Under consideration for publication in Math. Struct. in Comp. Science

Intuitionistic Phase Semantics is Almost Classical

Max I. Kanovich[1], Mitsuhiro Okada[2] and Kazushige Terui[3]

[1] *Department of Computer Science,*
Queen Mary, University of London,
Mile End Rd., London E1 4NS
mik@dcs.qmul.ac.uk
and Department of Computer and Information Science,
University of Pennsylvania,
maxkanov@cis.upenn.edu
[2] *Department of Philosophy, Faculty of Letters, Keio University,*
2-15-45 Mita, Minato-ku, Tokyo 108-8345, Japan.
mitsu@abelard.flet.keio.ac.jp
[3] *National Institute of Informatics,*
2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8430, Japan.
terui@nii.ac.jp

Received 23 September 2005

We study the relationship between classical phase semantics for classical linear logic (**LL**) and intuitionistic phase semantics for intuitionistic linear logic (**ILL**). We prove that (i) every intuitionistic phase space is a subspace of a classical phase space, and (ii) every intuitionistic phase space is phase isomorphic to an "almost classical" phase space. Here, by an "almost classical" phase space we mean a phase space having a double-negation-like closure operator. Based on these semantic considerations, we give a syntactic embedding of propositional **ILL** into **LL**.

1. Introduction

Linear logic (**LL**, (Girard 1987)) is a refinement of classical and intuitionistic logics. It inherits *constructivity* from intuitionistic logic and *duality* (exemplified by involutive negation and de Morgan laws) from classical logic through a careful handling of the structural inference rules. An intuitionistic version, called *intuitionistic linear logic* (**ILL**), is also considered (Lafont 1988; Abrusci 1990; Troelstra 1992). Although it lacks duality, it is sometimes handier than **LL** in some applications in functional/logic programming (see e.g. (Abramsky 1993; Hodas and Miller 1994)).

Given two forms of linear logic, it is natural to ask what is the relationship between them. A partial solution is given by Schellinx (Schellinx 1991). He shows by a syntactic argument that **LL** is conservative over **ILL** as far as the propositional formulas without $\perp$ and $\mathbf{0}$ are concerned. This should be contrasted with the traditional (non-linear) situation, where classical logic and intuitionistic logic are quite different even without absurdity (contradiction) and negation, as witnessed by Peirce's law: $((A \to B) \to A) \to A$. While Schellinx's approach is purely *syntactic*, we take a *semantic* approach to this problem in this paper.

There is a canonical semantics for linear logic that completely characterizes provability, that is *phase semantics* (Girard 1987; Girard 1995). It has been used to show various properties of linear logic, such as undecidability (Lafont 1996; Lafont and Scedrov 1996), cut-elimination, strong normalization (Okada 1996; Okada 1999; Okada 2002), and decidability of some fragments (Lafont 1997; Okada and Terui 1999). In our recent work, it is used to give a necessary and sufficient condition for structural rules to admit cut elimination (Terui 2005). Accordingly to the two forms of linear logic, there are two forms of phase semantics: classical and intuitionistic ones (see (Abrusci 1990; Troelstra 1992; Ono 1994; Okada 1996) for the latter).

The aim of this paper is to analyse the delicate distinction between **LL** and **ILL** in terms of phase semantics. Recall that an *intuitionistic phase space* is a commutative monoid M endowed with a *closure operator* $Cl : \wp(M) \longrightarrow \wp(M)$. A *classical phase space* $(M_c, \bot)$ is an intuitionistic phase space where the closure operator is defined by double-negation: $Cl(X) = X^{\bot\bot}$. Given a classical phase space $(M_c, \bot)$ and a submonoid $M_i \subseteq M_c$, one naturally obtains an intuitionistic phase space (M_i, Cl_i) by taking $Cl_i(X) = X^{\bot\bot} \cap M_i$ for $X \subseteq M_i$. Call the latter a *subspace* of $(M_c, \bot)$. In addition, one can imagine that the closure operator Cl_i thus defined happens to coincide with the double negation operator, namely $Cl_i(X) = X^{\bot\bot} \cap M_i = X^{\bot\bot}$, where $\bot$ is not necessarily a subset of M_i. In that case, we call (M_i, Cl_i) *quasi-classical*. With these notions, our main results can be stated as follows:

(i) Every intuitionistic phase space is a subspace of a classical phase space.

(ii) Every intuitionistic phase space is phase isomorphic to a quasi-classical phase space.

Here, by *phase isomorphic* we mean that two spaces have the same algebraic structure.

The result (i) on the one hand establishes a relationship between classical and intuitionistic phase spaces, and on the other hand gives a new definition of intuitionistic closure operator†: $Cl(X) = X^{\bot\bot} \cap M$. The result (ii) yields a completeness theorem for **ILL** with respect to a rather special class of phase models: quasi-classical phase models. Furthermore, these results suggest a new view of **ILL** as a "submonoid-restriction" of **LL**. In fact, it leads to a syntactic embedding of **ILL** to **LL**. Let $\varphi(p_0)$ be a formula whose intuitive meaning is that (the interpretation of) p_0 is a "submonoid" of (the interpretation of) $\top$. In the presence of the assumption $\varphi(p_0)$, a formula of the form $A \,\&\, p_0$ can be interpreted as a "submonoid-restriction" of A. Based on this idea, we define a translation A° for any formula A of **ILL**which is in full accordance with the subspace relation between intuitionistic and classical phase models. We then prove:

(iii) For every formula A of **ILL**, A is provable in **ILL** if and only if $\varphi(p_0) \multimap A^\circ$ is provable in **LL**.

It should be stressed that while Schellinx's conservation result is only concerned with a

† This idea was suggested by J.-Y. Girard.

fragment of **ILL** without $\bot$ and **0**, our result covers the full range of the propositional **ILL** formulas.

The rest of this paper is organized as follows. In Section 2, we recall the syntax and phase semantics for **ILL** and **LL**. In Section 3, we introduce the central notions of this paper: subspace, quasi-classical phase space and phase isomorphism. In Section 4, we prove our main result, called the three layered representation theorem, which integrates (i) and (ii) above. In Section 5, we turn to the syntax and give a faithful embedding of **ILL** into **LL**. In Section 6, we conclude the paper with a number of open problems.

2. Preliminaries

In this section, we recall the syntax of **ILL** and phase semantics for both **ILL** and **LL**. A proof of the completeness theorem for **ILL** is also outlined for later use. We refer to (Girard 1987; Girard 1995) for the syntax of **LL**.

Definition 2.1. *The set of* formulas *of* **ILL** *is defined as follows;*

— *Propositional variables* $p, q, r, \ldots$ *are formulas of* **ILL**.
— $\mathbf{1}$, $\bot$, $\top$, $\mathbf{0}$ *are formulas of* **ILL**.
— *If* A *and* B *are formulas of* **ILL**, *then so are* $A \otimes B$, $A \multimap B$, $A \,\&\, B$, $A \oplus B$ *and* $!A$.

Finite multisets of formulas are denoted by Greek capitals Γ, Δ, $\ldots$. *If* Γ *stands for* $A_1, \ldots, A_n$, *then* $!\Gamma$ *denotes* $!A_1, \ldots, !A_n$. *A* sequent *of* **ILL** *is of the form* $\Gamma \vdash \Delta$ *with* $|\Delta| \leq 1$. *The inference rules of* **ILL** *are given in Figure 1.*

There seems to be no natural way of introducing the inference rules for $⅋$ and ? in the intuitionistic framework, hence we do not count them as intuitionistic connectives. On the other hand, we keep constant $\bot$ (the unit for $⅋$) in **ILL**, because it has natural inference rules. However, having $\bot$ in **ILL** is not strictly necessary (and several authors exclude it from **ILL**), since it can be replaced with an arbitrary but fixed propositional variable.

Next, we introduce phase semantics for **ILL** in the style of (Troelstra 1992). The interpretation of exponentials $(!, ?)$ is due to (Girard 1995).

Let $(M, \cdot, \varepsilon)$ be a commutative monoid, with ε being its neutral element. For any $X, Y \subseteq M$, we define:

$$X \cdot Y := \{x \cdot y \mid x \in X, y \in Y\}.$$

Definition 2.2. *An* intuitionistic phase space $(M, \cdot, \varepsilon, Cl)$ *consists of a commutative monoid* $(M, \cdot, \varepsilon)$ *and a* closure operator Cl, *that is a mapping from the subsets of* M *to themselves such that for all* $X, Y \subseteq M$:

(Cl1) $X \subseteq Cl(X)$,
(Cl2) $Cl(Cl(X)) \subseteq Cl(X)$,
(Cl3) $X \subseteq Y \Longrightarrow Cl(X) \subseteq Cl(Y)$,
(Cl4) $Cl(X) \cdot Cl(Y) \subseteq Cl(X \cdot Y)$.

A set $X \subseteq M$ *is said to be* closed *if* $X = Cl(X)$. *The set of all closed sets in* $\mathcal{M}$ *is denoted by* $\mathcal{C}(\mathcal{M})$.

Identity and Cut:

$$\frac{}{A \vdash A}\ \mathit{Identity} \qquad \frac{\Gamma_1 \vdash A \quad A, \Gamma_2 \vdash \Delta}{\Gamma_1, \Gamma_2 \vdash \Delta}\ \mathit{Cut}$$

Multiplicatives:

$$\frac{A, B, \Gamma \vdash \Delta}{A \otimes B, \Gamma \vdash \Delta}\ \otimes l \qquad \frac{\Gamma_1 \vdash A \quad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}\ \otimes r \qquad \frac{\Gamma \vdash \Delta}{\mathbf{1}, \Gamma \vdash \Delta}\ \mathbf{1}l \qquad \frac{}{\vdash \mathbf{1}}\ \mathbf{1}r$$

$$\frac{\Gamma_1 \vdash A \quad B, \Gamma_2 \vdash \Delta}{A \multimap B, \Gamma_1, \Gamma_2 \vdash \Delta}\ \multimap l \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \multimap B}\ \multimap r \qquad \frac{}{\bot \vdash}\ \bot l \qquad \frac{\Gamma \vdash}{\Gamma \vdash \bot}\ \bot r$$

Additives:

$$\frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \oplus B, \Gamma \vdash \Delta}\ \oplus l \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}\ \oplus r_1 \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}\ \oplus r_2 \qquad \frac{}{\mathbf{0}, \Gamma \vdash \Delta}\ \mathbf{0}l$$

$$\frac{A, \Gamma \vdash \Delta}{A \,\&\, B, \Gamma \vdash \Delta}\ \& l_1 \qquad \frac{B, \Gamma \vdash \Delta}{A \,\&\, B, \Gamma \vdash \Delta}\ \& l_2 \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}\ \& r \qquad \frac{}{\Gamma \vdash \top}\ \top r$$

Exponentials:

$$\frac{A, \Gamma \vdash \Delta}{!A, \Gamma \vdash \Delta}\ !D \qquad \frac{!A, !A, \Gamma \vdash \Delta}{!A, \Gamma \vdash \Delta}\ !C \qquad \frac{\Gamma \vdash \Delta}{!A, \Gamma \vdash \Delta}\ !W \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ !r$$

Here $!\Gamma$ stands for a multisets of formulas of the form $!A_1, \ldots, !A_n$.

Fig. 1. Inference Rules of Intuitionistic Linear Logic (**ILL**)

Remark 2.3. *The first three axioms (Cl1)—(Cl3) together with* $Cl(\emptyset) = \emptyset$ *form the Kuratowsky's axiomatic definition of topological spaces. It is worthwhile to observe that under our considerations* $Cl(\emptyset)$ *is generally non-empty.*

Definition 2.4. *Given an intuitionistic phase space* $(M, \cdot, \varepsilon, Cl)$*, we define the following operations over sets* $X, Y \subseteq M$*:*

$$\begin{array}{rcl@{\qquad}rcl}
X \otimes Y & := & Cl(X \cdot Y) & 1 & := & Cl(\{\varepsilon\}) \\
X \,\&\, Y & := & X \cap Y & \top & := & M \\
X \oplus Y & := & Cl(X \cup Y) & 0 & := & Cl(\emptyset) \\
X \multimap Y & := & \{z \in M \mid \forall x \in X (x \cdot z \in Y)\} & & & \\
!X & := & Cl(X \cap I), \quad \text{where } I := \{x \in 1 \,|\, x \cdot x = x\}. & & &
\end{array}$$

These operations are defined exactly in the same way as in classical phase semantics. Notice that we do not introduce an operation (a set) associated to $\bot$; it will be treated through a valuation operator.

Example 2.5. *Let* $(N, +, 0)$ *be the additive monoid of natural numbers. For any* $X \subseteq N$*, define* $Cl_{id}(X)$ *and* $Cl_{\leq}(X)$ *as follows:*

$$\begin{array}{rcl}
Cl_{id}(X) & := & X, \\
Cl_{\leq}(X) & := & \{n \in N | \exists m \in X (n \leq m)\}.
\end{array}$$

Then one can easily see that both $(N,+,0,Cl_{id})$ *and* $(N,+,0,Cl_{\leq})$ *are intuitionistic phase spaces.*

Definition 2.6. *An* intuitionistic phase model $(M,\cdot,\varepsilon,Cl,v)$ *is an intuitionistic phase space* $(M,\cdot,\varepsilon,Cl)$ *with a* valuation v *that maps each propositional variable to a closed subset of* M*, and, in addition, assigns a closed subset of* M *to constant* $\bot$*.*

The valuation v *is naturally extended to all formulas of* **ILL** *using the operations in Definition 2.4. A formula* A *is* satisfied *in model* $(M,\cdot,\varepsilon,Cl,v)$ *if and only if* $\varepsilon \in v(A)$*.*

By definition, a closure operator determines the family of closed sets. Conversely, a closure operator is sometimes determined by a certain family of closed sets.

Definition 2.7. *Given an intuitionistic phase space* $(M,\cdot,\varepsilon,Cl)$*, a family* $\{G_\alpha\}_{\alpha\in\mathcal{A}}$ *of closed sets is called a* co-basis *if for any* $X\subseteq M$*:*

$$Cl(X) = \bigcap_{\{\alpha\in\mathcal{A}\mid X\subseteq G_\alpha\}} G_\alpha.$$

In fact, the (possibly uncountable) family of *all* closed sets forms a trivial co-basis. In some cases, however, one might like to have a countable (or even finite) co-basis. The next theorem states a sufficient condition for a family of sets to be a co-basis.

Theorem 2.8. Let $(M,\cdot,\varepsilon)$ be a commutative monoid and $\{G_\alpha\}_{\alpha\in\mathcal{A}}$ be a family of subsets of M. Suppose that for every G_α and $y\in M$, $\{y\}\multimap G_\alpha$ is also a member of the family $\{G_\alpha\}_{\alpha\in\mathcal{A}}$. Then operator Cl defined by:

$$Cl(X) := \bigcap_{\{\alpha\in\mathcal{A}\mid X\subseteq G_\alpha\}} G_\alpha,$$

satisfies (Cl1)—(Cl4) to be a closure operator on M.

Proof. (Cl1) — (Cl3) are all immediate. We claim that Cl above satisfies the following property:

(Cl4') $X\cdot Cl(Y)\subseteq Cl(X\cdot Y)$ for any $X,Y\subseteq M$.

(Cl4) is derived by using (Cl4') twice:

$$\begin{aligned} Cl(X)\cdot Cl(Y) &\subseteq Cl(Cl(X)\cdot Y) \\ &\subseteq Cl(Cl(X\cdot Y)) = Cl(X\cdot Y). \end{aligned}$$

To show (Cl4'), let $X\cdot Y\subseteq G_\beta$ with $\beta\in\mathcal{A}$. Then for any $x\in X$,

$$Y\subseteq\{x\}\multimap G_\beta.$$

Since $\{x\}\multimap G_\beta\in\{G_\alpha\}_{\alpha\in\mathcal{A}}$ by assumption,

$$\bigcap_{\{\alpha\in\mathcal{A}\mid Y\subseteq G_\alpha\}} G_\alpha \subseteq \{x\}\multimap G_\beta.$$

Therefore,

$$\{x\}\cdot\bigcap_{\{\alpha\in\mathcal{A}\mid Y\subseteq G_\alpha\}} G_\alpha \subseteq G_\beta$$

for any $x \in X$ and $G_\beta \supseteq X \cdot Y$, that is sufficient to show our claim. □

It is known that intuitionistic phase semantics is complete for **ILL**.

Theorem 2.9 (cf. (Abrusci 1990; Troelstra 1992)). For any formula A, A is provable in **ILL** if and only if A is satisfied in every intuitionistic phase model.

Moreover, this statement can be strengthened as follows; A is provable in **ILL** if and only if A is satisfied in every intuitionistic phase model that has a countable co-basis.

Proof (Outline). We define the *canonical model* $(M_0, \cdot, \varepsilon, Cl_0, v_0)$ *for* **ILL** as follows;

— M_0 consists of finite multisets Γ of **ILL** formulas. Here we ignore the multiplicity of formulas of the form $!B$; for instance we identify $!B, !B, !C, \Sigma$ with $!B, !C, !C, \Sigma$.
— $\Gamma \cdot \Delta = \Gamma, \Delta$.
— $\varepsilon = \emptyset$ (the empty multiset).
— $[\![\Gamma \vdash \Xi]\!] = \{\Delta \mid \Delta, \Gamma \vdash \Xi \text{ is provable in } \mathbf{ILL}\}$.

Let $\mathcal{G}$ be $\{[\![\Gamma \vdash \Xi]\!] \mid \Gamma \vdash \Xi \text{ is a sequent of } \mathbf{ILL}\}$. By definition $\mathcal{G}$ is countable. Furthermore, for every $[\![\Gamma \vdash \Xi]\!] \in \mathcal{G}$ and $\Delta \in M_0$, $\{\Delta\} \multimap [\![\Gamma \vdash \Xi]\!] = [\![\Delta, \Gamma \vdash \Xi]\!] \in \mathcal{G}$. Therefore, by Theorem 2.8, it induces a closure operator Cl_0 defined by

$$Cl_0(X) := \bigcap_{X \subseteq [\![\Gamma \vdash \Xi]\!]} [\![\Gamma \vdash \Xi]\!].$$

Finally, define

$$\begin{array}{rcl} v_0(q) & := & [\![\vdash q]\!]; \\ v_0(\perp) & := & [\![\vdash \perp]\!] \end{array}$$

to complete the definition of the canonical model for **ILL**.

It is readily seen that $I = \{x \in 1 \,|\, x \cdot x = x\}$ exactly consists of the elements of the form $!\Gamma$. The following lemma establishes the proof of Theorem 2.9:

Lemma 2.10. For any **ILL** formula B, we have $B \in v_0(B) \subseteq [\![\vdash B]\!]$. In particular, $\varepsilon \in v_0(B)$ implies that B is provable in **ILL**.

Proof. By induction on the complexity of B (see (Okada 1996)). □

Remark 2.11. *If one replaces the definition of* $[\![\Gamma \vdash \Xi]\!]$ *with*

$$[\![\Gamma \vdash \Xi]\!] = \{\Delta \mid \Delta, \Gamma \vdash \Xi \textit{ is } \text{cut-free } \textit{provable in } \mathbf{ILL}\},$$

one obtains the cut-elimination theorem at the same time (Okada 1996).

Finally, let us just mention phase semantics for **LL**.

Definition 2.12. *A* classical phase space $(M, \cdot, \varepsilon, \perp\!\!\!\perp)$ *consists of a commutative monoid* $(M, \cdot, \varepsilon)$ *and a distinguished subset* $\perp\!\!\!\perp$ *of* M.

We write $X^{\perp\!\!\!\perp}$ to denote $X \multimap \perp\!\!\!\perp$. A set $X \subseteq M$ is called a *closed set* (or a *fact*) if $X = X^{\perp\!\!\!\perp\perp\!\!\!\perp}$. In a classical phase space, one also defines the following operations over sets $X, Y \subseteq M$:

$$X ⅋ Y := (X^{\perp\!\!\!\perp} \cdot Y^{\perp\!\!\!\perp})^{\perp\!\!\!\perp} \qquad ?X := (X^{\perp\!\!\!\perp} \cap I)^{\perp\!\!\!\perp}.$$

Every classical phase space $(M, \cdot, \varepsilon, \perp)$ may be viewed as an intuitionistic phase space. Indeed, it is easily verified that the double negation operator

$$Cl(X) := X^{\perp\perp}$$

satisfies the conditions (Cl1)—(Cl4). On the contrary, it is *not* the case that every intuitionistic phase space can be viewed as a classical one.

Example 2.13. *The intuitionistic phase space $(N, +, 0, Cl_{\leq})$ given in Example 2.5 cannot be viewed as a classical phase space. Suppose that there is $\perp \subseteq N$ such that $Cl_{\leq}(X) = X^{\perp\perp}$ for all $X \subseteq N$. Since $\perp$ is a closed set, it must be $\emptyset$, N, or of the form $\{0, \ldots, k\}$ for some $k \in N$. The first two are clearly impossible (we would then have $X^{\perp\perp} = N$ for any nonempty X in both cases). The last one is also impossible, because it would imply that $k+1 \notin X^{\perp}$ for any nonempty X, and thus we would not have $\{k+1\}^{\perp\perp} = \{0, \ldots, k+1\}$. Hence there is no such $\perp$.*

3. Subspace, Quasi-Classical and Phase Isomorphism

In this section, we introduce three important notions: subspace, quasi-classical phase space and phase isomorphism. We also prove some basic properties of them.

Definition 3.1. *Let $\mathcal{M}_1 = (M_1, \cdot, \varepsilon, Cl_1)$ be an intuitionistic phase space. Then $\mathcal{M}_2 = (M_2, \cdot, \varepsilon, Cl_2)$ is called a* subspace *of $\mathcal{M}_1$ (written $\mathcal{M}_2 \sqsubseteq \mathcal{M}_1$) if*

— *$(M_2, \cdot, \varepsilon)$ is a submonoid of $(M_1, \cdot, \varepsilon)$, and*
— *$Cl_2(X) = Cl_1(X) \cap M_2$ for any $X \subseteq M_2$.*

It is then easy to verify the following:

Theorem 3.2. Every subspace of an intuitionistic phase space is an intuitionistic phase space.

Proof. Let $(M_2, \cdot, \varepsilon, Cl_2)$ be a subspace of $(M_1, \cdot, \varepsilon, Cl_1)$. It is easy to show that Cl_2 satisfies (Cl1) — (Cl3). The property (Cl4) is a consequence of the following facts:

$$\begin{array}{rcl} Cl_1(X) \cdot Cl_1(Y) & \subseteq & Cl_1(X \cdot Y) \\ M_2 \cdot M_2 & \subseteq & M_2 \end{array}$$

□

Corollary 3.3. Every subspace of a classical phase space is an intuitionistic phase space. Namely, if $(M_c, \cdot, \varepsilon, \perp)$ is a classical phase space and M is a submonoid of M_c, then $(M, \cdot, \varepsilon, Cl)$ with

$$Cl(X) := X^{\perp\perp} \cap M$$

for $X \subseteq M$ is an intuitionistic phase space.

Example 3.4. *Consider a classical phase space $(Z, +, 0, \perp)$, where $(Z, +, 0)$ is the additive monoid of integers and $\perp$ is the set of nonpositive integers. Then one can easily see that $X^{\perp\perp} = \{n \in Z | \exists m \in X (n \leq m)\}$ for any $X \subseteq Z$. Hence we have*

$X^{\perp\perp} \cap N = Cl_{\leq}(X)$ for every $X \subseteq N$. This means that $(N, +, 0, Cl_{\leq})$ is a subspace of $(Z, +, 0, \perp)$.

Does the converse of Corollary 3.3 also hold? That is to say, can *every* intuitionistic phase space be represented as a subspace of some classical phase space? A positive answer will be given in Section 4.

We now consider a subclass of the intuitionistic phase spaces which are of special interest.

Definition 3.5. *An intuitionistic phase space $(M, \cdot, \varepsilon, Cl)$ is called a* quasi-classical phase space *if it is a subspace of a classical phase space $(M_c, \cdot, \varepsilon, \perp)$ and*

$$Cl(X) = X^{\perp\perp}$$

for every $X \subseteq M$.

The closure operator Cl of a quasi-classical phase space $(M, \cdot, \varepsilon, Cl)$ looks almost like a classical double-negation operator, the only difference being that $\perp$ is not necessarily a subset of M. The following lemma gives a simple condition for a subspace of a classical phase space to be quasi-classical.

Lemma 3.6. Let $(M_c, \cdot, \varepsilon, \perp)$ be a classical phase space and let $(M, \cdot, \varepsilon, Cl)$ be a subspace of $(M_c, \cdot, \varepsilon, \perp)$. Then $(M, \cdot, \varepsilon, Cl)$ is quasi-classical if and only if $M^{\perp\perp} = M$.

Proof. If $(M, \cdot, \varepsilon, Cl)$ is quasi-classical, then $M^{\perp\perp} = Cl(M) = M$ by definition. Conversely, if $M^{\perp\perp} = M$, we have

$$X^{\perp\perp} \subseteq M^{\perp\perp} = M$$

for every $X \subseteq M$. Therefore, $Cl(X) = X^{\perp\perp} \cap M = X^{\perp\perp}$. □

Example 3.7. *The intuitionistic phase space $(N, +, 0, Cl_{id})$ in Example 2.5 is quasi-classical. To show this, define a classical phase space $(\tilde{N}, \tilde{+}, 0, \perp)$ as follows:*

— *$\tilde{N} := N \cup (N \times \wp(N)) \cup \{\surd\}$.*
— *For $n, m \in N$, $X, Y \subseteq N$ and $x \in \tilde{N}$, let $n \tilde{+} m := n + m$;*
  $n \tilde{+} (m, X) = (m, X) \tilde{+} n := (n + m, X)$;
  $(n, X) \tilde{+} (m, Y) := \surd$; $\surd \tilde{+} x = x \tilde{+} \surd := \surd$.
— *$\perp := \{(n, X) | n \in X\}$.*

Then, we have $(0, X) \in X^{\perp}$ for every $X \subseteq N$. Therefore, $X^{\perp\perp}$ does not contain an element of the form (n, Y) or $\surd$ (since $\surd \notin \perp$). Moreover, if $n \in X^{\perp\perp}$, then $(n, X) \in \perp$ (since $(0, X) \in X^{\perp}$), that means $n \in X$. Thus we have $X^{\perp\perp} = X = Cl_{id}(X)$. This shows that $(N, +, 0, Cl_{id})$ is quasi-classical.

One can similarly prove that $(N, +, 0, Cl_{\leq})$ is quasi-classical.

It is not known whether *every* intuitionistic phase space is quasi-classical or not. On the other hand, we shall see in the next section that every intuitionistic phase space is *phase isomorphic* to a quasi-classical one. Here, the notion of phase isomorphism is defined as follows.

Definition 3.8. *Let $\mathcal{M}_1 = (M_1, \cdot, \varepsilon, Cl_1)$ and $\mathcal{M}_2 = (M_2, \cdot, \varepsilon, Cl_2)$ be intuitionistic phase spaces. A function $F : \mathcal{C}(\mathcal{M}_1) \longrightarrow \mathcal{C}(\mathcal{M}_2)$ is called a* phase isomorphism *from $\mathcal{M}_1$ to $\mathcal{M}_2$ if it is bijective and preserves the operations and constants given in Definition 2.4. Namely, for any closed sets $X, Y \in \mathcal{C}(\mathcal{M}_1)$,*

$$\begin{array}{rcll} F(X \star_1 Y) & = & F(X) \star_2 F(Y) & \text{for } \star \in \{\otimes, \&, \oplus, \multimap\}; \\ F(!_1 X) & = & !_2 F(X); & \\ F(c_1) & = & c_2 & \text{for } c \in \{1, 0, \top\}. \end{array}$$

Here, the operations and constants in $\mathcal{M}_i$ are indexed by i (for $i = 1, 2$).

Example 3.9. *The intuitionistic phase space $(N, +, 0, Cl_{\leq})$ in Example 2.5 is phase isomorphic to $(Z, +, 0, Cl'_{\leq})$, where the latter closure operator $Cl'_{\leq}$ is defined by $Cl'_{\leq}(X) = Cl_{\leq}(X) \cup \{n | n < 0\}$ for any $X \subseteq Z$. In fact, there is a phase isomorphism F from the former to the latter, defined by $F(X) = X \cup \{n | n < 0\}$ for any closed set X.*

It is easy to see that if F is a phase isomorphism, then so is F^{-1}. As expected, we have:

Theorem 3.10. Let $\mathcal{M}_1 = (M_1, \cdot, \varepsilon, Cl_1)$ and $\mathcal{M}_2 = (M_2, \cdot, \varepsilon, Cl_2)$ be intuitionistic phase spaces and F be a phase isomorphism from $\mathcal{M}_1$ to $\mathcal{M}_2$. Given a valuation v_1 for $\mathcal{M}_1$, define a valuation v_2 for $\mathcal{M}_2$ by

$$\begin{array}{rcl} v_2(p) & := & F(v_1(p)); \\ v_2(\bot) & := & F(v_1(\bot)). \end{array}$$

Then, for any formula A, v_1 satisfies A ($\varepsilon \in v_1(A)$) if and only if v_2 satisfies A ($\varepsilon \in v_2(A)$).

Proof. It is readily seen that $v_2(A) = F(v_1(A))$ for any formula A. Now, observe that $\varepsilon \in v_1(A)$ iff $1_1 \subseteq v_1(A)$ iff $1_1 = v_1(A \,\&\, \mathbf{1})$. From this, we have:

$$\begin{array}{rcl} \varepsilon \in v_1(A) & \Longleftrightarrow & 1_1 = v_1(A \,\&\, \mathbf{1}) \\ & \Longleftrightarrow & F(1_1) = F(v_1(A \,\&\, \mathbf{1})) \\ & \Longleftrightarrow & 1_2 = v_2(A \,\&\, \mathbf{1}) \\ & \Longleftrightarrow & \varepsilon \in v_2(A). \end{array}$$

□

The above theorem in particular entails that there is an intuitionistic phase space which is *not* phase isomorphic to any classical phase space. Otherwise, any **ILL** formula satisfied in all classical phase models would be satisfied in all intuitionistic phase models too. However, we know that **LL** is not conservative over **ILL** (see Section 5).

4. Three Layered Representation

In this section, we prove our main theorem, the *three-layered representation theorem* (Theorem 4.1), which well summarizes the relationship between intuitionistic and classical phase semantics. According to the theorem, every intuitionistic phase space $\mathcal{M}$ is naturally endowed with a three-layered structure:

$$\mathcal{M} \sqsubseteq \mathcal{M}_q \sqsubseteq \mathcal{M}_c,$$

where $\mathcal{M}_q$ is quasi-classical and $\mathcal{M}_c$ is classical. Furthermore, there is a phase isomorphism from $\mathcal{M}_q$ to $\mathcal{M}$. The size of $\mathcal{M}_c$ is not unexpectedly large; in fact, it has a bound in terms of the size of $\mathcal{M}$ and a given co-basis for $\mathcal{M}$.

Theorem 4.1 (Three-Layered Representation). Let $\mathcal{M} = (M, \cdot, \varepsilon, Cl)$ be an intuitionistic phase space and let $\{G_\alpha\}_{\alpha \in \mathcal{A}}$ be a co-basis. Then there exist a quasi-classical phase space $\mathcal{M}_q = (M_q, \cdot, \varepsilon, Cl_q)$ and a classical phase space $\mathcal{M}_c = (M_c, \cdot, \varepsilon, \bot)$ satisfying the following;

(i) $\mathcal{M}$ is a subspace of $\mathcal{M}_q$, and $\mathcal{M}_q$ is a subspace of $\mathcal{M}_c$; specifically,

— for any $X \subseteq M$: $Cl(X) = X^{\bot\bot} \cap M$, and

— for any $X_q \subseteq M_q$: $Cl_q(X_q) = X_q^{\bot\bot}$.

(ii) There is a phase isomorphism from $\mathcal{M}_q$ to $\mathcal{M}$.

(iii) The cardinal number of M_c is bounded by $|M| \cdot |\mathcal{A}| \cdot \aleph_0$, with $|X|$ representing the cardinal number of X; in particular, whenever both M and $\mathcal{A}$ are countable, so is the resulting M_c.

Before proving the theorem, let us derive some consequences from it. The first one is the converse of Corollary 3.3:

Corollary 4.2. Every intuitionistic phase space is a subspace of a classical phase space.

Intuitionistic phase semantics is not just complete, but also *countably complete* for **ILL**; namely, for every unprovable formula A, there is a *countable* intuitionistic phase space which invalidates A. The next corollary ensures that this property remains true even though one takes into account the size of the outer classical phase space of which the intuitionistic phase space in question is a subspace.

Corollary 4.3. For any formula A of **ILL**, A is provable in **ILL** if and only if A is satisfied in every intuitionistic phase model $(\mathcal{M}, v)$ such that $\mathcal{M}$ is a subspace of a countable classical phase space.

Proof. By the property (iii) above and the second statement of Theorem 2.9. □

Finally, we have a completeness theorem for **ILL** with respect to the quasi-classical phase models:

Corollary 4.4. Every intuitionistic phase space is phase isomorphic to a quasi-classical phase space. As a consequence, a formula A is provable in **ILL** if and only if A is satisfied in every quasi-classical phase model.

Proof of Theorem 4.1. As a monoid M_c we take the set of all pairs of the form (x, Φ), where x is an element of M and Φ is a finite multiset of closed sets from $\{G_\alpha\}_{\alpha \in \mathcal{A}}$. The monoid product on M_c is defined as:

$$(x, \Phi) \cdot (y, \Xi) := (x \cdot y, \Phi \uplus \Xi),$$

where $\uplus$ represents the multiset union. It is readily seen that $(\varepsilon, \emptyset)$ is the neutral element of M_c. The construction ensures the requirement (iii) to be satisfied.

The original M is intended to be identified with the set $M \times \{\emptyset\}$, a submonoid of M_c. More precisely, we will use the following mapping:

$$\widetilde{X} := X \times \{\emptyset\},$$

which represents a natural isomorphism between M and $M \times \{\emptyset\}$. Accordingly, $\mathcal{M} = (M, \cdot, \varepsilon, Cl)$ is identified with $\widetilde{\mathcal{M}} = (\widetilde{M}, \cdot, \varepsilon, \widetilde{Cl})$, where $\widetilde{Cl}$ is defined by $\widetilde{Cl}(\widetilde{X}) := \widetilde{Cl(X)}$.

The desired $\bot \subseteq M_c$ is defined by

$$\begin{array}{rcl} \mathsf{0}_c & := & \{(x, \Phi) \mid x \in Cl(\emptyset), \Phi \text{ arbitrary}\}; \\ \bot & := & \{(x, \{G_\alpha\}) \mid x \in G_\alpha,\ \alpha \in \mathcal{A}\} \cup \mathsf{0}_c. \end{array}$$

This completes the definition of $\mathcal{M}_c = (M_c, \cdot, \varepsilon, \bot)$. One can easily verify that the set $\mathsf{0}_c$ is indeed the smallest closed set in $\mathcal{M}_c$. More specifically, we have:

Lemma 4.5. $\mathsf{0}_c \cdot M_c \subseteq \mathsf{0}_c$. Hence $\mathsf{0}_c \subseteq M_c^{\bot} = \emptyset^{\bot\bot}$.

Proof. Suppose that $(x, \Phi) \in \mathsf{0}_c$ and $(y, \Psi) \in M_c$. Since $x \in Cl(\emptyset)$, we have $x \cdot y \in Cl(\emptyset) \cdot Cl(\{y\}) \subseteq Cl(\emptyset \cdot \{y\}) = Cl(\emptyset)$. Hence $(x, \Phi) \cdot (y, \Psi) \in \mathsf{0}_c$. □

Let us now show that $\widetilde{\mathcal{M}}$ is a subspace of $\mathcal{M}_c$.

Lemma 4.6. For any $X \subseteq M$: $\widetilde{X}^{\bot\bot} \subseteq \widetilde{Cl(X)} \cup \mathsf{0}_c$.

Proof. First of all, observe that:

$$(\varepsilon, \{G_\alpha\}) \in \widetilde{X}^{\bot} \tag{1}$$

for any $G_\alpha \supseteq X$. Indeed, for any $(x, \emptyset) \in \widetilde{X}$ we have $x \in G_\alpha$. Therefore, $(\varepsilon, \{G_\alpha\}) \cdot (x, \emptyset) = (x, \{G_\alpha\}) \in \bot$.

Now, suppose that (z, Ψ) is an element of $\widetilde{X}^{\bot\bot}$, and $X \subseteq G_\alpha$ for some α. From (1), it follows that

$$(z, \Psi) \cdot (\varepsilon, \{G_\alpha\}) = (z, \Psi \uplus \{G_\alpha\}) \in \bot. \tag{2}$$

For empty Ψ, it means directly that $z \in G_\alpha$. Therefore,

$$z \in \bigcap_{\{\alpha \in \mathcal{A} \mid X \subseteq G_\alpha\}} G_\alpha = Cl(X),$$

and hence $(z, \Psi) \in \widetilde{Cl(X)}$.

For non-empty Ψ, (2) implies that $z \in Cl(\emptyset)$. Therefore, $(z, \Psi) \in \mathsf{0}_c$. □

Lemma 4.7. For any $X \subseteq M$: $\widetilde{Cl(X)} \cup \mathsf{0}_c \subseteq \widetilde{X}^{\bot\bot}$.

Proof. Since $\mathsf{0}_c \subseteq \emptyset^{\bot\bot} \subseteq \widetilde{X}^{\bot\bot}$ by Lemma 4.5, it suffices to show that $\widetilde{Cl(X)} \subseteq \widetilde{X}^{\bot\bot}$. Suppose that $(x, \emptyset) \in \widetilde{Cl(X)}$ and $(y, \Phi) \in \widetilde{X}^{\bot}$. Our goal is to show that $(y, \Phi) \cdot (x, \emptyset) = (y \cdot x, \Phi) \in \bot$. There are two cases to be considered.

— $\Phi = \{G_\alpha\}$ for some α. Then, for any $z \in X$,

$$(y, \{G_\alpha\}) \cdot (z, \emptyset) = (y \cdot z, \{G_\alpha\}) \in \bot\!\!\!\bot.$$

By the definition of $\bot\!\!\!\bot$, we obtain that $y{\cdot}z \in G_\alpha$. Therefore, $\{y\}{\cdot}X \subseteq G_\alpha$. By properties (Cl1)–(Cl4), $\{y\}{\cdot}Cl(X) \subseteq G_\alpha$. Since $x \in Cl(X)$, we have $y \cdot x \in G_\alpha$. Therefore, we conclude that:

$$(y, \Phi) \cdot (x, \emptyset) = (y \cdot x, \{G_\alpha\}) \in \bot\!\!\!\bot.$$

— Φ is not a singleton. For any $z \in X$,

$$(y, \Phi) \cdot (z, \emptyset) = (y \cdot z, \Phi) \in \bot\!\!\!\bot,$$

which means that $y \cdot z \in Cl(\emptyset)$. Therefore, $\{y\}{\cdot}X \subseteq Cl(\emptyset)$. By properties (Cl1)–(Cl4), $\{y\}{\cdot}Cl(X) \subseteq Cl(\emptyset)$. Since $x \in Cl(X)$, we have $y \cdot x \in Cl(\emptyset)$. Therefore, we conclude that:

$$(y, \Phi) \cdot (x, \emptyset) = (y \cdot x, \Phi) \in \mathsf{0}_c \subseteq \bot\!\!\!\bot.$$

□

Bringing together Lemmas 4.6 and 4.7, we obtain:

$$\widetilde{Cl(X)} \cup \mathsf{0}_c = \widetilde{X}^{\bot\!\!\!\bot\bot\!\!\!\bot} \tag{3}$$

for every $X \subseteq M$. Since $\widetilde{Cl(X)} \subseteq \widetilde{M}$ and $\mathsf{0}_c = \widetilde{Cl(\emptyset)} \cap \widetilde{M}$, we have:

$$\widetilde{Cl(X)} = \widetilde{X}^{\bot\!\!\!\bot\bot\!\!\!\bot} \cap \widetilde{M}. \tag{4}$$

This shows that $\widetilde{\mathcal{M}}$ is a subspace of $\sqsubseteq \mathcal{M}_c$. Now, define $\mathcal{M}_q = (M_q, \cdot, \varepsilon, Cl_q)$ by

$$\begin{aligned} M_q &:= \widetilde{M} \cup \mathsf{0}_c; \\ Cl_q(X_q) &:= (X_q)^{\bot\!\!\!\bot\bot\!\!\!\bot} \cap M_q \quad \text{for } X_q \subseteq M_q. \end{aligned}$$

Observe that M_q is a submonoid of M_c, because we have $\widetilde{M} \cdot \widetilde{M} \subseteq \widetilde{M}$, $\mathsf{0}_c \cdot \mathsf{0}_c \subseteq \mathsf{0}_c$ and $\widetilde{M} \cdot \mathsf{0}_c \subseteq \mathsf{0}_c$ by Lemma 4.5. Therefore, it is clear that $\widetilde{\mathcal{M}} \sqsubseteq \mathcal{M}_q \sqsubseteq \mathcal{M}_c$. Furthermore, we have:

Lemma 4.8. For any $X_q \subseteq M_q$: $X_q^{\bot\!\!\!\bot\bot\!\!\!\bot} = (X_q \cap \widetilde{M})^{\bot\!\!\!\bot\bot\!\!\!\bot}$.

Proof. Obviously, we have $(X_q \cap \widetilde{M})^{\bot\!\!\!\bot\bot\!\!\!\bot} \subseteq X_q^{\bot\!\!\!\bot\bot\!\!\!\bot}$. On the other hand,

$$\begin{aligned} X_q^{\bot\!\!\!\bot\bot\!\!\!\bot} &\subseteq ((X_q \cap \widetilde{M}) \cup \mathsf{0}_c)^{\bot\!\!\!\bot\bot\!\!\!\bot} \\ &\subseteq ((X_q \cap \widetilde{M}) \cup \emptyset^{\bot\!\!\!\bot\bot\!\!\!\bot})^{\bot\!\!\!\bot\bot\!\!\!\bot} \quad (by Lemma 4.5) \\ &= ((X_q \cap \widetilde{M}) \cup \emptyset)^{\bot\!\!\!\bot\bot\!\!\!\bot} \\ &= (X_q \cap \widetilde{M})^{\bot\!\!\!\bot\bot\!\!\!\bot}. \end{aligned}$$

□

Lemma 4.9. $Cl_q(X_q) = (X_q)^{\bot\!\!\!\bot\bot\!\!\!\bot}$ for any $X_q \subseteq M_q$. Therefore $\mathcal{M}_q$ is quasi-classical.

Proof. We have $M_q^{\perp\perp} = \widetilde{M}^{\perp\perp} = \widetilde{Cl(M)} \cup 0_c = \widetilde{M} \cup 0_c = M_q$, by Lemma 4.8 and (3). Therefore, our claim follows by Lemma 3.6. ☐

Finally, it remains to show that there is a phase isomorphism from $\mathcal{M}_q$ to $\widetilde{\mathcal{M}}$. For any $X_q \subseteq M_q$, let $F(X_q) := X_q \cap \widetilde{M}$.

Lemma 4.10. F is a bijection from $\mathcal{C}(\mathcal{M}_q)$ to $\mathcal{C}(\widetilde{\mathcal{M}})$.

Proof. For any closed set $X_q = X_q^{\perp\perp}$ in $\mathcal{M}_q$, we have

$$F(X_q) = X_q^{\perp\perp} \cap \widetilde{M} = (X_q \cap \widetilde{M})^{\perp\perp} \cap \widetilde{M} = \widetilde{Cl}(X_q \cap \widetilde{M}),$$

by Lemma 4.8 and (4). Hence F is a function from $\mathcal{C}(\mathcal{M}_q)$ to $\mathcal{C}(\widetilde{\mathcal{M}})$. Conversely, for any closed set $\widetilde{Y}$ in $\widetilde{\mathcal{M}}$, we have

$$\widetilde{Y} = \widetilde{Cl}(\widetilde{Y}) = \widetilde{Y}^{\perp\perp} \cap \widetilde{M} = F(\widetilde{Y}^{\perp\perp}),$$

by (4). Hence F is surjective. Finally, for any closed sets X_q, Y_q in $\mathcal{M}_q$, $F(X_q) = F(Y_q)$ implies $(X_q \cap \widetilde{M})^{\perp\perp} = (Y_q \cap \widetilde{M})^{\perp\perp}$. Hence $X_q = Y_q$ by Lemma 4.8. Therefore, F is injective. ☐

Lemma 4.11. F preserves the operations and constants given in Definition 2.4.

Proof. Since all cases are more or less similar, we consider just a few of them. In the sequel, the operations in $\mathcal{M}_q$ are indexed by q and X, Y range over $\mathcal{C}(\mathcal{M}_q)$.

First, note that $(X \cdot Y) \cap \widetilde{M} = (X \cap \widetilde{M}) \cdot (Y \cap \widetilde{M})$. Therefore,

$$\begin{aligned} F(X \otimes_q Y) &= (X \cdot Y)^{\perp\perp} \cap \widetilde{M} \\ &= ((X \cdot Y) \cap \widetilde{M})^{\perp\perp} \cap \widetilde{M} \quad \text{(by Lemma 4.8)} \\ &= ((X \cap \widetilde{M}) \cdot (Y \cap \widetilde{M}))^{\perp\perp} \cap \widetilde{M} \\ &= \widetilde{Cl}(F(X) \cdot F(Y)) \quad \text{(by (4))} \\ &= F(X) \otimes F(Y). \end{aligned}$$

Second, note that $Z \multimap_q Y = Z^{\perp\perp} \multimap_q Y$ for any $Z \subseteq M_q$ and $Y \in \mathcal{C}(\mathcal{M}_q)$. Therefore,

$$\begin{aligned} F(X \multimap_q Y) &= (X \multimap_q Y) \cap \widetilde{M} \\ &= ((X \cap \widetilde{M})^{\perp\perp} \multimap_q Y) \cap \widetilde{M} \quad \text{(by Lemma 4.8)} \\ &= ((X \cap \widetilde{M}) \multimap_q Y) \cap \widetilde{M} \\ &= (X \cap \widetilde{M}) \multimap (Y \cap \widetilde{M}) \\ &= F(X) \multimap F(Y). \end{aligned}$$

Finally, note that $I_q = \widetilde{I}$. Therefore,

$$\begin{aligned} F(!_q X) &= (X \cap I_q)^{\perp\perp} \cap \widetilde{M} \\ &= (X \cap I_q \cap \widetilde{M})^{\perp\perp} \cap \widetilde{M} \quad \text{(by Lemma 4.8)} \\ &= (X \cap \widetilde{M} \cap \widetilde{I})^{\perp\perp} \cap \widetilde{M} \\ &= \widetilde{Cl}(F(X) \cap \widetilde{I}) \quad \text{(by (4))} \\ &= !F(X). \end{aligned}$$

□

This establishes the requirement (ii). To get the precise form of Theorem 4.1 it remains to take into account our natural isomorphism between $\widetilde{M}$ and the original M.

Remark 4.12. *It can be read off from the definition of $\mathcal{M}_q$ that if $Cl(\emptyset) = \emptyset$, then $\mathcal{M}$ just coincides with $\mathcal{M}_q$. It follows that every intuitionistic phase space with $Cl(\emptyset) = \emptyset$ is quasi-classical.*

5. An Embedding of ILL into LL

Based on the semantic insights we have obtained so far, we now give a syntactic embedding of **ILL** into **LL**. Before doing so, we remark that **LL** is already conservative over **ILL** as far as the propositional formulas without **0** and $\perp$ are concerned:

Theorem 5.1 ((Schellinx 1991)). Let A be a formula of **ILL** which does not contain **0** or $\perp$. Then A is provable in **ILL** if and only if it is provable in **LL**.

Therefore, there is no need of translation for this fragment. On the other hand, in the presence of **0** or $\perp$ (or second order quantifiers), **LL** is not conservative over **ILL**, as witnessed by the following:

$$((p \multimap \perp) \multimap \perp) \multimap p;$$

$$(\top \multimap \mathbf{1}) \multimap ((p \multimap \mathbf{0}) \multimap \mathbf{0}) \multimap p.$$

These **ILL** formulas are provable in **LL** but not in **ILL**. Our embedding is intended to cover the full propositional logic **ILL** including **0** and $\perp$.

Definition 5.2. *Let p_0 be a distinguished propositional variable. we define*

$$\varphi(p_0) := (!p_0) \otimes !(p_0 \otimes p_0 \multimap p_0).$$

To each formula A of **ILL**, *we associate another formula A° of* **ILL** *as follows;*

$$\begin{array}{rcll} q^\circ & := & q \& p_0, & \textit{for a propositional variable } q \\ c^\circ & := & c \& p_0, & \textit{for } c \in \{\top, \perp\} \\ d^\circ & := & d, & \textit{for } d \in \{\mathbf{1}, \mathbf{0}\} \\ (A \multimap B)^\circ & := & (A^\circ \multimap B^\circ) \,\&\, p_0 & \\ (A \star B)^\circ & := & A^\circ \star B^\circ, & \textit{for } \star \in \{\otimes, \&, \oplus\} \\ (!A)^\circ & := & !A^\circ. & \end{array}$$

In addition, we define

$$(A_1, \dots, A_n)^\circ := A_1^\circ, \dots, A_n^\circ.$$

Lemma 5.3.

1 For every $n \geq 0$, the following is provable in **ILL**:

$$\varphi(p_0), \underbrace{p_0, \dots, p_0}_{n\ times} \vdash p_0.$$

2 The following is a derived rule in **ILL**:

$$\frac{\varphi(p_0), A_1 \,\&\, p_0, \dots, A_n \,\&\, p_0 \vdash B}{\varphi(p_0), A_1 \,\&\, p_0, \dots, A_n \,\&\, p_0 \vdash B \,\&\, p_0}.$$

Proof. 1. By induction on n. For $n = 0$ it follows from $!p_0 \vdash p_0$. The case $n = 1$ is immediate. For $n \geq 2$, one can derive $\varphi(p_0), p_0, p_0, \Gamma \vdash p_0$ from $\varphi(p_0), p_0, \Gamma \vdash p_0$ as follows:

$$\dfrac{\dfrac{\overline{p_0, p_0 \vdash p_0 \otimes p_0} \quad \varphi(p_0), p_0, \Gamma \vdash p_0}{\varphi(p_0), p_0 \otimes p_0 \multimap p_0, p_0, p_0, \Gamma \vdash p_0}}{\varphi(p_0), p_0, p_0, \Gamma \vdash p_0}.$$

2. From 1 one derives

$$\varphi(p_0), A_1 \,\&\, p_0, \dots, A_n \,\&\, p_0 \vdash p_0$$

by the $\&l$ rule. Hence the $\&r$ rule yields the desired sequent. □

Remark 5.4.

1 *Intuitively, the formula $\varphi(A)$ asserts that "(the interpretation of) A is a submonoid (of the domain of the phase space in question)." In fact, $\varphi(A)$ is satisfied in an intuitionistic phase model $(M, \cdot, \varepsilon, Cl, v)$ whenever $v(A)$ is a submonoid of M (cf. Lemma 5.10), although the converse is not always true. It is interesting to note that*

- *$\varphi(\mathbf{1})$ and $\varphi(\top)$ are provable in* **ILL**, *and*
- *$v(\mathbf{1})$ and $v(\top)$ are closed submonoids of M, the former being the smallest, and the latter being the largest.*

2 *The suffix $\&p_0$ may be viewed as S4-modality $\Box$ in the presence of the assumption $\varphi(p_0)$. The derived rule of Lemma 5.3(2) exactly corresponds to the right rule for $\Box$, while the $\&l$ rule corresponds to the left rule for $\Box$.*

Lemma 5.5. For every formula A of **ILL**, $\varphi(p_0) \vdash A^\circ \multimap (A^\circ \& p_0)$ and $\varphi(p_0) \vdash (A^\circ \& p_0) \multimap A^\circ$ are provable in **ILL**.

Proof. The latter is trivial, while the former is proved by induction on the complexity of A. □

Using this, the soundness of the embedding can be easily established.

Lemma 5.6. Let $\Gamma \vdash \Xi$ be a sequent of **ILL**. If $\Gamma \vdash \Xi$ is provable in **ILL**, then $\varphi(p_0), \Gamma^\circ \vdash \Xi^\circ$ is provable in **ILL**.

Proof. By induction on the length of the proof of $\Gamma \vdash \Xi$. For instance, when the last inference rule is $\multimap r$ of the form

$$\frac{A, C_1, \ldots, C_n \vdash B}{C_1, \ldots, C_n \vdash A \multimap B} \multimap r,$$

our claim can be shown as follows. By the induction hypothesis, we have

$$\varphi(p_0), A^\circ, C_1^\circ, \ldots, C_n^\circ \vdash B^\circ.$$

By the rules $\&l$ and $\multimap r$,

$$\varphi(p_0), C_1^\circ \mathbin{\&} p_0, \ldots, C_n^\circ \mathbin{\&} p_0 \vdash A^\circ \multimap B^\circ.$$

By Lemma 5.3,

$$\varphi(p_0), C_1^\circ \mathbin{\&} p_0, \ldots, C_n^\circ \mathbin{\&} p_0 \vdash (A^\circ \multimap B^\circ) \mathbin{\&} p_0.$$

Finally by Lemma 5.5, we obtain the desired sequent $\varphi(p_0), C_1^\circ, \ldots, C_n^\circ \vdash (A^\circ \multimap B^\circ) \mathbin{\&} p_0$. □

To prove the faithfulness of the embedding, we exploit the three layered structure of intuitionistic phase models. Since the general construction given in the previous section fails to satisfy the required properties (in particular, $v(\bot) = \boldsymbol{\bot} \cap M_q$), we explicitly build a canonical model endowed with a three layered structure.

First, define a classical phase space $\mathcal{M}_c = (M_c, \cdot, \varepsilon, \boldsymbol{\bot})$ by

$$\begin{array}{rcl} M_c & := & \{(\Gamma; \Delta) | \Gamma, \Delta \text{ are multisets of } \mathbf{ILL} \text{ formulas}\} \\ (\Gamma; \Delta) \cdot (\Sigma; \Pi) & := & (\Gamma, \Sigma; \Delta, \Pi) \\ \varepsilon & := & (\emptyset; \emptyset) \\ \mathsf{0}_c & := & \{(\Gamma; \Delta) | \Gamma \vdash \mathbf{0} \text{ is provable in } \mathbf{ILL}, \Delta \text{ arbitrary}\} \\ \boldsymbol{\bot} & := & \{(\Gamma; C) | \Gamma \vdash C \text{ is provable in } \mathbf{ILL}\} \cup \mathsf{0}_c. \end{array}$$

As in the proof of Theorem 2.9, the multiplicity of formulas of the form $!B$ is ignored. Note that if $(\Gamma; \Delta) \in \mathsf{0}_c$, then $(\Gamma, \Sigma; \Phi) \in \mathsf{0}_c$ for arbitrary Σ and Φ. In fact, if $\Gamma \vdash \mathbf{0}$ is provable, then so is $\Gamma, \Sigma \vdash \mathbf{0}$. Hence $(\Gamma, \Sigma; \Phi) \in \mathsf{0}_c$.

Second, define an intuitionistic phase space $\mathcal{M}_q = (M_q, \cdot, \varepsilon, Cl_q)$ by

$$\begin{array}{rcl} M & := & \{(\Gamma; \emptyset) | \Gamma \text{ is a multiset of } \mathbf{ILL} \text{ formulas}\} \\ M_q & := & M \cup \mathsf{0}_c \\ Cl_q(X) & := & X^{\boldsymbol{\bot}\boldsymbol{\bot}}. \end{array}$$

We then have:

Lemma 5.7. $\mathcal{M}_q$ is a quasi-classical subspace of $\mathcal{M}_c$.

Proof. $(M_q, \cdot, \varepsilon)$ is a submonoid of $(M_c, \cdot, \varepsilon)$, because $M \cdot M \subseteq M$, $M \cdot \mathsf{0}_c \subseteq \mathsf{0}_c$ and $\mathsf{0}_c \cdot \mathsf{0}_c \subseteq \mathsf{0}_c$.

To show that Cl_q is a well-defined closure operator on $(M_q, \cdot, \varepsilon)$, first observe that $(\emptyset; \top) \in M_q^{\boldsymbol{\bot}}$. Indeed, for any $(\Gamma; \emptyset) \in M$, we have

$$(\Gamma; \emptyset) \cdot (\emptyset; \top) = (\Gamma; \top) \in \boldsymbol{\bot},$$

and for any $(\Gamma;\Delta) \in 0_c$, we have

$$(\Gamma;\Delta)\cdot(\emptyset;\top) = (\Gamma;\Delta,\top) \in 0_c \subseteq \mathbf{\perp}.$$

Finally, we show that $M_q^{\mathbf{\perp}\mathbf{\perp}} \subseteq M_q$, that is sufficient to ensure that $X^{\mathbf{\perp}\mathbf{\perp}} \cap M_q = X^{\mathbf{\perp}\mathbf{\perp}}$. Let $(\Gamma;\Delta) \in M_q^{\mathbf{\perp}\mathbf{\perp}}$. If Δ is empty, $(\Gamma;\Delta) \in M_q$ holds trivially. Otherwise, ovserve that $(\Gamma;\Delta,\top) \in \mathbf{\perp}$, because $(\emptyset;\top) \in M_q^{\mathbf{\perp}}$. By the definition of $\mathbf{\perp}$, $(\Gamma;\Delta,\top) \in 0_c$. Hence $(\Gamma;\Delta) \in 0_c \subseteq M_q$. □

Define a canonical valuation v_q for $\mathcal{M}_q$ by

$$\begin{aligned} [\![A]\!] &:= \{(\Gamma;\emptyset) | \Gamma \vdash A \text{ is provable}\} \cup 0_c; \\ v_q(p) &:= [\![p]\!]; \\ v_q(\perp) &:= [\![\perp]\!]. \end{aligned}$$

One can show that $(\emptyset;A) \in [\![A]\!]^{\mathbf{\perp}}$, and thus $[\![A]\!]^{\mathbf{\perp}\mathbf{\perp}} \subseteq [\![A]\!]$ for any formula A. Therefore, v_q surely assigns a closed set to each propositional variable.

Furthermore, we have:

Lemma 5.8. $v_q(\perp) = \mathbf{\perp} \cap M_q$.

Proof. Observe the following:

$$\begin{aligned} (\Gamma;\emptyset) \in v_q(\perp) &\iff \Gamma \vdash \perp \text{ is provable} \\ &\iff \Gamma \vdash \quad \text{is provable} \\ &\iff (\Gamma;\emptyset) \in \mathbf{\perp}. \end{aligned}$$

□

On the other hand, the following can be proved by the standard method (see (Okada 1996)).

Lemma 5.9. For any formula A, $(A;\emptyset) \in v_q(A) \subseteq [\![A]\!]$. In particular, if $\varepsilon \in v_q(A)$, then $\vdash A$ is provable in **ILL**.

Define a valuation v_c for $\mathcal{M}_c$ by

$$\begin{aligned} v_c(p_0) &:= M_q; \\ v_c(r) &:= v_q(r), \end{aligned}$$

where r is a propositional variable other than p_0. Since $M_q = M_q^{\mathbf{\perp}\mathbf{\perp}}$ and $v_q(r) = v_q(r)^{\mathbf{\perp}\mathbf{\perp}}$, v_c is surely a valuation for $\mathcal{M}_c$.

Lemma 5.10.

1 $\varphi(p_0)$ is satisfied in $(M_c, \cdot, \varepsilon, \mathbf{\perp}, v_c)$.
2 $v_q(A) = v_c(A^\circ)$ for every formula A of **ILL**.

Proof. The first statement is easily verified based on the fact that M_q is a submonoid of M_c.

The second one is proved by induction on the complexity of A. For instance,

$$\begin{aligned} v_q(A \multimap B) &= v_q(A) \multimap_q v_q(B) = v_c(A^\circ) \multimap_q v_c(B^\circ) \\ &= (v_c(A^\circ) \multimap_c v_c(B^\circ)) \cap M_q = v_c((A^\circ \multimap B^\circ) \,\&\, p_0) = v_c((A \multimap B)^\circ). \end{aligned}$$

Here, it should be noted that $X \multimap_q Y = \{y \in M_q | \forall x \in X(x \cdot y \in Y)\}$ and $X \multimap_c Y = \{y \in M_c | \forall x \in X(x \cdot y \in Y)\}$. Hence $X \multimap_q Y = (X \multimap_c Y) \cap M_q$. We also have

$$v_q(\bot) = \boldsymbol{\bot} \cap M_q = v_c(\bot \,\&\, p_0) = v_c(\bot^\circ).$$

by Lemma 5.8. □

We finally end up with the main result of this section:

Theorem 5.11. A formula A of **ILL** is provable in **ILL** iff $\varphi(p_0) \multimap A^\circ$ is provable in **LL**.

Proof. Lemma 5.6 gives one direction. To show the other direction, suppose that $\varphi(p_0) \multimap A^\circ$ is provable in **LL**. By the soundness of classical phase semantics, $\varphi(p_0) \multimap A^\circ$ is satisfied in the classical phase model $(M_c, \cdot, \varepsilon, \boldsymbol{\bot}, v_c)$. Since $\varepsilon \in v_c(\varphi(p_0))$ by Lemma 5.10 (1), we have $\varepsilon \in v_c(A^\circ)$. Hence $\varepsilon \in v_q(A)$ by Lemma 5.10 (2). Therefore, A is provable in **ILL** by Lemma 5.9. □

6. Conclusion

LL and **ILL** are almost equivalent, though there are certainly some differences. To clarify the delicate relationship between them, we have made a phase semantic analysis. The outcome is a tight connection, perhaps a tighter one than expected, between intuitionistic and classical phase spaces. In particular, it has been shown that every intuitionistic phase space is a subspace of a classical one, and phase isomorphic to a quasi-classical one.

Besides the original motivation, these results give us a rather simple definition of intuitionistic closure operator as $Cl(X) = X^{\boldsymbol{\bot}\boldsymbol{\bot}} \cap M$. Remarkably, it does not involve any "big" (second-order, impredicative) quantifications in contrast to the closure operators in (Abrusci 1990; Okada 1996). We hope that our simpler definition will find an interesting application in the study of linear logic.

In Section 5, we have given an embedding of **ILL** into **LL**. It is interesting to observe that it is nothing but a linear analogue of Gödel's embedding of intuitionistic logic into modal logic S4 (in view of $\& p_0$ as a modal operator). A difference is that, while one has to introduce an extra modal operator $\Box$ in the case of intuitionistic logic, our modal operator $\& p_0$ is already definable in **ILL**. This is due to the rich expressive power of linear logic.

Finally, we leave the following questions as open problems:

1 Is every intuitionistic phase space quasi-classical? We conjecture it is not, but it seems difficult to give a concrete counterexample.
2 Does the embedding theorem (Theorem 5.11) extend to the second-order setting?
3 Do our main results hold for substructural logics other than linear logic?

Acknowledgements. We thank the anonymous referee for useful suggestions which help us improve the presentation of this paper.

References

Abramsky, S. (1993) Computational interpretations of linear logic. *Theoretical Computer Science*, 111, 3–57.

Abrusci, V. M. (1990) Sequent calculus for intuitionistic linear propositional logic. In P. P. Petkov (editor), *Mathematical Logic*, 223–242, Plenum Press.

Girard, J.-Y. (1987) Linear logic. *Theoretical Computer Science*, 50, 1–102.

Girard, J.-Y. (1995) Linear logic: Its syntax and semantics. In J.-Y. Girard, Y. Lafont, and L. Regnier (editors), *Advances in Linear Logic*, 1–42. Cambridge University Press.

Hodas, S. and Miller, D. (1994) Logic programming in a fragment of intuitionistic linear logic. *Information and Computation*, 110(2), 327–365.

Lafont, Y. (1988) The linear abstract machine. *Theoretical Computer Science*, 59, 157–180.

Lafont, Y. (1996) The undecidability of second order linear logic without exponentials. *Journal of Symbolic Logic*, 61(2), 541–548.

Lafont, Y. (1997) The finite model property for various fragments of linear logic. *Journal of Symbolic Logic*, 62(4), 1202–1208.

Lafont, Y. and Scedrov, A. (1996) The undecidability of second order multiplicative linear logic. *Information and Computation*, 125, 46–51.

Okada, M. (1996) Phase semantics for higher order completeness, cut-elimination and normalization proofs (extended abstract). In J.-Y. Girard, M. Okada, and A. Scedrov (editors), *ENTCS (Electronic Notes in Theoretical Computer Science) Vol.3: A Special Issue on the Linear Logic 96, Tokyo Meeting*. Elsevier-ENTCS.

Okada, M. (1999) Phase semantic cut-elimination and normalization proofs of first- and higher-order linear logic. *Theoretical Computer Science*, 227, 333–396.

Okada, M. (2002) A uniform semantic proof for cut-elimination and completeness of various first and higher order logics. *Theoretical Computer Science*, 281, 471–498.

Ono, H. Semantics for substructural logics. In K. Došen and P. Schröder-Heister (editors), *Substructural logics*, 259–291. Oxford University Press.

Okada, M. and Terui, K. (1999) The finite model property for various fragments of intuitionistic linear logic. *Journal of Symbolic Logic*, 64(2), 790–802.

Schellinx, H. (1991) Some syntactical observations on linear logic. *Journal of Logic and Computation*, 1(4), 537–559.

Terui, K. (2005) Which Structural Rules Admit Cut Elimination? — An Algebraic Criterion. Submitted.

Troelstra, A. S. (1992) *Lectures on Linear Logic*. CSLI Lecture Notes 29, Center for the Study of Language and Information, Stanford, California.