From Axioms to Rules

— A Coalition of Fuzzy, Linear and Substructural Logics

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Parties in Nonclassical Logics

Modal Logics

Default Logic

Intermediate Logics

(Padova) Basic Logic

Paraconsistent Logic

Linear Logic

Fuzzy Logics

Substructural Logics

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Our aim: Fruitful coalition of the 3 parties

Basic Requirements

- Substractural Logics: Algebraization

$$\Gamma \vdash_{\mathbf{L}} A \iff \Gamma \models_{\mathcal{V}(\mathbf{L})} A$$

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- Linear Logic: Cut Elimination

  A logic without cut elimination is like a car without engine (J.-Y. Girard)

Outcome

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- Give a uniform, semantic proof of cut-elimination via DM completion (Substructural Logics)
- To sum up: Every system of substructural and fuzzy logics defined by $\mathcal{P}_3$ axioms (acyclic $\mathcal{P}'_3$, in the absense of Weakening) admits a cut-admissible hypersequent calculus.

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  - given a sequent calculus $\mathbf{L}$ as input
  - generates a paper (with introduction and reference) that proves the cut elimination theorem for $\mathbf{L}$.
- The generated paper was submitted, and accepted.

Outline

1. Preliminary: Commutative Full Lambek Calculus $\mathbf{FLe}$ and Commutative Residuated Lattices
2. Background: Key concepts in Substructural, Fuzzy and Linear Logics
3. Substructural Hierarchy
4. From Axioms to Rules, Uniform Semantic Cut Elimination
5. Conclusion

Syntax of FLe $=$ IMALL

- Formulas: $A \otimes B$, $A \multimap B$, $A \,\&\, B$, $A \oplus B$, $\mathbf{1}$, $\bot$, $\top$, $\mathbf{0}$.
- Sequents: $\Gamma \Rightarrow \Pi$
  (Γ: multiset of formulas, Π: stoup with at most one formula)
- Inference rules:

$$\frac{\Gamma \Rightarrow A \quad A, \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi}\ Cut \qquad \frac{}{A \Rightarrow A}\ Identity$$

$$\frac{A, B, \Gamma \Rightarrow \Pi}{A \otimes B, \Gamma \Rightarrow \Pi}\ \otimes l \qquad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \otimes B}\ \otimes r$$

$$\frac{\Gamma \Rightarrow A \quad B, \Delta \Rightarrow \Pi}{\Gamma, A \multimap B, \Delta \Rightarrow \Pi}\ \multimap l \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \multimap B}\ \multimap r$$

Syntax of FLe = IMALL

$$\frac{A, \Gamma \Rightarrow \Pi \quad B, \Gamma \Rightarrow \Pi}{A \oplus B, \Gamma \Rightarrow \Pi}\ \oplus l \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \oplus A_2}\ \oplus r \qquad \frac{}{\mathbf{0}, \Gamma \Rightarrow \Pi}\ \mathbf{0}l$$

$$\frac{A_i, \Gamma \Rightarrow \Pi}{A_1 \,\&\, A_2, \Gamma \Rightarrow \Pi}\ \& l \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \,\&\, B}\ \& r \qquad \frac{}{\Gamma \Rightarrow \top}\ \top r$$

$$\frac{\Gamma \Rightarrow \Pi}{\mathbf{1}, \Gamma \Rightarrow \Pi}\ \mathbf{1}l \qquad \frac{}{\Rightarrow \mathbf{1}}\ \mathbf{1}r \qquad \frac{}{\bot \Rightarrow}\ \bot l \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \bot}\ \bot r$$

Notational Correspondence

	$M - Conj$		$M - Imp$		$A - Conj$		$A - Disj$	
$Linear Logic$	$\otimes$	$\mathbf{1}$	$\multimap$	$\bot$	$\&$	$\top$	$\oplus$	$\mathbf{0}$
$Fuzzy Logics$	$\odot$	e	$\rightarrow$	f	$\wedge$	$\top$	$\vee$	$\bot$

Commutative Residuated Lattices

- A (bounded pointed) commutative residuated lattice is

$$\mathbf{P} = \langle P, \&, \oplus, \otimes, \multimap, \top, \mathbf{0}, \mathbf{1}, \bot \rangle$$

  1. $\langle P, \&, \oplus, \top, \mathbf{0} \rangle$ is a lattice with $\top$ greatest and $\mathbf{0}$ least
  2. $\langle P, \otimes, \mathbf{1} \rangle$ is a commutative monoid.
  3. For any $x, y, z \in P$, $x \otimes y \leq z \iff y \leq x \multimap z$
  4. $\bot \in P$.

- $\mathbf{P} \models A$ if $\mathbf{1} \leq f(A)$ for any valuation f.
- $\mathcal{CRL}$: the variety of commutative residuated lattices.

Algebraization

- A (commutative) substructural logic $\mathbf{L}$ is an extension of $\mathbf{FLe}$ with axioms $\Phi_{\mathbf{L}}$.
- $\mathcal{V}(\mathbf{L})$: the subvariety of $\mathcal{CRL}$ corresponding to $\mathbf{L}$

$$\mathcal{V}(\mathbf{L}) = \{\mathbf{P} \in \mathcal{CRL} : \mathbf{P} \models A \text{ for any } A \in \Phi_{\mathbf{L}}\}$$

- Theorem: For every substructural logic $\mathbf{L}$,

$$\Gamma \vdash_{\mathbf{L}} A \iff \Gamma \models_{\mathcal{V}(\mathbf{L})} A$$

Algebraization

- Is it trivial? Yes, but the consequences are not.
- Syntax: existential

$$\vdash_{\mathbf{L}} A \iff \exists \pi.(\pi \text{ is a proof of } A)$$

- Semantics: universal

$$\models_{\mathcal{V}(\mathbf{L})} A \iff \forall \mathbf{P} \in \mathcal{V}(\mathbf{L}).(\mathbf{P} \models A)$$

- Consequence: Semantics mirrors Syntax

Syntax	Semantics
Argument	Argument
Property	Property

Interporation and Amalgamation

- $Form(X)$: formulas over variables $\alpha, \beta, \dots \in X$.
- $\mathbf{L}$ admits Interpolation: Suppose $A \in Form(X)$ and $B \in Form(Y)$. If $\vdash_{\mathbf{L}} A \multimap B$, then there is $I \in Form(X \cap Y)$ such that

$$\vdash_{\mathbf{L}} A \multimap I \qquad \vdash_{\mathbf{L}} I \multimap B$$

- A class $\mathcal{V}$ of algebras admits Amalgamation if for any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{V}$ with embeddings f_1, f_2, there are $\mathbf{D} \in \mathcal{V}$ and embeddings g_1, g_2 such that

$$\begin{tikzcd}
 & \mathbf{B} \arrow[dr, dotted, "g_1"] & \\
\mathbf{A} \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & \mathbf{D} \\
 & \mathbf{C} \arrow[ur, dotted, "g_2"'] &
\end{tikzcd}$$

Interporation and Amalgamation

- Theorem (Maximova): For any intermediate logic $\mathbf{L}$,

$$\mathbf{L} \text{ admits interpolation} \iff \mathcal{V}(\mathbf{L}) \text{ admits amalgamation}$$

- Extended to substructural logics by Wroński, Kowalski, Galatos-Ono, etc.
- Syntax is to split, semantics is to join.

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Linearization

- Logically, this amounts to adding the axiom of linearity:

$$(lin) \quad (A \multimap B) \oplus (B \multimap A)$$

- Example: Gödel logic = $\mathbf{IL} + (lin)$
- Complete w.r.t. the valuations $f : Form \longrightarrow [0,1]$ s.t.

$$\begin{aligned}
f(\top) &= 1 \\
f(\mathbf{0}) &= 0 \\
f(A \,\&\, B) &= min(f(A), f(B)) \\
f(A \oplus B) &= max(f(A), f(B)) \\
f(A \multimap B) &= \begin{cases} f(B) & \text{if } f(A) > f(B) \\ 1 & \text{otherwise} \end{cases}
\end{aligned}$$

Linearization

- Other Fuzzy Logics:

$$
\begin{aligned}
\text{Uninorm Logic} &= \mathbf{FLe} + (prelin) \\
\text{Monoidal T-norm Logic} &= \mathbf{FLew} + (lin) \\
\text{Basic Logic} &= \mathbf{FLew} + (lin) + (div) \\
\text{Łukasiewicz Logic} &= \mathbf{FLew} + (lin) + (div) + (A^{\perp\perp} \multimap A
\end{aligned}
$$

$$
\begin{array}{ll}
(prelin) & (A \multimap B)_{\&\mathbf{1}} \oplus (B \multimap A)_{\&\mathbf{1}} \\
(div) & (A \,\&\, B) \multimap (A \otimes (A \multimap B))
\end{array}
$$

- If axioms are added, cut elimination is lost. We need to find corresponding rules.

Hypersequent Calculus

- Hypersequent calculus (Avron 87)
- Hypersequent: $\Gamma_1 \Rightarrow \Pi_1 \mid \cdots \mid \Gamma_n \Rightarrow \Pi_n$
- Intuition: $(\otimes\Gamma_1 \multimap \Pi_1)_{\&\mathbf{1}} \oplus \cdots \oplus (\otimes\Gamma_n \multimap \Pi_n)_{\&\mathbf{1}}$
- $\mathbf{HFLe}$ consists of

Rules of $\mathbf{FLe}$ | Ext-Weakening | Ext-Contraction

$$\frac{G \mid A, \Gamma \Rightarrow B}{G \mid \Gamma \Rightarrow A \multimap B} \qquad \frac{G}{G \mid \Gamma \Rightarrow \Pi} \qquad \frac{G \mid \Gamma \Rightarrow \Pi \mid \Gamma \Rightarrow \Pi}{G \mid \Gamma \Rightarrow \Pi}$$

- Communication Rule:

$$\frac{G \mid \Gamma_1, \Delta_1 \Rightarrow \Pi_1 \quad G \mid \Gamma_2, \Delta_2 \Rightarrow \Pi_2}{G \mid \Gamma_2, \Delta_1 \Rightarrow \Pi_1 \mid \Gamma_1, \Delta_2 \Rightarrow \Pi_2}\ (com)$$

Hypersequent Calculus

- $\mathbf{HFLe} + (com)$ proves (lin).

$$\frac{H \mid \Gamma_1, \Delta_1 \Rightarrow \Pi_1 \quad H \mid \Gamma_2, \Delta_2 \Rightarrow \Pi_2}{H \mid \Gamma_2, \Delta_1 \Rightarrow \Pi_1 \mid \Gamma_1, \Delta_2 \Rightarrow \Pi_2} \ (com)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \quad B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow A}\ (com)}{\Rightarrow A \multimap B \mid \ \Rightarrow B \multimap A}\ (\multimap r)}{\Rightarrow (A \multimap B) \oplus (B \multimap A) \mid \ \Rightarrow (A \multimap B) \oplus (B \multimap A)}\ (\oplus r)}{\Rightarrow (A \multimap B) \oplus (B \multimap A)}\ (EC)$$

- $\mathbf{HIL} + (com)$ = Gödel Logic. Enjoys cut elimination (Avron 92).
- Similarly for Monoidal T-norm and Uninorm Logics (cf. Metcalfe-Montagna 07)
- Semantics is to narrow, Syntax is to widen.

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Conservativity

- Infinitary extension $\mathbf{L}^\infty$ of $\mathbf{L}$: $\&_{i\in I} A_i$

$$\frac{\Gamma \Rightarrow A_i \quad \text{for any } i \in I}{\Gamma \Rightarrow \&_{i\in I} A_i} \qquad \frac{A_i, \Gamma \Rightarrow \Pi \quad \text{for some } i \in I}{\&_{i\in I} A_i, \Gamma \Rightarrow \Pi}$$

- $\mathbf{L}^\infty$ is a conservative extension of $\mathbf{L}$ if

$$\Gamma \vdash_{\mathbf{L}^\infty} A \Longrightarrow \Gamma \vdash_{\mathbf{L}} A$$

  for any set $\Gamma \cup \{A\}$ of finite formulas.

- Theorem: For any substructural logic $\mathbf{L}$, $\mathbf{L}^\infty$ is a conservative extension of $\mathbf{L}$ iff any $\mathbf{P} \in \mathcal{V}(\mathbf{L})$ can be embedded into a complete algebra $\mathbf{P}^\infty \in \mathcal{V}(\mathbf{L})$.

- Syntax is to eliminate, Semantics is to enrich.

Dedekind Completion of Rationals

- For any $X \subseteq \mathbb{Q}$,

$$\begin{aligned} X^{\triangleright} &= \{y \in \mathbb{Q} : \forall x \in X . x \leq y\} \\ X^{\triangleleft} &= \{y \in \mathbb{Q} : \forall x \in X . y \leq x\} \end{aligned}$$

- X is closed if $X = X^{\triangleright\triangleleft}$
- $(\mathbb{Q}, +, \cdot)$ can be embedded into $(\mathcal{C}(\mathbb{Q}), +, \cdot)$ with

$$\mathcal{C}(\mathbb{Q}) = \{X \subseteq \mathbb{Q} : X \text{ is closed}\}$$

- Dedekind completion extends to various ordered algebras (MacNeille).

Dedekind-MacNeille Completion

- Theorem: Every $\mathbf{P} \in \mathcal{CRL}$ can be embedded into a complete $\mathbf{P}^\infty \in \mathcal{CRL}$, where $\mathbf{P}^\infty = (\mathcal{C}(\mathbf{P}), \&, \oplus, \otimes, \multimap, \top, \mathbf{0}, \mathbf{1}, \bot)$

$$\begin{array}{rcl} X \,\&\, Y & = & X \cap Y \\ X \oplus Y & = & (X \cup Y)^{\rhd\lhd} \\ X \otimes Y & = & \{xy : x \in X, y \in Y\}^{\rhd\lhd} \\ X \multimap Y & = & \{y : \forall x \in X xy \in Y\} \end{array}$$

$$\begin{array}{rclrcl} \top & = & P & \mathbf{0} & = & \emptyset^{\rhd\lhd} \\ \mathbf{1} & = & \{1\}^{\rhd\lhd} & \bot & = & \{\bot\}^{\rhd\lhd} \end{array}$$

(Ono 93; cf. Abrusci 90, Sambin 93)

- Some axioms (eg. distributivity) are not preserved by DM completion (cf. Kowalski-Litak 07).

Cut Elimination via Completion

- Syntactic argument:

$$\text{Cut-ful Proofs} \quad \overset{\text{elimination procedure}}{\Longrightarrow} \quad \text{Cut-free Proofs}$$

- Semantic argument:

$$\text{CRL} \quad \overset{\text{Quasi-DM completion}}{\Longleftarrow} \quad \text{‘Intransitive’ CRL}$$

Cut Elimination via Completion

- Due to (Okada 96). Algebraically reformulated by (Belardinelli-Ono-Jipsen 01).
- $MUL =$ the set of multisets $\Gamma, \Delta, \ldots$ of formulas
- $SEQ =$ the set of sequents $\Sigma \Rightarrow \Pi$
- For $\Gamma \in MUL$ and $\Sigma \Rightarrow \Pi \in SEQ$,

$$\Gamma \sqsubset \Sigma \Rightarrow \Pi \text{ iff } \Gamma, \Sigma \Rightarrow \Pi \text{ is cut-free provable in } \mathbf{FLe}$$

- For $X \subseteq MLT$ and $Y \subseteq SEQ$,

$$\begin{aligned} X^{\triangleright} &= \{(\Sigma \Rightarrow \Pi) \in SEQ : \forall \Gamma \in X.\, \Gamma \sqsubset (\Sigma \Rightarrow \Pi)\} \\ Y^{\triangleleft} &= \{\Gamma \in MUL : \forall (\Sigma \Rightarrow \Pi) \in Y.\, \Gamma \sqsubset (\Sigma \Rightarrow \Pi)\} \end{aligned}$$

Cut Elimination via Completion

- $\rhd, \lhd$ induce a complete CRL $\mathbf{P}_{cf}$, and

$$F(A) = \{X : A \in X \subseteq A^{\rhd\lhd},\ X \text{ closed}\}$$

is a quasi-homomorphism $Form \longrightarrow \mathcal{P}(\mathbf{P}_{cf})$:

$$\begin{array}{rcll} * & \in & F(*) & \text{for } * \in \{\mathbf{1}, \bot, \top, \mathbf{0}\} \\ F(x) \star F(y) & \subseteq & F(x \star y) & \text{for } \star \in \{\otimes, \multimap, \&, \oplus\} \end{array}$$

- If the valuation $f(\alpha) = \alpha^{\lhd}$ validates A, then A is cut-free provable in $\mathbf{FLe}$.
- Again, not all axioms are preserved by quasi-DM completion.
- Which axioms are preserved by quasi-DM completion?

Kouan 2: What is Completeness?

- Completeness: to establish a correspondence between syntax and semantics.
- Gödel Completeness: $\Gamma \vdash_{\mathbf{L}} A \Longleftrightarrow \Gamma \models_{\mathcal{V}(\mathbf{L})} A$ is trivial in the algebraic setting.
- Meta-Completeness: to describe the Syntax-Semantics Mirror correspondence as precisely as possible.

Syntax	Semantics
Interpolation	Amalgamation
Hypersequentialization	Linearization
Conservativity	Completion
Cut-elimination	Quasi-DM completion

Kouan 3: Is Semantic Cut-Elimination Weak?

- Proof of Cut-Elimination

$$\frac{\vdash_{\mathbf{FLe}} A \stackrel{\text{sound}}{\Longrightarrow} \mathbf{P}_{cf} \models A \qquad \mathbf{P}_{cf} \models A \stackrel{\text{complete}}{\Longrightarrow} \vdash^{cf}_{\mathbf{FLe}} A}{\vdash_{\mathbf{FLe}} A \Longrightarrow \vdash^{cf}_{\mathbf{FLe}} A}$$

- We have just replaced object cuts with a big META-CUT.
- Another criticism: it does not give a cut-elimination procedure.
- Conjecture: If we eliminate the meta-cut, a concrete (object) cut-elimination procedure emerges.

When Cut-Elimination Holds?

- Cut-Elimination holds when $\mathbf{FLe}$ is extended with a natural structural rule:
  - Contraction: $A \multimap A \otimes A$ $$\frac{A, A, \Gamma \Rightarrow \Pi}{A, \Gamma \Rightarrow \Pi}$$
- It fails when extended with an unnatural one:
  - Broccoli: $A \otimes A \multimap A$ $$\frac{A, \Gamma \Rightarrow \Pi}{A, A, \Gamma \Rightarrow \Pi}$$
- What is 'natural'?

Girard's test

- (Girard 99) proposes a test for naturality of structural rules.
  - A structural rule passes Girard's test if, in every 'phase structure' $(M, \triangleright, \triangleleft)$, it propagates from atomic closed sets $\{x\}^{\triangleright\triangleleft}$ to all closed sets $X^{\triangleright\triangleleft}$.
  - "If the rule holds for rationals, it also holds for all reals."
- Contraction passes it:

$$\forall x \in M.\ \{x\}^{\triangleright\triangleleft} \subseteq \{x \cdot x\}^{\triangleright\triangleleft} \implies \forall X: \text{ closed } (X \subseteq X \otimes X).$$

- Broccoli fails it:

$$\forall x \in M.\ \{x \cdot x\}^{\triangleright\triangleleft} \subseteq \{x\}^{\triangleright\triangleleft} \not\Longrightarrow \forall X: \text{ closed } (X \otimes X \subseteq X).$$

Irony

- Broccoli is equivalent to Mingle: $A \otimes B \multimap A \oplus B$

$$\frac{\Gamma, \Sigma_1, \Delta \Rightarrow \Pi \quad \Gamma, \Sigma_2, \Delta \Rightarrow \Pi}{\Gamma, \Sigma_1, \Sigma_2, \Delta \Rightarrow \Pi}$$

- Mingle passes Girard's test.

Broccoli and Mingle

- Broccoli does not admit cut elimination:

$$
\dfrac{\dfrac{\overline{\beta \Rightarrow \beta}}{\beta \Rightarrow \alpha \oplus \beta} \qquad \dfrac{\dfrac{\overline{\alpha \Rightarrow \alpha}}{\alpha \Rightarrow \alpha \oplus \beta} \quad \dfrac{\overline{\alpha \oplus \beta \Rightarrow \alpha \oplus \beta}}{\alpha \oplus \beta, \alpha \oplus \beta \Rightarrow \alpha \oplus \beta}\, Exp}{\alpha, \alpha \oplus \beta \Rightarrow \alpha \oplus \beta}\, Cut}{\alpha, \beta \Rightarrow \alpha \oplus \beta}\, Cut
$$

- When Broccoli is replaced with Mingle, the above cut can be eliminated:

$$
\dfrac{\dfrac{\overline{\alpha \Rightarrow \alpha}}{\alpha \Rightarrow \alpha \oplus \beta} \quad \dfrac{\overline{\beta \Rightarrow \beta}}{\beta \Rightarrow \alpha \oplus \beta}}{\alpha, \beta \Rightarrow \alpha \oplus \beta}\, Min
$$

Characterization of Cut-Elimination

- Additive structural rules:

$$\frac{\Gamma, \vec{X}_1, \Delta \Rightarrow C \quad \cdots \quad \Gamma, \vec{X}_n, \Delta \Rightarrow C}{\Gamma, \vec{X}_0, \Delta \Rightarrow C}\ R$$

  such that $\{\vec{X}_1, \ldots, \vec{X}_n\} \subseteq \{\vec{X}_0\}$.

- Theorem (Terui 07): For any additive structural rule (r), $\mathbf{FLe} + (r)$ admits cut-elimination iff (r) passes Girard's test.
- Key fact: (r) passes Girard's test $\Longrightarrow$ (r) is preserved by (quasi-)DM completion.
- (Ciabattoni-Terui 06) considerably extends this result.

Questions

- Which axiom can be transformed into structural rules? Usually checked one by one. Is there a more systematic way?
- Which structural rules can be transformed into good ones admitting cut-elimination? (eg. Broccoli $\Longrightarrow$ Mingle)
- How does the situation change if we adopt hypersequent calculus? (eg. Linearity $\Longrightarrow$ Communication)
- Does hypersequent calculus admit semantic cut-elimination? All known proofs are syntactic, tailored to each specific logic (exception: Metcalfe-Montagna 07), and quite complicated. Semantic one would lead to a uniform, conceptually simpler proof.
- $\Longrightarrow$ (Ciabattoni-Galatos-Terui 08)

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Polarity

- Key notion in Linear Logic since its inception (Girard 87)
- Plays a central role in Efficient Proof Search (Andreoli 90), Constructive Classical Logic (Girard 91), Polarized Linear Logic (Laurent), Game Semantics, Ludics.

Polarity

- Positive connectives $\mathbf{1}, \mathbf{0}, \otimes, \oplus$ have invertible left rules:

$$\frac{A \otimes B, \Gamma \Rightarrow \Pi}{\overline{A, B, \Gamma \Rightarrow \Pi}}$$

and propagate closure operator:

$$X^{\triangleright\triangleleft} \bullet Y^{\triangleright\triangleleft} \subseteq (X \bullet Y)^{\triangleright\triangleleft} = X \otimes Y$$

- Negative connectives $\top, \bot, \&, \multimap$ have invertible right rules:

$$\frac{\Gamma \Rightarrow A_1 \,\&\, A_2}{\overline{\Gamma \Rightarrow A_i}}$$

and distribute closure operator:

$$(X \cap Y)^{\triangleright\triangleleft} \subseteq X^{\triangleright\triangleleft} \cap Y^{\triangleright\triangleleft}$$

Polarity

- Connectives of the same polarity associate well.
- Positives:

$$A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$$

$$A \otimes \mathbf{1} = A \qquad A \otimes \mathbf{0} = \mathbf{0} \qquad A \oplus \mathbf{0} = A$$

- Negatives:

$$\begin{aligned} A \multimap (B \,\&\, C) &= (A \multimap B) \,\&\, (A \multimap C) \\ (A \oplus B) \multimap C &= (A \multimap C) \,\&\, (B \multimap C) \end{aligned}$$

$$A \,\&\, \top = A \qquad A \multimap \top = \top \qquad \mathbf{1} \multimap A = A$$

(polarity reverses on the LHS of an implication)

Substructural Hierarchy

$$\begin{tikzcd}
{} & & {} \\
\mathcal{P}_3 \arrow[u, dotted] & & \mathcal{N}_3 \arrow[u, dotted] \\
\mathcal{P}_2 \arrow[u] \arrow[urr] & & \mathcal{N}_2 \arrow[u] \arrow[ull] \\
\mathcal{P}_1 \arrow[u] \arrow[urr] & & \mathcal{N}_1 \arrow[u] \arrow[ull] \\
\mathcal{P}_0 \arrow[u] \arrow[urr] \arrow[rr, dash] & & \mathcal{N}_0 \arrow[u] \arrow[ull]
\end{tikzcd}$$

- The sets $\mathcal{P}_n, \mathcal{N}_n$ of formulas defined by:

(0) $\mathcal{P}_0 = \mathcal{N}_0 =$ the set of atomic formulas

(P1) $\mathcal{N}_n \subseteq \mathcal{P}_{n+1}$

(P2) $A, B \in \mathcal{P}_{n+1} \implies A \oplus B, A \otimes B, \mathbf{1}, \mathbf{0} \in \mathcal{P}_{n+1}$

(N1) $\mathcal{P}_n \subseteq \mathcal{N}_{n+1}$

(N2) $A, B \in \mathcal{N}_{n+1} \implies A \,\&\, B, \bot, \top \in \mathcal{N}_{n+1}$

(N3) $A \in \mathcal{P}_{n+1}, B \in \mathcal{N}_{n+1} \implies A \multimap B \in \mathcal{N}_{n+1}$

Substructural Hierarchy

- Due to lack of Weakening, $\mathcal{P}_n$ is too strong. It is also convenient to consider a subclass $\mathcal{P}'_n \subseteq \mathcal{P}_n$:
  - $A \in \mathcal{N}_n \Longrightarrow A \,\&\, \mathbf{1} \in \mathcal{P}'_{n+1}$
  - $A, B \in \mathcal{P}'_{n+1} \;\Longrightarrow\; A \oplus B, A \otimes B, \mathbf{1}, \mathbf{0} \in \mathcal{P}'_{n+1}$
- Intuition:
  - $\mathcal{P}_0$: Formulas
  - $\mathcal{P}_1$: Disj of multisets Γ
  - $\mathcal{N}_1$: Conj of sequents $\Gamma \Rightarrow \Pi$
  - $\mathcal{P}'_2$: Conj of hypersequents $S_1 \mid \cdots \mid S_n$
  - $\mathcal{N}_2$: Conj of structural rules
  - $\mathcal{P}'_3$: Conj of hyperstructural rules

Examples

Class	Axiom	Name
$\mathcal{N}_2$	$\alpha \multimap \mathbf{1}$, $\bot \multimap \alpha$	weakening
	$\alpha \multimap \alpha \otimes \alpha$	contraction
	$\alpha \otimes \alpha \multimap \alpha$	expansion
	$\otimes \alpha^n \multimap \otimes \alpha^m$	knotted axioms ($n, m \geq 0$)
	$(\alpha \mathbin{\&} \alpha^\bot)^\bot$	weak contraction
$\mathcal{P}_2$	$\alpha \oplus \alpha^\bot$	excluded middle
	$(\alpha \multimap \beta) \oplus (\beta \multimap \alpha)$	linearity
$\mathcal{P}_3'$	$((\alpha \multimap \beta) \mathbin{\&} \mathbf{1}) \oplus ((\beta \multimap \alpha) \mathbin{\&} \mathbf{1})$	prelinearity
$\mathcal{P}_3$	$\alpha^\bot \oplus \alpha^{\bot\bot}$	weak excluded middle
	$\bigoplus_{i=0}^{k}(p_i \multimap \bigoplus_{j\neq i} p_j)$	Kripke models with width $\leq k$

$A^\bot = A \multimap \bot$

Examples

- Axioms in $\mathcal{N}_3$:
  - Łukasiewicz axiom $((\alpha \multimap \beta) \multimap \beta) \multimap ((\beta \multimap \alpha) \multimap \alpha)$
  - Distributivity $(\alpha \mathbin{\&} (\beta \oplus \gamma)) \multimap ((\alpha \mathbin{\&} \beta) \oplus (\alpha \mathbin{\&} \gamma))$
  - Divisibility $(\alpha \mathbin{\&} \beta) \multimap (\alpha \otimes (\alpha \multimap \beta))$

From Axioms to Rules

- A structural rule is

$$\frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \cdots \quad \Upsilon_n \Rightarrow \Psi_n}{\Upsilon_0 \Rightarrow \Psi_0}$$

where Υ_i, Ψ_i are sets of metavariables ($|\Psi_i| \leq 1$).

- A hyperstructural rule is

$$\frac{G \mid \Upsilon_1' \Rightarrow \Psi_1' \quad \cdots \quad G \mid \Upsilon_n' \Rightarrow \Psi_n'}{G \mid \Upsilon_1 \Rightarrow \Psi_1 \mid \cdots \mid \Upsilon_m \Rightarrow \Psi_m}$$

- Theorem:
  1. Any $\mathcal{N}_2$-axiom is equivalent to (a set of) structural rules in $\mathbf{FLe}$.
  2. Any $\mathcal{P}_3'$-axiom is equivalent to hyperstructural rules in $\mathbf{HFLe}$.
  3. Any $\mathcal{P}_3$-axiom is equivalent to hyperstructural rules in $\mathbf{HFLew}$.

Example

- Weak nilpotent minimum (Esteva-Godo 01):

$$(\alpha \otimes \beta)^{\perp} \oplus (\alpha \,\&\, \beta \multimap \alpha \otimes \beta)$$

- Its $\mathcal{P}_3'$ version: $(\alpha \otimes \beta)^{\perp}_{\&\mathbf{1}} \oplus (\alpha \,\&\, \beta \multimap \alpha \otimes \beta)_{\&\mathbf{1}}$ is equivalent to

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- The procedure is automatic (in contrast to the usual practice).

- Similarity with principle of reflection (automatic derivation of inference rules for a logical connective from its defining equation; Sambin-Battilotti-Faggian 00).

Towards Cut Elimination

- Not all rules admit cut elimination. They have to be completed.
- In absence of Weakening, cyclic rules are problematic:

$$\frac{\alpha, \gamma \Rightarrow \beta \quad \beta \Rightarrow \alpha}{\alpha, \gamma \Rightarrow \beta}$$

- Theorem:
  1. Any acyclic hyperstructural rule can be transformed into an equivalent one in $\mathbf{HFLe}$ that enjoys cut elimination.
  2. Any hyperstructural rule can be transformed into an equivalent one in $\mathbf{HFLew}$ that enjoys cut elimination.

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- We have obtained:

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- $$\frac{\begin{array}{l} G \mid \gamma \Rightarrow \alpha \quad G \mid \gamma \Rightarrow \beta \quad G \mid \alpha, \beta \Rightarrow \delta \\ G \mid \Gamma \Rightarrow \alpha \end{array}}{G \mid \Gamma, \textcolor{red}{\beta} \Rightarrow \; \mid \gamma \Rightarrow \delta}$$

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- $$\frac{\begin{array}{ccc} G \mid \gamma \Rightarrow \alpha & G \mid \gamma \Rightarrow \beta & G \mid \alpha, \beta \Rightarrow \delta \\ G \mid \Gamma \Rightarrow \alpha & G \mid \Delta \Rightarrow \beta & G \mid \Lambda \Rightarrow \gamma \end{array}}{G \mid \Gamma, \Delta \Rightarrow \;\mid \Lambda \Rightarrow {\color{red}\delta}}$$

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- $$\frac{\begin{array}{c} G \mid \gamma \Rightarrow \alpha \quad G \mid \gamma \Rightarrow \beta \quad G \mid \alpha, \beta \Rightarrow \color{red}{\delta} \\ G \mid \Gamma \Rightarrow \alpha \quad G \mid \Delta \Rightarrow \beta \quad G \mid \Lambda \Rightarrow \gamma \quad G \mid \color{red}{\delta}, \Sigma \Rightarrow \Pi \end{array}}{G \mid \Gamma, \Delta \Rightarrow \ \mid \Lambda, \Sigma \Rightarrow \Pi}$$

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- $$\frac{\begin{array}{lll} G \mid \gamma \Rightarrow \alpha & G \mid \gamma \Rightarrow \beta & G \mid \alpha, \beta, \Sigma \Rightarrow \Pi \\ G \mid \Gamma \Rightarrow \alpha & G \mid \Delta \Rightarrow \beta & G \mid \Lambda \Rightarrow \gamma \end{array}}{G \mid \Gamma, \Delta \Rightarrow \ \mid \Lambda, \Sigma \Rightarrow \Pi}$$

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Uniform Cut Elimination

- The resulting rule

$$\frac{\begin{array}{cc} G \mid \Gamma, \Delta, \Sigma \Rightarrow \Pi & G \mid \Gamma, \Lambda, \Sigma \Rightarrow \Pi \\ G \mid \Lambda, \Delta, \Sigma \Rightarrow \Pi & G \mid \Lambda, \Lambda, \Sigma \Rightarrow \Pi \end{array}}{G \mid \Gamma, \Delta \Rightarrow \ \mid \Lambda, \Sigma \Rightarrow \Pi}$$

  satisfies
  - Strong subformula property: Any formula occuring on the LHS (resp. RHS) of a premise also occurs on the LHS (resp. RHS) of the conclusion
  - Conclusion-linearity: No metavariable occurs in the conclusion twice.
  - Coupling: ...

Uniform Cut Elimination

- Theorem: If a hyperstructural rule (r) satisfies the above conditions, then $\mathbf{HFLe} + (r)$ admits cut-elimination.
- Proof: The above are sufficient conditions for a rule to be preserved by quasi-DM completion.
- Semantics allows for a uniform proof.

Main Results

- Theorem:
  1. Any acyclic $\mathcal{N}_2$-axiom is equivalent in $\mathbf{FLe}$ to (a set of) structural rules enjoying cut-elimination.
  2. Any $\mathcal{N}_2$-axiom is equivalent in $\mathbf{FLew}$ to (a set of) structural rules enjoying cut-elimination.
  3. Any acyclic $\mathcal{P}_3'$-axiom is equivalent in $\mathbf{HFLe}$ to hyperstructural rules enjoying cut-elimination.
  4. Any $\mathcal{P}_3$-axiom is equivalent in $\mathbf{HFLew}$ to hyperstructural rules enjoying cut-elimination.
- Our results automatically yield:

  Esteva-Godo's logic

  $\mathbf{FLew}$ + (linearity) + (weak nilpotent minimum)

  admits a cut-admissible hypersequent calculus.

Examples

Class	Axiom	Name
$\mathcal{N}_2$	$\alpha \multimap \mathbf{1}$, $\bot \multimap \alpha$	weakening
	$\alpha \multimap \alpha \otimes \alpha$	contraction
	$\alpha \otimes \alpha \multimap \alpha$	expansion
	$\otimes \alpha^n \multimap \otimes \alpha^m$	knotted axioms ($n, m \geq 0$)
	$(\alpha \mathbin{\&} \alpha^\bot)^\bot$	weak contraction
$\mathcal{P}_2$	$\alpha \oplus \alpha^\bot$	excluded middle
	$(\alpha \multimap \beta) \oplus (\beta \multimap \alpha)$	linearity
$\mathcal{P}'_3$	$((\alpha \multimap \beta) \mathbin{\&} \mathbf{1}) \oplus ((\beta \multimap \alpha) \mathbin{\&} \mathbf{1})$	prelinearity
$\mathcal{P}_3$	$\alpha^\bot \oplus \alpha^{\bot\bot}$	weak excluded middle
	$\bigoplus_{i=0}^{k}(p_i \multimap \bigoplus_{j \neq i} p_j)$	Kripke models with width $\leq k$

$A^\bot = A \multimap \bot$

Conclusion

- Our coalition successfully combines various ideas:
  - Polarity, Girard's test from Linear Logic
  - Hypersequent calculus from Fuzzy Logic
  - DM completion from Substructural Logic

  to establish uniform cut-elimination for extensions of $\mathbf{FLe}$ with $\mathcal{P}_3$ axioms.
- Research directions:
  - Computational meaning of axioms in Fuzzy Logic. Eg. Peirce's law corresponds to call/cc in Functional Programming. What about Linearity $(\alpha \multimap \beta) \oplus (\beta \multimap \alpha)$?
  - Better understanding of Syntax-Semantics Mirror

Conclusion

$$
\begin{tikzcd}
{} & {} \\
\mathcal{P}_3 \arrow[u, dotted] & \mathcal{N}_3 \arrow[u, dotted] \\
\mathcal{P}_2 \arrow[u] \arrow[ur] & \mathcal{N}_2 \arrow[u] \arrow[ul] \\
\mathcal{P}_1 \arrow[u] \arrow[ur] & \mathcal{N}_1 \arrow[u] \arrow[ul] \\
\mathcal{P}_0 \arrow[u] \arrow[ur] \arrow[r, dash] & \mathcal{N}_0 \arrow[u] \arrow[ul]
\end{tikzcd}
$$

- Uniform treatment of axioms in $\mathcal{N}_3$: Łukasiewicz axiom, divisibility, cancellativity, distributivity, etc. (Known calculi are tailord to each specific logic; cf. Metcalfe-Olivietti-Gabbay 04)
- How high can we climb up?