How and when axioms can be transformed into good structural rules

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Introduction

- Our Setting: Full Lambek calculus [Ono 90]
  = Intuitionistic logic $-$ structural rules
  = Noncommutative intuitionistic linear logic
  (without exponentials)

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$$\alpha \multimap \mathbf{1} \qquad \alpha \multimap \alpha \otimes \alpha \qquad \alpha \otimes \beta \multimap \beta \otimes \alpha$$

correspond to structural rules

$$\frac{\Gamma, \Delta \Rightarrow \Pi}{\Gamma, A, \Delta \Rightarrow \Pi}\,(w) \quad \frac{\Gamma, A, A, \Delta \Rightarrow \Pi}{\Gamma, A, \Delta \Rightarrow \Pi}\,(c) \quad \frac{\Gamma, A, B, \Delta \Rightarrow \Pi}{\Gamma, B, A, \Delta \Rightarrow \Pi}\,(e)$$

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- What about $\delta \oplus (\delta \multimap \bot \,\&\, \gamma \multimapinv \delta) \multimap \gamma \otimes \delta$?

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- Q3: Do all structural rules admit cut-elimination?
  - No.

$$\frac{\Gamma, A, A, \Delta \Rightarrow \Pi}{\Gamma, A, \Delta \Rightarrow \Pi}\ (c) \qquad \frac{\Gamma, \Sigma, \Sigma, \Delta \Rightarrow \Pi}{\Gamma, \Sigma, \Delta \Rightarrow \Pi}\ (seq - c)$$

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- Q6: Is there anything else?

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- Q6: Is there anything else?
  - [This talk] Those equivalent to acyclic rules.

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1. Introduction
2. Substructural hierarchy and structuralization
3. Acyclic structural rules and simplification
4. Semantic cut-elimination: an introduction
5. Simple rules admit conservativity and cut-elimination
6. Conservativity implies acyclicity
7. Conclusion

Syntax of Full Lambek Calculus

- Formulas: $A \mathbin{\&} B$, $A \oplus B$, $A \otimes B$, $A \multimap B$, $B \circ\!\!- A$, $\top$, $\bot$, $\mathbf{0}$, $\mathbf{1}$.
- Sequents: $\Gamma \Rightarrow \Pi$
  (Γ: sequence of formulas, Π: stoup with at most one formula)
- Inference rules:

$$\frac{\Gamma \Rightarrow A \quad \Delta_1, A, \Delta_2 \Rightarrow \Pi}{\Delta_1, \Gamma, \Delta_2 \Rightarrow \Pi}\ Cut \qquad \frac{}{A \Rightarrow A}\ Identity$$

$$\frac{\Gamma_1, A, B, \Gamma_2 \Rightarrow \Pi}{\Gamma_1, A \otimes B, \Gamma_2 \Rightarrow \Pi}\ \otimes l \qquad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \otimes B}\ \otimes r$$

$$\frac{\Gamma \Rightarrow A \quad \Delta_1, B, \Delta_2 \Rightarrow \Pi}{\Delta_1, \Gamma, A \multimap B, \Delta_2 \Rightarrow \Pi}\ \multimap l \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \multimap B}\ \multimap r$$

$$\frac{\Gamma \Rightarrow A \quad \Delta_1, B, \Delta_2 \Rightarrow \Pi}{\Delta_1, B \circ\!\!- A, \Gamma, \Delta_2 \Rightarrow \Pi}\ \circ\!\!- l \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow B \circ\!\!- A}\ \circ\!\!- r$$

Syntax of Full Lambek Calculus

$$\frac{\Gamma_1, A, \Gamma_2 \Rightarrow \Pi \quad \Gamma_1, B, \Gamma_2 \Rightarrow \Pi}{\Gamma_1, A \oplus B, \Gamma_2 \Rightarrow \Pi} \oplus l \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \oplus B} \oplus r_1 \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \oplus B}$$

$$\frac{\Gamma_1, A, \Gamma_2 \Rightarrow \Pi}{\Gamma_1, A \,\&\, B, \Gamma_2 \Rightarrow \Pi} \,\&l_1 \qquad \frac{\Gamma_1, B, \Gamma_2 \Rightarrow \Pi}{\Gamma_1, A \,\&\, B, \Gamma_2 \Rightarrow \Pi} \,\&l_2 \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow}{\Gamma \Rightarrow A \,\&\, B}$$

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \Pi}{\Gamma_1, \mathbf{1}, \Gamma_2 \Rightarrow \Pi} \,\mathbf{1}l \qquad \frac{}{\Rightarrow \mathbf{1}} \,\mathbf{1}r$$

$$\frac{}{\bot \Rightarrow} \perp l \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \bot} \perp r$$

$$\frac{}{\Gamma_1, \mathbf{0}, \Gamma_2 \Rightarrow \Pi} \,\mathbf{0}l \qquad \frac{}{\Gamma \Rightarrow \top} \top r$$

What is a structural rule?

- Examples:

$$\frac{\Gamma, A, A, \Delta \Rightarrow \Pi}{\Gamma, A, \Delta \Rightarrow \Pi}\ (c) \qquad \frac{\Gamma, A, A, \Delta \Rightarrow}{\Gamma, A, \Delta \Rightarrow}\ (wc)$$

$$\frac{\Gamma, \Sigma, \Sigma, \Delta \Rightarrow \Pi}{\Gamma, \Sigma, \Delta \Rightarrow \Pi}\ (seq - c) \qquad \frac{\Gamma, \Sigma_1, \Delta \Rightarrow \Pi \quad \Gamma, \Sigma_2, \Delta \Rightarrow \Pi}{\Gamma, \Sigma_1, \Sigma_2 \Delta \Rightarrow \Pi}\ (min)$$

- Ingredients:
  - Metavariables for formulas: $A, B, C, \ldots$
  - Metavariables for sequences: $\Gamma, \Delta, \Sigma, \ldots$
  - Metavariables for stoups: $\Pi, \ldots$

What is a structural rule?

- A structural rule is

$$\frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \dots \quad \Upsilon_n \Rightarrow \Psi_n}{\Upsilon_0 \Rightarrow \Psi_0}$$

  where

  - $\Upsilon_0, \dots, \Upsilon_n$: sequences of A's and Γ's.
  - $\Psi_0, \dots, \Psi_n$: one A or Π or empty.

Substructural hierarchy

- The sets $\mathcal{P}_n, \mathcal{N}_n$ of formulas defined by:

(0) $\mathcal{P}_0 = \mathcal{N}_0 =$ the set of atomic formulas

(P1) $\mathcal{N}_n \subseteq \mathcal{P}_{n+1}$

(P2) $A, B \in \mathcal{P}_{n+1} \implies A \oplus B, A \otimes B, \mathbf{1}, \mathbf{0} \in \mathcal{P}_{n+1}$

(N1) $\mathcal{P}_n \subseteq \mathcal{N}_{n+1}$

(N2) $A, B \in \mathcal{N}_{n+1} \implies A \,\&\, B, \bot, \top \in \mathcal{N}_{n+1}$

(N3) $A \in \mathcal{P}_{n+1}, B \in \mathcal{N}_{n+1} \implies A \multimap B, B \circ\!\!- A \in \mathcal{N}_{n+1}$

Substructural Hierarchy

$$
\begin{tikzcd}
{} & {} \\
\mathcal{P}_2 \arrow[u, dotted] & \mathcal{N}_2 \arrow[u, dotted] \\
\mathcal{P}_1 \arrow[u] \arrow[ur] & \mathcal{N}_1 \arrow[u] \arrow[ul] \\
\mathcal{P}_0 \arrow[u] \arrow[ur] \arrow[r, dash] & \mathcal{N}_0 \arrow[u] \arrow[ul]
\end{tikzcd}
$$

- Alternation of positive layers ($\mathbf{1}, \mathbf{0}, \oplus, \otimes$) and negative ones ($\bot, \top, \&, \multimap, \circ\!\!-$)
- $\alpha \multimap \mathbf{1},\ \alpha \multimap \alpha \otimes \alpha,\ \alpha \otimes \beta \multimap \beta \otimes \alpha \in \mathcal{N}_2$
- $(\alpha \multimap \beta) \oplus (\beta \multimap \alpha) \in \mathcal{P}_2$
- $((\alpha \multimap \beta) \multimap \beta) \multimap (\beta \multimap \alpha) \multimap \alpha \in \mathcal{N}_3$
- $\alpha^{\bot} \oplus \alpha^{\bot\bot} \in \mathcal{P}_3$

From axioms to structural rules

- Theorem: Any axiom in $\mathcal{N}_2$ corresponds to a set of structural rules.
- Key Lemma: An axiom $A_1, \ldots, A_n \Rightarrow B$ is equivalent to

$$\frac{\alpha_1 \Rightarrow A_1 \quad \ldots \quad \alpha_n \Rightarrow A_n}{\alpha_1, \ldots, \alpha_n \Rightarrow B}$$

and also to

$$\frac{B \Rightarrow \beta}{A_1, \ldots, A_n \Rightarrow \beta}$$

with $\alpha_1, \ldots, \alpha_n, \beta$ fresh.

From axioms to structural rules

$$\delta \oplus (\delta \multimap \bot \,\&\, \gamma \circ\!\!- \delta) \multimap \gamma \otimes \delta \in \mathcal{N}_2$$

$$\delta \multimap \gamma \otimes \delta \qquad (\delta \multimap \bot \,\&\, \gamma \circ\!\!- \delta) \multimap \gamma \otimes \delta$$

$$\delta \Rightarrow \gamma \otimes \delta \qquad (\delta \multimap \bot \,\&\, \gamma \circ\!\!- \delta) \Rightarrow \gamma \otimes \delta$$

$$\frac{\gamma \otimes \delta \Rightarrow \beta}{\delta \Rightarrow \beta} \qquad \frac{\gamma \otimes \delta \Rightarrow \beta}{\delta \multimap \bot \,\&\, \gamma \circ\!\!- \delta \Rightarrow \beta} \qquad KeyLemma$$

$$\frac{\gamma, \delta \Rightarrow \beta}{\delta \Rightarrow \beta} \qquad \frac{\gamma, \delta \Rightarrow \beta}{\delta \multimap \bot \,\&\, \gamma \circ\!\!- \delta \Rightarrow \beta}$$

$$OK \qquad \frac{\gamma, \delta \Rightarrow \beta \quad \alpha \Rightarrow \delta \multimap \bot \,\&\, \gamma \circ\!\!- \delta}{\alpha \Rightarrow \beta} \qquad KeyLemma$$

From axioms to structural rules

$$\frac{\gamma, \delta \Rightarrow \beta \quad \alpha \Rightarrow \delta \multimap \bot \mathbin{\&} \gamma \circ\!\!- \delta}{\alpha \Rightarrow \beta}$$

$$\frac{\gamma, \delta \Rightarrow \beta \quad \alpha \Rightarrow \delta \multimap \bot \quad \alpha \Rightarrow \gamma \circ\!\!- \delta}{\alpha \Rightarrow \beta}$$

$$\frac{\gamma, \delta \Rightarrow \beta \quad \delta, \alpha \Rightarrow \quad \alpha, \delta \Rightarrow \gamma}{\alpha \Rightarrow \beta}$$

- $\delta \oplus (\delta \multimap \bot \mathbin{\&} \gamma \circ\!\!- \delta) \multimap \gamma \otimes \delta$ is equivalent to

$$\frac{C, D \Rightarrow B}{D \Rightarrow B} \qquad \frac{C, D \Rightarrow B \quad D, A \Rightarrow \quad A, D \Rightarrow C}{A \Rightarrow B}$$

From axioms to structural rules

- Theorem: Any axiom in $\mathcal{N}_2$ corresponds to a set of structural rules. More generally, any axiom in $\mathcal{N}_{n+2}$ corresponds to a (non-structural) rule in which only $\mathcal{N}_n$ formulas appear:

$$\mathcal{N}_{n+2} \quad \xRightarrow{\text{structural rules}} \quad \mathcal{N}_n$$

- Partial converse result: Any weakly acyclic structural rule corresponds to an axiom in $\mathcal{N}_2$.
- Question: Does $\mathcal{P}_3$, to which the linearity axiom belongs, correspond to structural rules in hypersequent calculus? What about $\mathcal{N}_3$?

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Acyclicity

- Given a structural rule

$$\frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \ldots \quad \Upsilon_n \Rightarrow \Psi_n}{\Upsilon \Rightarrow \Psi}$$

  draw edges between metavariables occurring in the premises (upper sequents):
  - $\alpha \longrightarrow \beta$ if $\alpha \in \Upsilon_i$ and $\beta \in \Psi_i$ for some $1 \leq i \leq n$
  - identify two occurrences of the same metavariable
- A structural rule is acyclic if the resulting DAG is.
- In other words, if $\Upsilon_l, \alpha, \Upsilon_r \Rightarrow \alpha$ never belongs to the cut closure of the premises.

Acyclicity

- Given a structural rule, we draw a DAG on the metavariables occurring in the premises.
- A structural rule is acyclic if the DAG is.
- The following structural rule is not acyclic:

$$\frac{\alpha, \beta \Rightarrow \gamma \quad \gamma \Rightarrow \delta \quad \delta \Rightarrow \alpha}{\alpha, \beta, \gamma \Rightarrow \gamma}$$

$$\begin{tikzcd} \alpha \arrow[d] & \beta \arrow[dl] \\ \gamma \arrow[r] & \delta \arrow[ul] \end{tikzcd}$$

while the following is acyclic:

$$\frac{\alpha, \beta \Rightarrow \gamma \quad \beta, \gamma \Rightarrow \delta}{\alpha, \beta, \gamma \Rightarrow \gamma}$$

$$\begin{tikzcd} \alpha \arrow[d] & \beta \arrow[dl] \arrow[d] \\ \gamma \arrow[r] & \delta \end{tikzcd}$$

Simplification procedure

- Given an acyclic rule, we apply:
  1. Conclusion separation
  2. Premise separation
  3. Removing redundant premises
  4. Sequencing
  5. Contexing
  6. Linearization
- The resulting rule is simple. Always admits cut-elimination.

Simp (1): conclusion separation

- A rule with shared variables in the conclusion

$$\frac{B, \Gamma \Rightarrow}{{\color{red}B}, \Gamma \Rightarrow {\color{red}B}}\,(r)$$

is problematic:

$$\dfrac{\dfrac{B \Rightarrow B}{A \,\&\, B \Rightarrow B} \qquad \dfrac{\dfrac{B \Rightarrow B \quad \bot \Rightarrow}{B, B \multimap \bot \Rightarrow}}{B, B \multimap \bot \Rightarrow B}\,(r)}{A \,\&\, B, B \multimap \bot \Rightarrow B}\,(cut)$$

$${\color{red}\not\Downarrow}$$

$$\dfrac{\dfrac{\dfrac{B \Rightarrow B \quad \bot \Rightarrow}{B, B \multimap \bot \Rightarrow}}{A \,\&\, B, B \multimap \bot \Rightarrow}}{A \,\&\, B, B \multimap \bot \Rightarrow B}\ {\color{red}???}$$

Simp (1): conclusion separation

- Conclusion separation:

$$\frac{B, \Gamma \Rightarrow}{B, \Gamma \Rightarrow B}\ (r) \quad \Longrightarrow \quad \frac{B, \Gamma \Rightarrow \quad B \Rightarrow C}{B, \Gamma \Rightarrow C}\ (r')$$

with C fresh. Now the cut can be removed.

$$\frac{\dfrac{\dfrac{B \Rightarrow B \quad \bot \Rightarrow}{B, B \multimap \bot \Rightarrow}}{A \,\&\, B, B \multimap \bot \Rightarrow} \quad \dfrac{B \Rightarrow B}{A \,\&\, B \Rightarrow B}}{A \,\&\, B, B \multimap \bot \Rightarrow B}\ (r')$$

Simp (2): premise separation

- A rule with shared variable in the premises

$$\frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow A \quad A, \Sigma, A \Rightarrow \Pi}{\Gamma, \Delta, \Sigma \Rightarrow \Pi}$$

is also problematic. It can be transformed into

$$\frac{\Gamma, \Sigma, \Gamma \Rightarrow \Pi \quad \Gamma, \Sigma, \Delta \Rightarrow \Pi \quad \Delta, \Sigma, \Gamma \Rightarrow \Pi \quad \Delta, \Sigma, \Delta \Rightarrow \Pi}{\Gamma, \Delta, \Sigma \Rightarrow \Pi}$$

- Acyclicity is crucial. The procedure does not apply to a premise like $A, \Gamma \Rightarrow A$.
- As a result, variables are separated: LHS-variables and RHS-variables are disjoint.

Simp (3): removing redundant premises

Simp (4): sequencing

- Contraction alone (without Exchange) does not admit cut elimination:

$$
\dfrac{\dfrac{\overline{A \Rightarrow A} \quad \overline{B \Rightarrow B}}{A, B \Rightarrow A \otimes B} \qquad \dfrac{\dfrac{\overline{A \otimes B \Rightarrow A \otimes B} \quad \overline{A \otimes B \Rightarrow A \otimes B}}{\color{blue}{A \otimes B, A \otimes B} \Rightarrow (A \otimes B) \otimes (A \otimes B)}}{\color{blue}{A \otimes B} \Rightarrow (A \otimes B) \otimes (A \otimes B)}\;\color{blue}{Cntr}}{A, B \Rightarrow (A \otimes B) \otimes (A \otimes B)}\;\color{red}{Cut}
$$

$$
\color{red}{\not\Downarrow}
$$

$$
\dfrac{\dfrac{\dfrac{\overline{A \Rightarrow A} \quad \overline{B \Rightarrow B}}{A, B \Rightarrow A \otimes B} \qquad \dfrac{\overline{A \Rightarrow A} \quad \overline{B \Rightarrow B}}{A, B \Rightarrow A \otimes B}}{\color{red}{A, B, A, B} \Rightarrow (A \otimes B) \otimes (A \otimes B)}}{\color{red}{A, B} \Rightarrow (A \otimes B) \otimes (A \otimes B)}\;\color{red}{???}
$$

Simp (4): sequencing

- Sequencing [Terui 07]:

$$\frac{\Gamma, A, A, \Delta \Rightarrow \Pi}{\Gamma, A, \Delta \Rightarrow \Pi}\ (c) \quad \Longrightarrow \quad \frac{\Gamma, \Sigma, \Sigma, \Delta \Rightarrow \Pi}{\Gamma, \Sigma, \Delta \Rightarrow \Pi}\ (seq - c)$$

- Now the cut can be eliminated:

$$\dfrac{\dfrac{\dfrac{}{A \Rightarrow A} \quad \dfrac{}{B \Rightarrow B}}{A, B \Rightarrow A \otimes B} \quad \dfrac{\dfrac{}{A \Rightarrow A} \quad \dfrac{}{B \Rightarrow B}}{A, B \Rightarrow A \otimes B}}{\dfrac{A, B, A, B \Rightarrow (A \otimes B) \otimes (A \otimes B)}{A, B \Rightarrow (A \otimes B) \otimes (A \otimes B)}\ (seq - c)}$$

Simp (5): contexing

- The following version of mix is problematic:

$$\frac{\Gamma \Rightarrow \quad \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi}\ (mix)$$

- For example,

$$\dfrac{\dfrac{\bot \Rightarrow \quad A \multimap B \Rightarrow A \multimap B}{\bot, A \multimap B \Rightarrow A \multimap B}\ (mix) \qquad \begin{array}{c}\vdots\\ A, A \multimap B \Rightarrow B\end{array}}{A, \bot, A \multimap B \Rightarrow B}\ (cut)$$

$$\not\Downarrow$$

$$\dfrac{\begin{array}{c}\vdots\ ??\\ A, \bot \Rightarrow\end{array} \quad \begin{array}{c}\vdots\ ??\\ A \multimap B \Rightarrow B\end{array}}{A, \bot, A \multimap B \Rightarrow B}\ (mix)$$

Simp (5): contexing

- Contexing:

$$\frac{\Gamma \Rightarrow \quad \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi} \ (mix) \quad \Longrightarrow \quad \frac{\Gamma \Rightarrow \quad \Sigma_l, \Delta, \Sigma_r \Rightarrow \Pi}{\Sigma_l, \Gamma, \Delta, \Sigma_r \Rightarrow \Pi} \ (mix')$$

- Now the cut can be eliminated:

$$\frac{\bot \Rightarrow \quad A, A \multimap B \Rightarrow B}{A, \bot, A \multimap B \Rightarrow B} \ (mix')$$

Simp (6): linearization

- A rule with multiple occ. of the same metavariable in conclusion

$$\frac{\Gamma, \Sigma, \Delta \Rightarrow \Pi}{\Gamma, \Sigma, \Sigma, \Delta \Rightarrow \Pi}\ (exp)$$

is problematic:

$$\dfrac{\dfrac{\overline{B \Rightarrow B}}{B \Rightarrow A \oplus B} \qquad \dfrac{\dfrac{\overline{A \Rightarrow A}}{A \Rightarrow A \oplus B} \qquad \dfrac{\overline{A \oplus B \Rightarrow A \oplus B}}{A \oplus B, A \oplus B \Rightarrow A \oplus B}\ Exp}{A, A \oplus B \Rightarrow A \oplus B}\ Cut}{A, B \Rightarrow A \oplus B}\ Cut$$

$$\not\Downarrow$$

$$\dfrac{\dfrac{\overline{A \Rightarrow A}}{A \Rightarrow A \oplus B} \qquad \dfrac{\overline{B \Rightarrow B}}{B \Rightarrow A \oplus B}}{A, B \Rightarrow A \oplus B}\ ???$$

Simp (6): linearization

- Linearization [Terui 07]:

$$\frac{\Gamma, \Sigma, \Delta \Rightarrow \Pi}{\Gamma, \Sigma, \Sigma, \Delta \Rightarrow \Pi}\ (exp) \quad \Longrightarrow \quad \frac{\Gamma, \Sigma, \Delta \Rightarrow C \quad \Gamma, \Theta, \Delta \Rightarrow C}{\Gamma, \Sigma, \Theta, \Delta \Rightarrow C}\ (min)$$

- Now the cut can be eliminated:

$$\dfrac{\dfrac{\overline{A \Rightarrow A}}{A \Rightarrow A \oplus B} \quad \dfrac{\overline{B \Rightarrow B}}{B \Rightarrow A \oplus B}}{A, B \Rightarrow A \oplus B}\ (min)$$

Simple structural rules

- Theorem: Every acyclic structural rule is equivalent to a simple structural rule of the form:

$$\frac{\Upsilon_1 \Rightarrow \quad \ldots \quad \Upsilon_m \Rightarrow \quad \Sigma_l, \Upsilon_{m+1}, \Sigma_r \Rightarrow \Pi \quad \ldots \Sigma_l, \Upsilon_n, \Sigma_r \Rightarrow \Pi}{\Sigma_l, \Upsilon_0, \Sigma_r \Rightarrow \Pi}$$

  or

$$\frac{\Upsilon_1 \Rightarrow \quad \ldots \quad \Upsilon_m \Rightarrow}{\Upsilon_0 \Rightarrow}$$

  where Υ_0 consists only of metavariables for sequences (not formulas) and each occurs at most once in Υ_0.

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Conservativity and cut-elimination

- Given a logic L, one can consider its extension L^ω with infinitary $\bigwedge$, $\bigvee$.
- To show L^ω is a conservative extension of L, one proves:

  Any L-algebra can be completed, i.e., embedded into a complete L-algebra.

- In fact, given a formula A without $\bigwedge$, $\bigvee$,

$$\begin{aligned}
\vdash_{L^\omega} A \quad &\Longleftrightarrow \quad \forall \mathbf{B}\text{: complete } L\text{-alg. } \mathbf{B} \models A \\
&\Longleftrightarrow \quad \forall \mathbf{B}\text{: } L\text{-alg. } \mathbf{B} \models A \\
&\Longleftrightarrow \quad \vdash_L A.
\end{aligned}$$

Conservativity and cut-elimination

- For intuitionistic and modal logics, completion (Stone, Jonsson-Tarski) can be most effectively done via Kripke frames.
- Given a Heyting algebra $\mathbf{B}$, define $\mathbf{B}_+ = (W, R)$ (the dual Kripke frame) by

$$\begin{array}{rcl} W & = & \text{the set of prime filters of } \mathbf{B} \\ uRv & = & u \subseteq v. \end{array}$$

- Let $(\mathbf{B})_+^+$ = the complete Heyting algebra associated to (W, R).
- There is an embedding $\mathbf{B} \longrightarrow (\mathbf{B})_+^+$.

Conservativity and cut-elimination

- Cut-elimination is stronger than conservativity.
- To show cut-elimination, one proves a stronger completion:

  Any "intransitive" (cut-free) L-structure can be "quasi-embedded" into an L-algebra. [Okada 96, Belardinelli-Jipsen-Ono 04, Galatos-Jipsen 07]

- From IL to **FL**:

IL	**FL**
Heyting algebras	Residuated lattices
Kripke frames	Residuated frames (GJ07)
S-J-T completion	Dedekind-MacNeille completion

Residuated lattices

- A (pointed) residuated lattice is

$$\mathbf{P} = \langle P, \wedge, \vee, \otimes, \multimap, \multimapinv, 1, b \rangle$$

1. $\langle P, \wedge, \vee \rangle$ is a lattice.
2. $\langle P, \otimes, 1 \rangle$ is a monoid.
3. For any $x, y, z \in P$,

$$x \otimes y \leq z \iff x \leq z \multimapinv y \iff y \leq x \multimap z.$$

4. $b \in P$.

- A valuation is a homomorphism $f : \mathbf{Fm} \longrightarrow \mathbf{P}$ ($\mathbf{Fm}$: the absolutely free algebra of formulas).
- A is true under $f \iff 1 \leq f(A)$.

Residuated frames

- Given a set P of atoms, define

$$\begin{array}{rcll} P^* & = & \text{the free monoid generated by } P & \text{(sequences)} \\ P^{con} & = & P^* \times P^* \times (P \cup \{\varepsilon\}) & \text{(contexts)} \end{array}$$

- $(x_l, x_r, u) \in P^{con}$ is written $(x_l, \_, x_r \Rightarrow u)$. $(\varepsilon, \varepsilon, a)$ written a.
- A (simple) residuated frame $\mathbf{W} = (P, \sqsubset)$: $\sqsubset$ is a binary relation between P^* and P^{con} such that

$$\begin{array}{rcl} xy \sqsubset (z_l, \_, z_r \Rightarrow u) & \Longleftrightarrow & y \sqsubset (z_l x, \_, z_r \Rightarrow u) \\ & \Longleftrightarrow & x \sqsubset (z_l, \_, y z_r \Rightarrow u) \end{array}$$

for any $x, y \in P^*$ and $(z_l, \_, z_r \Rightarrow u) \in P^{con}$.

Residuated frames

- Example 1 (the dual frame):
  If $\mathbf{P} = (P, \wedge, \vee, \otimes, \multimap, \circ\!\!-, 1, b)$ is a pointed residuated lattice, then $\mathbf{P}_+ = (P, \sqsubset)$ is a simple residuated frame, where

$$x \sqsubset (z_l, \_, z_r \Rightarrow u) \iff z_l \cdot x \cdot z_r \leq_{\mathbf{P}} u$$

- Example 2 (cut-free sequent calculus):
  $(\mathbf{FL}^{cf})_+ = (Fm, \sqsubset)$ is a simple residuated frame, where

  $\Gamma \sqsubset (\Sigma_l, \_, \Sigma_r \Rightarrow \Pi) \iff \Sigma_l, \Gamma, \Sigma_r \Rightarrow \Pi$ is cut-free provable in $\mathbf{FL}$

From frames to algebras

- Let $\mathbf{W} = (W, \sqsubset)$ be a simple residuated frame. For any $X \subseteq W^*$ and $U \subseteq W^{con}$,

$$
\begin{aligned}
X^{\rhd} &= \{u \in W^{con} | \forall x \in X (x \sqsubset u)\} && \text{(upperbounds of } X\text{)} \\
U^{\lhd} &= \{x \in W^{*} | \forall u \in U (x \sqsubset u)\} && \text{(lowerbounds of } U\text{)}
\end{aligned}
$$

- Eg. in $(\mathbf{FL}^{cf})_+$,

$$A^{\lhd} = \{A\}^{\lhd} = \{\Gamma : \Gamma \Rightarrow A \text{ is cut-free provable } \}$$

- $\mathcal{C}(\mathbf{W})$ denotes the set of all closed subsets $X = X^{\rhd\lhd}$ of $\mathbf{W}$.

From frames to algebras

- Theorem: If $\mathbf{W} = (W, \sqsubset)$ is a simple frame, then the dual algebra

$$\mathbf{W}^+ = (\mathcal{C}(\mathbf{W}), \&, \oplus, \otimes, \multimap, \circ\!-, \mathbf{1}, \perp)$$

is a complete pointed residuated lattice, where

$$\begin{array}{rcl}
X \,\&\, Y & = & X \cap Y \\
X \oplus Y & = & (X \cup Y)^{\rhd\lhd} \\
X \otimes Y & = & \{xy : x \in X, y \in Y\}^{\rhd\lhd} \\
X \multimap Y & = & \{y : \forall x \in X xy \in Y\} \\
Y \circ\!- X & = & \{y : \forall x \in X yx \in Y\} \\
\mathbf{1} & = & \{\varepsilon\}^{\rhd\lhd} \\
\perp & = & \{\varepsilon\}^{\lhd}
\end{array}$$

Conservativity

- Theorem [GJ07]: If $\mathbf{P}$ is a pointed residuated lattice, then $f(a) = a^{\lhd}$ is an embedding $\mathbf{P} \longrightarrow (\mathbf{P})^{+}_{+}$ (Dedekind-MacNeille Completion).
- Corollary: $\mathbf{FL}^{\omega}$ is conservative over $\mathbf{FL}$.

Cut-elimination

- Given two algebras $\mathbf{P}, \mathbf{Q}$ in the language of $\mathbf{FL}$, a quasi-homomorphism is a function $P \longrightarrow \mathcal{P}(Q)$ such that

$$\begin{array}{rcll} c_{\mathbf{Q}} & \in & F(c_{\mathbf{P}}) & \text{for } c \in \{b, 1\}, \\ F(a) \star_{\mathbf{Q}} F(b) & \subseteq & F(a \star_{\mathbf{P}} b) & \text{for } \star \in \{\otimes, \multimap, \circ\!-, \wedge, \vee\}, \end{array}$$

  where $X \star_{\mathbf{Q}} Y = \{x \star_{\mathbf{Q}} y | x \in X, y \in Y\}$.
- Theorem [GJ07]: Consider the frame $(\mathbf{FL}^{cf})_+$. The function $F(A) = \{X = X^{\rhd\lhd} : A \in X \subseteq A^{\lhd}\}$ is a quasi-homomorphism $\mathbf{FL}^{cf} \longrightarrow (\mathbf{FL}^{cf})^+_+$.
- F is analogous to
  - Schütte's semi-valuation
  - Girard's reducibility candidates

Cut-elimination

- Recall that $A^{\lhd} = \{\Gamma : \Gamma \Rightarrow A$ is cut-free provable in $\mathbf{FL}\}$.
- F being quasi-homomorphism, there is a valuation f on $(\mathbf{FL}_R^{cf})_+^+$ such that $f(A) \in F(A)$, i.e., $A \in f(A) \subseteq A^{\lhd}$.
- If A is true under f,

$$1 \leq f(A)$$

$$\{\varepsilon\}^{\rhd\lhd} \subseteq f(A)$$

$$\varepsilon \in f(A)$$

$$\varepsilon \in A^{\lhd}$$

$\Rightarrow A$ is cut-free provable in $\mathbf{FL}$.

- Validity in $(\mathbf{FL}_R^{cf})_+^+$ implies cut-free provability.

Cut-elimination

- Proof of cut-elimination

$$\frac{\vdash_{\mathbf{FL}} A \overset{\text{sound}}{\Longrightarrow} (\mathbf{FL}^{cf})^+_+ \models A \qquad (\mathbf{FL}^{cf})^+_+ \models A \overset{\text{complete}}{\Longrightarrow} \vdash^{cf}_{\mathbf{FL}} A}{\vdash_{\mathbf{FL}} A \Longrightarrow \vdash^{cf}_{\mathbf{FL}} A}$$

Cut-elimination

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- We have just replaced object cuts with a big META-CUT.

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- Another criticism: it does not give a cut-elimination procedure.

Cut-elimination

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- We have just replaced object cuts with a big META-CUT.
- Another criticism: it does not give a cut-elimination procedure.
- Conjecture: If we eliminate the meta-cut, a concrete (object) cut-elimination procedure emerges.

Structural rules in residuated frames

- A simple residuated frame $(P, \sqsubset)$ satisfies

$$\frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \ldots \quad \Upsilon_n \Rightarrow \Psi_n}{\Upsilon \Rightarrow \Psi}\ (r)$$

  if

$$\frac{\Upsilon_1 \sqsubset \Psi_1 \quad \ldots \quad \Upsilon_n \sqsubset \Psi_n}{\Upsilon \sqsubset \Psi}\ (r)$$

  holds, where in the latter A stands for $a \in P$, Γ for $x \in P^*$ and Π for $u \in (P \cup \{\varepsilon\})$.

- Example 1: If $\mathbf{P}$ satisfies a set R of structural rules, then $\mathbf{P}_+$ satisfies R.

- Example 2: $(\mathbf{FL}_R^{cf})_+$ satisfies R.

Structural rules in residuated frames

$$\begin{array}{ccccc} \mathbf{P} & \Longrightarrow & \mathbf{P}_+ & \Longrightarrow & \mathbf{P}_+^+ \\ & \text{preserves } R & & ??? & \\ \mathbf{FL}_R^{cf} & \Longrightarrow & (\mathbf{FL}_R^{cf})_+ & \Longrightarrow & (\mathbf{FL}_R^{cf})_+^+ \end{array}$$

- Theorem: If a rule (r) is simple, then it is preserved by $\mathbf{W} \mapsto \mathbf{W}^+$.
- Corollary: Let R be a set of simple structural rules.
  1. R is preserved by Dedekind-MacNeille completion.
  2. $\mathbf{FL}_R^\omega$ is a conservative extension of $\mathbf{FL}_R$.
  3. $\mathbf{FL}_R^\omega$ enjoys cut-elimination.

Outline

1. Introduction
2. Substructural hierarchy and structuralization
3. Acyclic structural rules and simplification
4. Semantic cut-elimination: an introduction
5. Simple rules admit conservativity and cut-elimination
6. Conservativity implies acyclicity
7. Conclusion

Conservativity implies acyclicity

- Consider a cyclic rule

$$\frac{\alpha, \beta \Rightarrow \beta}{\beta, \alpha \Rightarrow \beta}\ (we)$$

  and suppose that $\mathbf{FL}_R^\omega$ with $R = R_0 \cup \{(we)\}$ is conservative over $\mathbf{FL}_R$.

- We show that (we) is equivalent to an acyclic rule in $\mathbf{FL}_{R_0}$.
- (we) is equivalent to

$$\frac{\alpha, \beta \Rightarrow \beta \quad \gamma \Rightarrow \beta}{\gamma, \alpha \Rightarrow \beta}\ (we')$$

Conservativity implies acyclicity

- Let $\overline{\beta}$ be a 'solution' of $\alpha \otimes \beta \leq \beta$ smaller than β:

$$\overline{\beta} = \bigwedge_{0 \leq n} \alpha^n \multimap \beta.$$

- Then in $\mathbf{FL}_R^\omega$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \alpha, \overline{\beta} \Rightarrow \overline{\beta}\end{array} \qquad \begin{array}{c}\{\alpha^k, \gamma \Rightarrow \beta : 0 \leq k\}\\ \vdots\\ \gamma \Rightarrow \overline{\beta}\end{array}}{\gamma, \alpha \Rightarrow \overline{\beta}}\ (we') \qquad \begin{array}{c}\vdots\\ \overline{\beta} \Rightarrow \beta\end{array}}{\gamma, \alpha \Rightarrow \beta}\ (Cut)$$

- Since $\mathbf{FL}_R^\omega$ is conservative over $\mathbf{FL}_R$, we have in $\mathbf{FL}_R$:

$$\begin{array}{c}\{\alpha^k, \gamma \Rightarrow \beta : 0 \leq k\}\\ \vdots\\ \gamma, \alpha \Rightarrow \beta\end{array}$$

Conservativity implies acyclicity

- Since $\mathbf{FL}_R$ is finitary, there is n such that

$$\begin{array}{c} \gamma \Rightarrow \beta \quad \alpha, \gamma \Rightarrow \beta \quad \alpha^2, \gamma \Rightarrow \beta \quad \ldots \quad \alpha^n, \gamma \Rightarrow \beta \\ \vdots \\ \gamma, \alpha \Rightarrow \beta \end{array}$$

- $R = R_0 \cup \{(we)\}$ is equivalent to $R_0 \cup \{(we'')\}$:

$$\frac{\gamma \Rightarrow \beta \quad \alpha, \gamma \Rightarrow \beta \quad \alpha^2, \gamma \Rightarrow \beta \quad \ldots \quad \alpha^n, \gamma \Rightarrow \beta}{\gamma, \alpha \Rightarrow \beta}\ (we'')$$

- (we'') is acyclic.
- Theorem: If $\mathbf{FL}_R^\omega$ is conservative over $\mathbf{FL}_R$, then R is equivalent to a set of acyclic structural rules.

Main results

- Any axiom in $\mathcal{N}_2$ can be transformed into a set of structural rules.
- For any set R of structural rules, the following are equivalent.
  1. R is equivalent to a set of acyclic rules.
  2. R is equivalent to a set of simple rules.
  3. R is equivalent to a set R' of rules such that $\mathbf{FL}_R^\omega$ enjoys (a stronger form of) cut-elimination.
  4. R is preserved by Dedekind-MacNeille completion.
  5. $\mathbf{FL}_R^\omega$ is a conservative extension of $\mathbf{FL}_R$.

Conclusion

$$
\begin{tikzcd}
{} & & {} \\
\mathcal{P}_2 \arrow[u, dotted] & & \mathcal{N}_2 \arrow[u, dotted] \\
\mathcal{P}_1 \arrow[u] \arrow[urr] & & \mathcal{N}_1 \arrow[u] \arrow[ull] \\
\mathcal{P}_0 \arrow[u] \arrow[urr] \arrow[rr, dash] & & \mathcal{N}_0 \arrow[u] \arrow[ull]
\end{tikzcd}
$$

- Structural rules in single-conclusion (RHS) calculi capture $\mathcal{N}_2$.
- Acyclicity = Conservativity
- To conquer $\mathcal{P}_3, \mathcal{N}_3$, to which more interesting axioms belong, one would need more sophisticated calculi (eg. hypersequent calculi).
- Question: How high can we go up?