B}$ as above, except that we set $0_{\boldsymbol{B}} = 0_{\boldsymbol{A}} \wedge 1 = a$. $\boldsymbol{B}$ is an $\mathrm{FL_{ei}}$-algebra of type $1 \not\leq 0$ with a unique subcover a of 1. We again claim (*), for which the only nontrivial case is when $u = 0$. Let v be an evaluation into $\boldsymbol{B}$. If $v(t) \leq 0_{\boldsymbol{A}}$, then $v(t) \neq 1$ and so $v(t) \leq a = 0_{\boldsymbol{B}}$ since a is the unique subcover of 1. Conversely, if $v(t) \leq 0_{\boldsymbol{B}}$, then $v(t) \leq 0_{\boldsymbol{A}}$ by our definition of $0_{\boldsymbol{B}}$. As before, this ensures $\boldsymbol{B} \in \mathsf{K}$. $\Box$

For the next step of our construction, we will need the notion of conucleus (see [10]). Recall that an interior operator σ on an FL-algebra $\boldsymbol{A}$ is a map $\sigma\colon A \to A$ which is contracting ($\sigma(x) \leq x$), idempotent ($\sigma(\sigma(x)) = \sigma(x)$) and monotone ($x \leq y$ implies $\sigma(x) \leq$

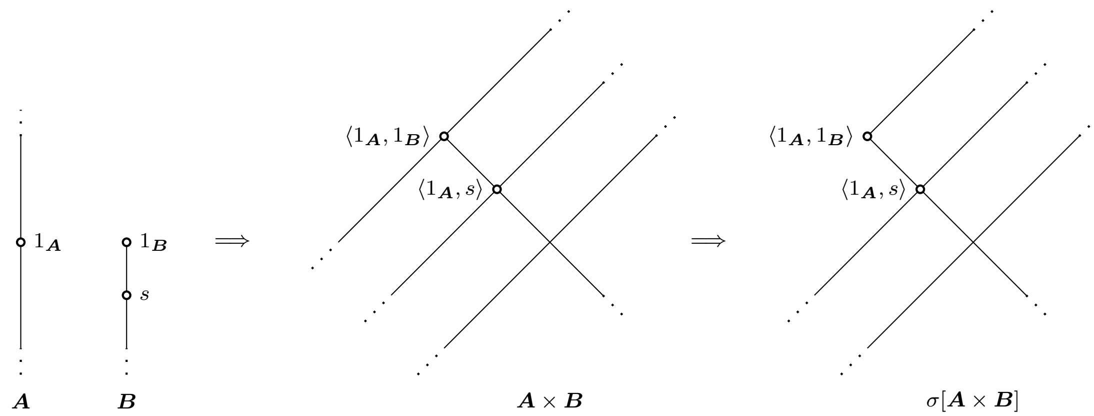

Figure 3: The structure of $\sigma[\boldsymbol{A}\times\boldsymbol{B}]$.

$\sigma(y))$. If $\sigma(1) = 1$ and $\sigma(x)\sigma(y) \leq \sigma(xy)$ for all $x, y \in A$, then σ is called a *conucleus*. Given an FL-algebra $\boldsymbol{A}$ and a conucleus σ on $\boldsymbol{A}$, the algebra $\sigma[\boldsymbol{A}] = \langle \sigma[A], \wedge_\sigma, \vee, \cdot, \backslash_\sigma, /_\sigma, \sigma(0), 1\rangle$ is an FL-algebra, where $x \wedge_\sigma y = \sigma(x \wedge y)$, $x\backslash_\sigma y = \sigma(x\backslash y)$ and $x/_\sigma y = \sigma(x/y)$. The algebra $\sigma[\boldsymbol{A}]$ is called a *conuclear contraction* of $\boldsymbol{A}$.

Given an FL-algebra $\boldsymbol{A}$, we denote by A^+ its positive cone, i.e., $A^+ = \{a \in A \mid 1 \leq a\}$. Note that A^+ forms a sub-ℓ-monoid of $\boldsymbol{A}$, namely it forms a subalgebra of $\boldsymbol{A}$ with respect to the language $\{\cdot, \wedge, \vee, 1\}$.

Let $\boldsymbol{B}$ be an $\mathrm{FL_i}$-algebra of the same type as $\boldsymbol{A}$ with a unique subcover s of 1. We define an operator σ on $A \times B$ as follows:

$$\sigma(a, b) = \begin{cases} \langle a, s\rangle & \text{if } a \notin A^+ \text{ and } b = 1, \\ \langle a, b\rangle & \text{otherwise.} \end{cases}$$

It yields:

$$\sigma[A \times B] = A^+ \times \{1\} \cup A \times (B \setminus \{1\})$$

Figure 3 visualizes the construction of $\sigma[A \times B]$.

Lemma 3.8 *The operator σ is a conucleus on $\boldsymbol{A}\times\boldsymbol{B}$ such that $\sigma[A\times B]$ forms a subalgebra of $\boldsymbol{A}\times\boldsymbol{B}$ with respect to the language $\{\cdot, \wedge, \vee, 1, 0\}$. Moreover, $\sigma[\boldsymbol{A}\times\boldsymbol{B}]$ is well-connected and $\boldsymbol{A}$ is a homomorphic image of $\sigma[\boldsymbol{A}\times\boldsymbol{B}]$.*

Proof: It is straightforward to verify that σ is an interior operator and $\sigma(1, 1) = \langle 1, 1\rangle$. Further, we have to check that $\sigma(a, x)\sigma(b, y) \leq \sigma(ab, xy)$. Clearly $\sigma(a, x)\sigma(b, y) \leq \langle ab, xy\rangle$ since σ is contracting. The only nontrivial case is $ab \notin A^+$ and $xy = 1$ because $\sigma(ab, xy) = \langle ab, s\rangle$ in this case. Since A^+ is closed under the multiplication, we get $a \notin A^+$ or $b \notin A^+$, say $a \notin A^+$. Further, we have $x = y = 1$ since $\boldsymbol{B}$ is integral. Thus $\sigma(a, x)\sigma(b, y) = \langle a, s\rangle \cdot \sigma(b, y) \leq \langle ab, s\rangle = \sigma(ab, xy)$. Thus σ is a conucleus.

Next we verify that $\sigma[\boldsymbol{A}\times\boldsymbol{B}]$ is a subalgebra of $\boldsymbol{A}\times\boldsymbol{B}$ with respect to the language $\{\cdot, \wedge, \vee, 1, 0\}$. The image of any conucleus is closed under the multiplication and join.

Also, $\sigma[A \times B] = A^+ \times \{1\} \cup A \times (B \setminus \{1\})$ is clearly closed under the meet. Finally, $0_{\boldsymbol{A}\times\boldsymbol{B}} = \langle 0_{\boldsymbol{A}}, 0_{\boldsymbol{B}} \rangle$ belongs to $\sigma[A \times B]$ since $\boldsymbol{A}$ and $\boldsymbol{B}$ are of the same type.

Now we check that $\sigma[\boldsymbol{A} \times \boldsymbol{B}]$ is well-connected. Let $\langle a, x\rangle, \langle b, y\rangle \in \sigma[A \times B]$ such that $\langle a, x\rangle \vee \langle b, y\rangle \geq \langle 1, 1\rangle$, i.e., $a \vee b \geq 1$ and $x \vee y = 1$. Since $\boldsymbol{B}$ is well-connected, we get $x = 1$ or $y = 1$ (say $x = 1$). Then $a \in A^+$. Consequently, $\langle 1, 1\rangle \leq \langle a, x\rangle$.

Let $f\colon \sigma[A \times B] \to A$ be a mapping defined $f(a, x) = a$. Then f is clearly a surjective homomorphism since σ keeps the first component unchanged. Indeed, for example f preserves $\backslash_\sigma$ since

$$f(\langle a, x\rangle \backslash_\sigma \langle b, y\rangle) = f(\sigma(a\backslash b, x\backslash y)) = a\backslash b = f(a, x)\backslash f(b, y)\,.$$

□

We are now ready to prove the main result of this subsection:

Theorem 3.9 *Let K be a quasivariety of FL-algebras defined by ℓ-monoidal quasi-identities. Then K has the DP.*

Proof: We have to check that the condition (*) holds for K, i.e., we have to find for every $\boldsymbol{A} \in \mathsf{K}$ a well-connected algebra $\boldsymbol{C} \in \mathsf{K}$ such that $\boldsymbol{A}$ is a homomorphic image of $\boldsymbol{C}$. If $\boldsymbol{A}$ is trivial then it is obvious because the trivial algebra is well-connected. Hence assume that $\boldsymbol{A}$ is nontrivial. By Lemma 3.7, K contains an $\mathrm{FL}_{\mathrm{ei}}$-algebra $\boldsymbol{B}$ of the same type as $\boldsymbol{A}$ with a unique subcover of 1. We claim that $\boldsymbol{C} = \sigma[\boldsymbol{A} \times \boldsymbol{B}]$ has the desired properties. Indeed, $\boldsymbol{C}$ is well-connected and $\boldsymbol{A}$ is a homomorphic image of $\boldsymbol{C}$ by Lemma 3.8. Moreover, since $\boldsymbol{C}$ is a subalgebra of $\boldsymbol{A} \times \boldsymbol{B} \in \mathsf{K}$ with respect to the language $\{\cdot, \wedge, \vee, 1, 0\}$ and quasi-identities defining K are in the same language, it follows that $\boldsymbol{C} \in \mathsf{K}$. □

Hence by Theorem 3.4 we obtain:

Corollary 3.10 *Let* $\mathbf{L}$ *be an extension of* $\mathbf{FL}$ *by ℓ-monoidal inference rules. Then* $\mathbf{L}$ *has the DP.*

Typical examples of inference rules, where Corollary 3.10 is applicable, are the structural rules (e), (c), (i), (o). Thus every extension $\mathbf{FL}_S$ for $S \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$ enjoys the DP. Another example of ℓ-monoidal inference rule, where Corollary 3.10 can be used, is for instance the rule

$$\frac{\Rightarrow \varphi \cdot \psi}{\Rightarrow \varphi}\ (r)$$

Unlike the structural rules (e), (c), (i), (o), the rule (r) does not define an axiomatic extension of $\mathbf{FL}$ because its corresponding quasi-identity

$$1 \leq xy \Longrightarrow 1 \leq x \qquad (r^{\cdot})$$

defines a proper subquasivariety of FL (i.e., a quasivariety which is not a variety).

3.3. $\mathcal{M}_2$ axioms

Theorem 3.9 deals with quasivarieties of FL-algebras axiomatized in the language $\{\cdot, \wedge, \vee, 1, 0\}$. However, sometimes an identity in a richer language can be expressed as a quasi-identity in a smaller language. An example is $1 \leq \neg(x \wedge \neg x)$ which involves divisions but is equivalent to $xx \leq 0 \Longrightarrow x \leq 0$. For another example, the identities $xy/y = x = y\backslash yx$ axiomatizing cancellative FL-algebras (i.e., FL-algebras whose monoidal reduct is cancellative) are equivalent to the quasi-identities $xz = yz \Longrightarrow x = y$ and $zx = zy \Longrightarrow x = y$. More generally, the following class of identities corresponds to ℓ-monoidal quasi-identities. The definition below is inspired by the class $\mathcal{N}_2$ in the *substructural hierarchy*, which well corresponds to structural inference rules [5, 6].

Definition 3.11 (Class $\mathcal{M}_2$) Fix an infinite set $\mathcal{V}$ of variables. Given a set T of terms, let T° be the least set of terms that includes T and is closed under the operations $\{\cdot, \wedge, \vee, 1\}$. In particular, $\mathcal{V}^\circ$ is the set of terms in the language $\{\cdot, \wedge, \vee, 1\}$. Likewise, let $T^\bullet$ be the least set of terms that satisfies the following closure properties:

- $0 \in T^\bullet$, $\mathcal{V}^\circ \subseteq T^\bullet$;
- if $t, u \in T^\bullet$ then $t \wedge u \in T^\bullet$;
- if $t \in T^\circ$ and $u \in T^\bullet$, then $t\backslash u, u/t \in T^\bullet$.

We define $\mathcal{M}_1 = \mathcal{V}^\bullet$ and $\mathcal{M}_2 = \mathcal{M}_1^\bullet$. We say that an identity $t \leq u$ belongs to $\mathcal{M}_2$ if $t \in \mathcal{M}_1^\circ$ and $u \in \mathcal{M}_2$, namely $t\backslash u \in \mathcal{M}_2$. An axiom belongs to $\mathcal{M}_2$ just in case it does as a term of FL-algebras.

To get an intuition on how $\mathcal{M}_2$ terms and identities look like, let us observe:

- every term in $\mathcal{M}_1$ is equivalent to a finite meet of terms of the form $t_1\backslash(u/t_2)$, where u is either 0 or in the language $\{\cdot, \wedge, \vee, 1\}$, and t_1, t_2 are in the language $\{\cdot, \wedge, \vee, 1\}$.
- every term in $\mathcal{M}_2$ is equivalent to a finite meet of terms of the form $t_1\backslash(u/t_2)$, where u is either 0 or in the language $\{\cdot, \wedge, \vee, 1\}$, and $t_1, t_2 \in \mathcal{M}_1^\circ$;
- every identity in $\mathcal{M}_2$ is equivalent to a finite set of identities of the form $t \leq u$, where u is either 0 or in the language $\{\cdot, \wedge, \vee, 1\}$, and $t \in \mathcal{M}_1^\circ$.

For instance, $xy/y \in \mathcal{M}_1$, so $(xy/y) \leq x$ is an $\mathcal{M}_2$ identity. Therefore cancellativity can be expressed by $\mathcal{M}_2$ identities. See Figure 4 for some typical $\mathcal{M}_2$ axioms. On the other hand, the following axioms do *not* fall into the class $\mathcal{M}_2$:

$\alpha \vee \neg\alpha$	excluded middle
$(\alpha \to \beta) \vee (\beta \to \alpha)$	prelinearity
$\alpha(\alpha\backslash 1)$	ℓ-group
$\alpha \wedge \beta \to \alpha(\alpha\backslash\alpha \wedge \beta)$, $\quad \alpha \wedge \beta \to (\alpha \wedge \beta/\alpha)\alpha$	divisibility

In fact, extensions of **FL** by the first three axioms do not satisfy the DP. On the other hand, $\mathbf{FL}_\mathrm{i}$ with the divisibility axiom satisfies the DP.

Axiom	Name
$\alpha\beta \to \beta\alpha$	exchange (e)
$\alpha \to 1$	integrality, left weakening (i)
$0 \to \alpha$	right weakening (o)
$\alpha \to \alpha\alpha$	contraction (c)
$\alpha^n \to \alpha^m$	knotted axioms $(n, m \geq 0)$
$\neg(\alpha \wedge \neg\alpha)$	no-contradiction
$(\alpha\beta/\beta) \to \alpha,\ (\alpha\backslash\alpha\beta) \to \beta$	cancellativity
$\alpha \wedge (\beta \vee \gamma) \to (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$	distributivity
$((\alpha \wedge \beta) \vee \gamma) \wedge \beta \to (\alpha \wedge \beta) \vee (\gamma \wedge \beta)$	modularity
$\alpha\beta \wedge \alpha\gamma \to \alpha(\beta \wedge \gamma)$	$(\cdot, \wedge)$-distributivity
$\alpha \wedge (\beta\gamma) \to (\alpha \wedge \beta)(\alpha \wedge \gamma)$	$(\wedge, \cdot)$-distributivity

Figure 4: Some $\mathcal{M}_2$ axioms

Remark 3.12 There is another way to look at the class $\mathcal{M}_2$. Every $t \in \mathcal{M}_2$ is a substitution instance of a term t_0 in the class $\mathcal{N}_2$ [6], where terms in the language $\{\cdot, \wedge, \vee, 1\}$ are substituted for variables in t_0.

Although $\mathcal{M}_2$ identities involve divisions, they can be removed by unfolding identities into quasi-identities. More precisely, we have:

Theorem 3.13 *Every identity in $\mathcal{M}_2$ is equivalent in* FL *to a set of ℓ-monoidal quasi-identities.*

Proof: Consider the following transformation rules defined on identities of the form $t \leq u$:

$$\begin{array}{rcl} t \leq u_1 \wedge u_2 & \mapsto & t \leq u_1, \quad t \leq u_2 \\ t \leq u_1 \backslash u_2 & \mapsto & u_1 t \leq u_2 \\ t \leq u_2 / u_1 & \mapsto & t u_1 \leq u_2 \end{array}$$

Recall that an $\mathcal{M}_2$ term is built by suitably applying $\backslash, /$ and $\wedge$ to either 0 or a term in the language $\{\cdot, \wedge, \vee, 1\}$. Hence if we successively apply the above rules to an identity in $\mathcal{M}_2$, we obtain an equivalent set of identities of the form $t \leq u_0$, where $t \in \mathcal{M}_1^\circ$ and u_0 is either 0 or in the language $\{\cdot, \wedge, \vee, 1\}$. So there is a term $t_0 = t_0(x_1, \ldots, x_n)$ in the language $\{\cdot, \wedge, \vee, 1\}$ and $u_1, \ldots, u_n \in \mathcal{M}_1$ such that $x_1, \ldots, x_n$ are distinct fresh variables and $t = t_0(u_1, \ldots, u_n)$. Observe that $t \leq u_0$ is equivalent to the quasi-identity:

$$x_1 \leq u_1 \ \mathsf{and} \ \ldots \ \mathsf{and} \ x_n \leq u_n \Longrightarrow t_0 \leq u_0. \qquad (q)$$

Indeed, (q) implies $t \leq u_0$ by substitution of u_i for x_i $(1 \leq i \leq n)$. Conversely, assumptions $x_1 \leq u_1$ and $\ldots$ and $x_n \leq u_n$ imply $t_0(x_1, \ldots, x_n) \leq t_0(u_1, \ldots, u_n) = t$. Hence in conjunction with $t \leq u_0$ we obtain the conclusion $t_0 \leq u_0$.

Finally by applying the above transformation rules to the assumptions $x_1 \leq u_1, \ldots, x_n \leq u_n$, we obtain a set of ℓ-monoidal quasi-identities. $\Box$

Corollary 3.14 *Every extension of* **FL** *by* $\mathcal{M}_2$*-axioms has the DP.*

In particular, axioms in Figure 4 preserve the DP when added to **FL**.

3.4. Involutive logics

In the previous sections we have proved the DP for extensions of **FL** by ℓ-monoidal quasi-identities and $\mathcal{M}_2$ axioms. We can also prove the DP for rule extensions of **InFL** and $\mathbf{InFL_e}$ if the extending quasi-identities use only the language $\mathcal{L} = \{\wedge, \vee, 1\}$. The removal of $\cdot$ is necessary, since $\mathbf{InFL_c}$, whose corresponding variety is defined by (c) $x \leq x \cdot x$, does not have the DP. On the other hand, notice that (w) $x \leq 1$ is in the language $\mathcal{L}$, and $\mathbf{InFL_w}$ indeed satisfies the DP.

Let K be a nontrivial subquasivariety of $\mathsf{Q}(\mathbf{InFL})$ or $\mathsf{Q}(\mathbf{InFL_e})$ relatively axiomatized by a set Q of quasi-identities in the language $\mathcal{L}$. Given an algebra $\boldsymbol{A} \in \mathsf{K}$, we will show that there is a well-connected algebra $\boldsymbol{C}$ such that $\boldsymbol{A}$ is a homomorphic image of $\boldsymbol{C}$. Recall that the 3-element MV-chain is the algebra $\boldsymbol{L}_3 = \langle L_3, \min, \max, \cdot, \to, 0, 1\rangle$, where $L_3 = \{0, 1/2, 1\}$, $x \cdot y = \max(x + y - 1, 0)$ and $x \to y = \min(1 - x + y, 1)$.

Lemma 3.15 *The 3-element MV-chain* $\boldsymbol{L}_3$ *belongs to the quasivariety* K*.*

Proof: First, recall that $\boldsymbol{L}_3$ is an $\mathrm{InFL_{ew}}$-algebra. Thus it suffices to show that $\boldsymbol{L}_3$ satisfies all the quasi-identities from Q. Let $\boldsymbol{B}$ be a nontrivial algebra from K. Then by Lemma 3.6 there is an element $a \in B$ such that $a < 1$. Since K is closed under direct products, the algebra $\boldsymbol{B} \times \boldsymbol{B}$ belongs to K as well. The 3-element chain $C = \{\langle a, a\rangle < \langle a, 1\rangle < \langle 1, 1\rangle\}$ forms a subalgebra of $\boldsymbol{B} \times \boldsymbol{B}$ with respect to the language $\mathcal{L}$, i.e., the chain C satisfies all the quasi-identities from Q. Consequently, $\boldsymbol{L}_3 \in \mathsf{K}$ since the $\{\wedge, \vee, 1\}$-reduct of $\boldsymbol{L}_3$ is isomorphic to the 3-element chain C. □

Lemma 3.16 *Let* $\boldsymbol{A} \in \mathsf{K}$*. There is a well-connected algebra* $\boldsymbol{C} \in \mathsf{K}$ *such that* $\boldsymbol{A}$ *is a homomorphic image of* $\boldsymbol{C}$*.*

Proof: To construct the well-connected algebra $\boldsymbol{C}$, we will use the same construction as in [24]. $\boldsymbol{C}$ is constructed from the algebra $\boldsymbol{A} \times \boldsymbol{L}_3$ which belongs to K by Lemma 3.15. The universe of $\boldsymbol{C}$ is defined as follows (see Figure 5):

$$C = A^+ \times \{1\} \cup A \times \{1/2\} \cup \{a \in A \mid a \leq 0_{\boldsymbol{A}}\} \times \{0\}\,.$$

The operations are defined as follows:

$$\langle a, b\rangle \cdot \langle c, d\rangle = \begin{cases} \langle ac, 1/2\rangle & \text{if } bd = 0 \text{ and } a \not\leq 0_{\boldsymbol{A}}, \\ \langle ac, bd\rangle & \text{otherwise,} \end{cases}$$

$$\langle a, b\rangle\backslash\langle c, d\rangle = \begin{cases} \langle a\backslash c, 1/2\rangle & \text{if } b \to d = 1 \text{ and } a \not\geq 1_{\boldsymbol{A}}, \\ \langle a\backslash c, b \to d\rangle & \text{otherwise.} \end{cases}$$

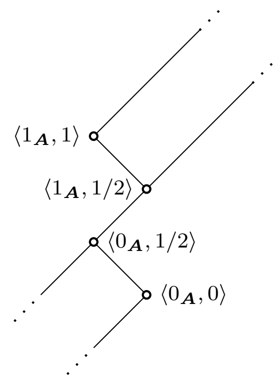

Figure 5: The structure of $\boldsymbol{C}$.

The right division / is defined analogously. It is proved in [24] that $\boldsymbol{C}$ is a well-connected InFL-algebra such that $\boldsymbol{A}$ is its homomorphic image. It is also easy to see that $\boldsymbol{C}$ is an InFL$_\mathrm{e}$-algebra if $\boldsymbol{A}$ is. Further, observe that $\boldsymbol{C}$ is a subalgebra of $\boldsymbol{A} \times \boldsymbol{L}_3$ with respect to the language $\mathcal{L}$. Thus $\boldsymbol{C}$ belongs to K as well. □

Remark 3.17 The algebra $\boldsymbol{C}$ from the previous lemma can be constructed similarly as the well-connected algebra from Lemma 3.8 in two steps. First, consider the algebra $\boldsymbol{B} = \sigma[\boldsymbol{A} \times \boldsymbol{L}_3]$, where σ is the conucleus as in Lemma 3.8. Then $\boldsymbol{C}$ can be seen as a nuclear retraction of $\boldsymbol{B}$, namely $\boldsymbol{C} = \gamma[\boldsymbol{B}]$ for the nucleus

$$\gamma(a, b) = \begin{cases} \langle a, 1/2 \rangle & \text{if } a \not\leq 0_{\boldsymbol{A}} \text{ and } b = 0, \\ \langle a, b \rangle & \text{otherwise.} \end{cases}$$

Now using Lemma 3.16 and Theorem 3.4 we will get the following corollary.

Corollary 3.18 *Every extension of* **InFL** *and* **InFL**$_\mathrm{e}$ *by inference rules in the language* $\{\wedge, \vee, 1\}$ *has the DP.*

4. PSPACE-hardness

It is well known that the satisfiability of closed quantified Boolean formulas in the conjunctive normal form (CNF) is a PSPACE-complete problem (see [21]). The same is true also for closed quantified Boolean formulas in the disjunctive normal form (DNF) since PSPACE = coPSPACE.

Now we introduce a precise definition of quantified Boolean formula which is suitable for our purposes. A *quantified Boolean formula* (QBF) A built up from variables $x_1, \dots, x_n$ is a formula of the form $Q_k x_k \cdots Q_1 x_1 (D_1 \vee \cdots \vee D_m)$, where $Q_i \in \{\exists, \forall\}$, $0 \leq k \leq n$ ($k = 0$ means that A is quantifier-free), and D_i's are conjunctions of literals $x_1, \dots, x_n, \neg x_1, \dots, \neg x_n$ such that no variable repeats in D_i. Thus each D_i can be viewed as a set of literals. Given a $\{0, 1\}$-valued evaluation e, the value $e(A)$ depends only on the evaluation of free variables $x_{k+1}, \ldots, x_n$. If A is closed (i.e., $k = n$), then A is either true or false no matter what e is.

Let $\mathbf{L}$ be a consistent substructural logic satisfying the DP. Given a QBF A and a $\{0,1\}$-valued evaluation e, we will define a sequent $e' \Rightarrow A'$ such that $e(A) = 1$ iff $\vdash_{\mathbf{L}} e' \Rightarrow A'$. We use the same translation of the propositional part of A as in [15]. Our coding of quantifiers was inspired by [26].

Remark 4.1 We stress here that we cannot use the coding of quantifiers from [15]. Our coding is going to work in any consistent substructural logic having the DP, in particular in the extension $\mathbf{L}$ of $\mathbf{FL}$ by the $\mathcal{M}_2$ axiom $\alpha\beta \wedge \alpha\gamma \rightarrow \alpha(\beta \wedge \gamma)$ (see Corollary 3.14). On the other hand, it is easy to show that the translation from [15] of the false QBF $\exists x \forall y (x \wedge \neg y) \vee (\neg x \wedge y)$ is provable in $\mathbf{L}$.

First, for each variable x_j we introduce a new variable $\bar{x}_j$ which will play the role of the literal $\neg x_j$. The translation of e is the sequence of variables $e' = z_{k+1}, \ldots, z_n$, where for each $k+1 \leq j \leq n$ we have

$$z_j = \begin{cases} x_j & \text{if } e(x_j) = 1, \\ \bar{x}_j & \text{if } e(x_j) = 0. \end{cases}$$

Next we define the translation A' of a QBF A. We proceed inductively on the number of quantifiers in A. Assume that A is quantifier-free, i.e., $A = D_1 \vee \cdots \vee D_m$. Then $A' = D'_1 \vee \cdots \vee D'_m$, where $D'_i = y_1 \cdots y_n$ and

$$y_j = \begin{cases} x_j & \text{if } x_j \in D_i, \\ \bar{x}_j & \text{if } \neg x_j \in D_i, \\ x_j \vee \bar{x}_j & \text{otherwise.} \end{cases}$$

Finally, we describe the coding of quantifiers. Assume that $A = \forall x_k\, B$. Then

$$A' = (x_k \vee \bar{x}_k) \backslash B' .$$

If $A = \exists x_k\, B$, then

$$A' = (x_k \backslash q_k \vee \bar{x}_k \backslash q_k) / (B' \backslash q_k) ,$$

where q_k is a fresh variable.

Now we are going to prove that the coding defined above works correctly. We start with the quantifier-free part.

Lemma 4.2 *Let $\mathbf{L}$ be a consistent substructural logic, A a quantifier-free Boolean formula and e a $\{0,1\}$-valued evaluation. Then the following are equivalent:*

1. $e(A) = 1$.
2. $\vdash_{\mathbf{L}} e' \Rightarrow A'$.
3. $e' \vdash_{\mathbf{L}} A'$ *(where e' is considered to be a set).*

Proof: (1⇒2): Suppose that $e(A) = 1$. Then there is D_i such that $e(D_i) = 1$. Then it is easy to see that $y_j = z_j = x_j$ if $x_j \in D_i$, $y_j = z_j = \bar{x}_j$ if $\neg x_j \in D_i$, and $y_j = x_j \vee \bar{x}_j$ otherwise. In all cases we have $\vdash_{\mathbf{L}} z_j \Rightarrow y_j$. Consequently, we will obtain that $\vdash_{\mathbf{L}} e' \Rightarrow D_i'$ by the rule $(\Rightarrow \cdot)$. Then $\vdash_{\mathbf{L}} e' \Rightarrow A'$ follows by the rule $(\Rightarrow\vee)$.

(2⇒3): By applying the (cut) rule to $e' \Rightarrow A'$ with the axioms $\Rightarrow z_i$ $(z_i \in e')$.

(3⇒1): Assume that $e(A) = 0$. We have to show that $e' \not\vdash_{\mathbf{L}} A'$. Let $\boldsymbol{C}$ be any nontrivial algebra from $\mathsf{Q}(\mathbf{L})$. We will define an evaluation v into $\boldsymbol{C}$ such that $v(A') < 1$ and $v(z_1) = \cdots = v(z_n) = 1$. By Lemma 3.6 there is $a \in C$ such that $a < 1$. Let $f\colon \{0,1\} \to \{a,1\}$ be a mapping such that $f(0) = a$ and $f(1) = 1$. Then the evaluation v is defined by $v(x_j) = f(e(x_j))$ and $v(\bar{x}_j) = f(e(\neg x_j))$. Observe that $v(z_j) = 1$. Consider $D_i' = y_1 \cdots y_n$. Then for each y_j we have $v(y_j) = f(e(x_j))$ if $x_j \in D_i$, $v(y_j) = f(e(\neg x_j))$ if $\neg x_j \in D_i$, and $v(y_j) = v(x_j \vee \bar{x}_j) = 1$ otherwise. ¿From $e(A) = 0$, it follows that for all D_i's we have $e(D_i) = 0$. By the observation above there is y_j such that $v(y_j) = a$. Thus $v(D_i') = v(y_1) \cdots v(y_n) \leq a$. Since $v(D_i') \leq a$ for all D_i's, we get $v(A') \leq a < 1$. □

Lemma 4.3 *Let* $\mathbf{L}$ *be a consistent substructural logic having the DP,* $0 \leq k \leq n$*,* A *a QBF with free variables* $x_{k+1}, \ldots, x_n$ *and* e *be a* $\{0,1\}$*-valued evaluation. Then the following are equivalent:*

1. $e(A) = 1$.
2. $\vdash_{\mathbf{L}} e' \Rightarrow A'$.
3. $e' \vdash_{\mathbf{L}} A'$.

Proof: We proceed by induction on k. If $k = 0$ then the lemma follows from Lemma 4.2. Assume that $k > 0$, i.e., $A = Qx_k\, B$ for $Q \in \{\forall, \exists\}$ and a QBF B with free variables $x_k, \ldots, x_n$. Let e_0 be the $\{0,1\}$-valued evaluation such that $e_0(x_j) = e(x_j)$ for $j \neq k$ and $e_0(x_k) = 0$. Analogously e_1 is the $\{0,1\}$-valued evaluation such that $e_1(x_j) = e(x_j)$ for $j \neq k$ and $e_1(x_k) = 1$.

(1⇒2): Assume that $Q = \forall$. Then $e(A) = 1$ implies $e_0(B) = e_1(B) = 1$. Thus by induction hypothesis we have $\vdash_{\mathbf{L}} \bar{x}_k, e' \Rightarrow B'$ and $\vdash_{\mathbf{L}} x_k, e' \Rightarrow B'$. By $(\vee\Rightarrow)$ we obtain $\vdash_{\mathbf{L}} \bar{x}_k \vee x_k, e' \Rightarrow B'$. Consequently, $\vdash_{\mathbf{L}} e' \Rightarrow A'$ by $(\Rightarrow\backslash)$.

Now suppose that $Q = \exists$. Then at least one of $e_0(B), e_1(B)$ equals 1, say $e_0(B)$. Thus by induction hypothesis we have $\vdash_{\mathbf{L}} \bar{x}_k, e' \Rightarrow B'$. Applying $(\Rightarrow\backslash)$, we get $\vdash_{\mathbf{L}} e' \Rightarrow \bar{x}_k\backslash B'$. Since $\bar{x}_k\backslash B', B'\backslash q_k \Rightarrow \bar{x}_k\backslash q_k$ is a provable sequent in $\mathbf{L}$, we get $\vdash_{\mathbf{L}} e', B'\backslash q_k \Rightarrow \bar{x}_k\backslash q_k$ by the cut rule. Then $\vdash_{\mathbf{L}} e', B'\backslash q_k \Rightarrow (\bar{x}_k\backslash q_k) \vee (x_k\backslash q_k)$ by $(\Rightarrow\vee)$. Consequently, $\vdash_{\mathbf{L}} e' \Rightarrow A'$ by $(\Rightarrow/)$.

(2⇒3): Similarly as before.

(3⇒1): Assume that $Q = \forall$. Then $e' \vdash_{\mathbf{L}} (\bar{x}_k \vee x_k)\backslash B'$ implies $e' \vdash_{\mathbf{L}} \bar{x}_k \Rightarrow B'$ and $e' \vdash_{\mathbf{L}} x_k \Rightarrow B'$ because $(\Rightarrow\backslash)$ and $(\vee\Rightarrow)$ are invertible rules, and so $e_0' \vdash_{\mathbf{L}} B'$ and $e_1' \vdash_{\mathbf{L}} B'$. Thus $e_0(B) = e_1(B) = 1$ by induction hypothesis which shows that $e(A) = 1$.

Now suppose that $Q = \exists$. Then $e' \vdash_{\mathbf{L}} A'$ implies $e' \vdash_{\mathbf{L}} (x_k\backslash B') \vee (\bar{x}_k\backslash B')$ because we can substitute B' for q_k. It follows from Lemma 3.2 that $e' \vdash_{\mathbf{L}} x_k\backslash B'$ or $e' \vdash_{\mathbf{L}} \bar{x}_k\backslash B'$. Without any loss of generality assume $e' \vdash_{\mathbf{L}} x_k\backslash B'$. Then $e' \vdash_{\mathbf{L}} x_k \Rightarrow B'$ as well since $(\Rightarrow\backslash)$

is invertible, and so $e'_1 \vdash_{\mathbf{L}} B'$. Consequently, $e_1(B) = 1$ by induction hypothesis. Thus $e(A) = 1$. $\square$

Remark 4.4 Note that the correctness of our coding of existential quantifier relies on the fact that any substructural logic above **FL** proves the sequent $(\alpha\backslash\beta) \cdot (\beta\backslash\gamma) \Rightarrow \alpha\backslash\gamma$, which is not provable in the nonassociative Lambek calculus (cf. Remark 2.1). The rest of the proof works also in the nonassociative case.

The latter lemma shows that given a closed QBF A, we have A is true iff $\vdash_{\mathbf{L}} A'$ since e' is the empty sequence in this case. We have thus established the PSPACE-hardness of substructural logics with the DP.

In addition, observe that the DP is used only to show that the coding of existential quantifier works. We can therefore translate any universally quantified Boolean formula A into an **FL**-formula A' such that A is true iff $\vdash_{\mathbf{L}} A'$ without assuming the DP. By noting that deciding universally quantified Boolean formulas is coNP-hard, we obtain the following theorem:

Theorem 4.5 *Let* **L** *be a consistent substructural logic. The decision problem for* **L** *is coNP-hard. If* **L** *further satisfies the DP, then it is PSPACE-hard.*

Corollary 4.6 *Let* **L** *be a consistent extension of* **FL** *by ℓ-monoidal inference rules and/or $\mathcal{M}_2$ axioms. Then the decision problem for* **L** *is PSPACE-hard.*

The same is true also for every consistent extension of **InFL** *or* $\mathbf{InFL_e}$ *by inference rules in the language* $\{\wedge, \vee, 1\}$.

In particular, extensions of **FL** by axioms in Figure 4 are all PSPACE-hard.

While the DP is a sufficient condition for PSPACE-hardness, it is not a necessary one. A counterexample is the logic **LQ** obtained by extending intuitionistic logic with the law of weak excluded middle $\neg\alpha \vee \neg\neg\alpha$. **LQ** does not satisfy the DP but still is PSPACE-complete (see e.g. [4]).

5. Membership in PSPACE

In this section, we briefly discuss the problem of membership in PSPACE. In contrast to PSPACE-hardness, there does not seem to be an established algebraic method for proving membership in PSPACE that works for substructural logics. So let us argue in proof theory.

It is obvious that **FL** is in PSPACE. To show this, it is sufficient to observe:

1. The sequent calculus enjoys cut elimination.
2. For every inference rule other than (cut), each of the premises contains strictly less symbols than the conclusion.

Hence given a sequent $\Gamma \Rightarrow \varphi$, the cut-free bottom-up proof search yields a proof search tree of height bounded by the size of $\Gamma \Rightarrow \varphi$. Therefore by an obvious alternating algorithm one can decide whether $\Gamma \Rightarrow \varphi$ is provable in APTIME = PSPACE.

The same argument works for $\mathbf{FL}_S$ and $\mathbf{InFL}_S$ for every $S \subseteq \{e, i, o\}$. More generally, let $\mathbf{L}$ be a rule extension of $\mathbf{FL}$ by finitely many rules. To prove that $\mathbf{L}$ is in PSPACE, it is sufficient to show that $\mathbf{L}$ satisfies the properties 1 and 2 above.

As to property 1, the paper [6] extensively studies under which condition adding a structural rule to $\mathbf{FL}$ preserves cut elimination. So let us recall the relevant part of [6] (see also [5]).

For the current purpose, a *structural rule* is an inference rule of the form

$$\frac{\Upsilon_1 \Rightarrow \Xi_1 \quad \cdots \quad \Upsilon_n \Rightarrow \Xi_n}{\Upsilon_0 \Rightarrow \Xi_0}$$

where each Υ_i is a sequence of symbols from $\{\Gamma, \Delta, \Sigma, \dots\}$, and each Ξ_i is either empty or consists of a symbol from $\{\varphi, \varphi' \dots\}$. Here we stress that $\Gamma, \Delta, \dots$ and $\varphi, \varphi', \dots$ are considered to be *formal symbols* in this context, not notations standing for concrete sequences of formulas. Each $\Upsilon_i \Rightarrow \Xi_i$ $(1 \leq i \leq n)$ is called a *premise*, and $\Upsilon_0 \Rightarrow \Xi_0$ the *conclusion* of the structural rule. We denote by $Symb(\Upsilon_i)$ the set of symbols occurring in Υ_i.

Examples of structural rules are the *mingle* rule (m), the *weak contraction* rule (wc) and the *knotted* rules (knot^n_m):

$$\frac{\Sigma_1, \Gamma, \Sigma_2 \Rightarrow \varphi \quad \Sigma_1, \Delta, \Sigma_2 \Rightarrow \varphi}{\Sigma_1, \Gamma, \Delta, \Sigma_2 \Rightarrow \varphi}\ (\mathrm{m}) \qquad \frac{\Gamma, \Gamma \Rightarrow}{\Gamma \Rightarrow}\ (\mathrm{wc})$$

$$\frac{\{\Sigma_1, \Gamma_{i_1}, \dots, \Gamma_{i_m}, \Sigma_2 \Rightarrow \varphi\}_{i_1, \dots, i_m \in \{1, \dots, n\}}}{\Sigma_1, \Gamma_1, \dots, \Gamma_n, \Sigma_2 \Rightarrow \varphi}\ (\mathrm{knot}^n_m)$$

Note that $(\mathrm{knot}^1_2) = (\mathrm{c})$, $(\mathrm{knot}^1_0) = (\mathrm{i})$ and $(\mathrm{knot}^2_1) = (\mathrm{m})$. Just as (e), (c), (i) and (o) are expressed by axiomatic schemata, most of structural rules can be expressed by axiomatic schemata of special form. For instance, (m) is equivalent to an axiomatic schema $\alpha \cdot \alpha \to \alpha$ in $\mathbf{FL}$, (wc) is to $\neg(\alpha \wedge \neg\alpha)$, and (knot^n_m) is to $\alpha^n \to \alpha^m$. These axiomatic schemata belong to the class $\mathcal{N}_2$ in the substructural hierarchy of [5, 6]. It is shown that every $\mathcal{N}_2$-axiom is equivalent to a structural rule, though the converse does not hold.

Now consider a structural rule in one of the following forms, where $0 \leq m \leq n$ and symbols Σ_1 and Σ_2 are distinct:

$$\frac{\Sigma_1, \Upsilon_1, \Sigma_2 \Rightarrow \varphi \quad \cdots \quad \Sigma_1, \Upsilon_m, \Sigma_2 \Rightarrow \varphi \quad \Upsilon_{m+1} \Rightarrow \quad \cdots \quad \Upsilon_n \Rightarrow}{\Sigma_1, \Upsilon_0, \Sigma_2 \Rightarrow \varphi} \qquad \frac{\Upsilon_1 \Rightarrow \quad \cdots \quad \Upsilon_n \Rightarrow}{\Upsilon_0 \Rightarrow}$$

Such a rule is said to be *analytic* if the following conditions are further satisfied:

Linearity Each $\Gamma \in Symb(\Upsilon_0)$ occurs exactly once in Υ_0 and is different from Σ_1, Σ_2.

Inclusion $Symb(\Upsilon_1) \cup \cdots \cup Symb(\Upsilon_n) \subseteq Symb(\Upsilon_0)$.

Observe that (e), (c), (i), (o), (m) and (knot^n_m) are analytic rules of the first type, while (wc) is of the second type.

We have the following general result.

Theorem 5.1 ([6]) *Let* $\mathbf{L}$ *be an extension of* $\mathbf{FL}$ *by analytic structural rules. Then* $\mathbf{L}$ *enjoys cut elimination, i.e., if a sequent* $\Gamma \Rightarrow \varphi$ *is provable in* $\mathbf{L}$*, then* $\Gamma \Rightarrow \varphi$ *can be proved in* $\mathbf{L}$ *without (cut).*

We now move on to the property 2 above. A structural rule

$$\frac{\Upsilon_1 \Rightarrow \Xi_1 \quad \cdots \quad \Upsilon_n \Rightarrow \Xi_n}{\Upsilon_0 \Rightarrow \Xi_0}\ (\mathrm{r})$$

is said to be *shrinking* if the following condition is satisfied:

- Let $S = \{\Gamma_1, \ldots, \Gamma_n, \varphi_1, \ldots, \varphi_m\}$ be an arbitrary set of symbols. Remove from (r) all the occurrences of symbols in S. Then *either* a premise identical with the conclusion arises, *or* each of the premises contains strictly fewer occurrences of symbols than the conclusion. This holds for any choice of S.

For instance, (e), (c), (wc) and (knot^n_m) with $m \geq n$ are not shrinking, since if we take $S = \emptyset$, the number of symbols in each of the premises is no less than the number of symbols in the conclusion. For another example, the structural rule

$$\frac{\Delta_1, \Gamma \Rightarrow \quad \Delta_2, \Gamma \Rightarrow}{\Gamma, \Delta_1, \Delta_2 \Rightarrow}$$

is not shrinking either, since by taking $S = \{\Delta_2\}$, it becomes

$$\frac{\Delta_1, \Gamma \Rightarrow \quad \Gamma \Rightarrow}{\Gamma, \Delta_1 \Rightarrow}$$

and the left premise violates the condition. On the other hand, (i), (o), (m) and (knot^n_m) with $m < n$ are shrinking.

Now, let (r) be a shrinking structural rule. When (r) is used in bottom-up proof search, each symbol Γ is instantiated with a concrete (possibly empty) sequence of formulas. If (after instantiation) a premise identical with the conclusion arises, then (r) is redundant; it does not reduce the task of proving the conclusion at all. Otherwise, each of the premises has strictly less symbols than the conclusion. Hence adding (r) to $\mathbf{FL}$ or $\mathbf{FL}_\mathrm{e}$ preserves the property 2 above.

Finally, notice that structural rules are ℓ-monoidal, so adding them to $\mathbf{FL}$ or $\mathbf{FL}_\mathrm{e}$ preserves the DP. Altogether, we obtain the following result.

Theorem 5.2 *Let* $\mathbf{L}$ *be an extension of* $\mathbf{FL}$ *or* $\mathbf{FL}_\mathrm{e}$ *with a finite set of analytic, shrinking structural rules. Then the decision problem for* $\mathbf{L}$ *is PSPACE-complete.*

For example, any extension of **FL** or $\mathbf{FL}_e$ by rules (i), (o), (m) and (knot^n_m) with $m < n$ is PSPACE-complete.

Of course the condition is far from a necessary one. An immediate counterexample is intuitionistic logic, which involves the contraction rule (c) that is not shrinking, but is PSPACE-complete [25].

The paper [7] studies cut elimination for rule extensions of $\mathbf{InFL}_e$ in one sided (hyper)sequent calculus. With analytic rules defined as in [7] (where an analytic rule is instead called a completed rule), we have essentially the same theorem for extensions of $\mathbf{InFL}_e$ with analytic, shrinking rules. We strongly believe that we will be able to prove the same for extensions of **InFL**, once effects of adding structural rules to **InFL** have been studied along the line of [5, 6, 7].

6. Conclusion

We have shown that a wide class of substructural logics above **FL** satisfies the disjunction property, and thus the decision problems for them are PSPACE-hard. Our methodology is mainly algebraic, in contrast to the existing works that are largely proof theoretic. We hope that our algebraic method will bring new insight into the complexity issue of substructural logics.

We have also shown that some of the PSPACE-hard logics are indeed PSPACE-complete. While the current argument is a standard proof theoretic one, it would be interesting to find an algebraic method that works for membership in PSPACE.

Concerning future research directions, recall that the DP is not a necessary condition for PSPACE-hardness, a counterexample being **LQ**, intuitionistic logic with weak excluded middle. Hence it is natural to look for a weaker form of the DP which is sufficient for PSPACE-hardness and captures a wider class of substructural logics, including **LQ**.

Refining our result in this direction is of particular interest because of the apparent dichotomy phenomenon. By the result of this paper, we now know that a great number of substructural logics are PSPACE-hard. We also know that many others are coNP-complete (recall that all consistent substructural logics are at least coNP-hard). This class includes classical logic and most of major many-valued logics such as (finite- or infinite-valued) Gödel logics, Łukasiewicz logics, product logic and Hájek's basic logic [1]; see [4] for some coNP-complete superintuitionistic logics. On the other hand, we do not know any substructural logic that is neither coNP-complete nor PSPACE-hard.[1] Hence a natural question arises:

Dichotomy problem: Is there a substructural logic which is neither coNP-complete nor PSPACE-hard?

This is a fundamentally important problem, which is reminiscent of the famous dichotomy

[1] Here we exclude fragments of substructural logics without $\cdot, \backslash$ or $\vee$; for instance we know that the multiplicative fragment of linear logic is NP-complete [16].

conjecture in constraint satisfaction problems [9]. For this problem, even a partial solution would be very interesting.

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