Intuitionistic phase spaces are almost classical

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Why study phase semantics? (1)

- Useful.
  - Semantic cut-elimination (Okada 96)
  - Undecidability of MALL2 (Lafont 96)
  - Decidability of LL/ILL with weakening/contraction via finite model property (Lafont 96, Okada-Terui 99)
  - Denotational completeness (Girard 98)
  - Verification of concurrent constraint programs (Fages, Ruet, Soliman 98)

Why study phase semantics? (2)

- Models not only provability, but also counter-proofs.
- Conter-proofs: possibly infinite trees, defined dually to proofs, not reaching axioms. E.g.,

$$\dfrac{\dfrac{\dfrac{\vdash \alpha, \beta^\perp}{\vdash \alpha \,\&\, \beta, \beta^\perp} \quad \vdash \alpha, \beta^\perp}{\vdash (\alpha \,\&\, \beta) \oplus \alpha, \beta^\perp}}{\vdash (\alpha \,\&\, \beta) \oplus \alpha, \alpha^\perp \,\&\, \beta^\perp}$$

Why study phase semantics? (3)

- Theorem: Any formula has either a proof or a counter-proof.
- Theorem (Terui 98): To each counter-proof π of a formula A, one can associate a phase model $\pi^\bullet$ such that $\pi^\bullet \not\models A$.

$$\begin{tikzcd}
\text{Proofs} \arrow[r, Leftrightarrow, "dual"] \arrow[d, Rightarrow] & \text{Counter-proofs} \arrow[d, Rightarrow] \\
\text{Cliques} \arrow[r, Leftrightarrow, "?"] & \text{Phase models}
\end{tikzcd}$$

Why study phase semantics? (4)

- Similar to classical logic proofs, from the viewpoint of computational complexity.
- Classical logic provability: coNP-complete (Cook 71).
- MLL provability: NP-complete (Kanovitch 92).
- Syntax-semantics twist between CL and MLL:

Classical logic		MLL
Proofs	$\approx$	Phase models
Boolean valuations	$\approx$	Proofs

Intuitionistic LL is almost classical

- Intuitionistic connectives: $\mathbf{1}, \perp, \top, \mathbf{0}, \otimes, \multimap, \oplus, \&, !$
- Theorem (Schellinx 91): A propositional formula A of ILL without $\mathbf{0}$ nor $\perp$ is provable in ILL iff it is provable in LL.
- Should be contrasted with the CL/IL case:

$$\text{CL} \vdash \quad ((\alpha \to \beta) \to \alpha) \to \alpha \quad \not\dashv \text{IL}$$

- Schellinx' Theorem fails for $\mathbf{0}$ or $\perp$:

$$\text{LL} \vdash \qquad \alpha^{\perp\perp} \multimap \alpha \qquad \not\dashv \text{ILL}$$
$$\text{LL} \vdash \quad (\top \multimap \mathbf{1}) \multimap \alpha^{\mathbf{00}} \multimap \alpha \quad \not\dashv \text{ILL}$$

- Syntactically, LL and ILL are almost equivalent. However, semantically, they look so different...

Phase semantics for LL

- Classical phase space: $(M, \perp)$ such that
  - M: a commutative monoid
  - $\perp \subseteq M$.
- $X \subseteq M$ is closed if $X^{\perp\perp} = X$.
- Formulas interpreted by closed sets:

$$
\begin{aligned}
(A \otimes B)^\bullet &= (A^\bullet \cdot B^\bullet)^{\perp\perp} \\
(A \oplus B)^\bullet &= (A^\bullet \cup B^\bullet)^{\perp\perp}, \quad \text{etc.}
\end{aligned}
$$

Phase semantics for ILL (1)

- Intuitionistic phase space: (M, Cl) such that
  - M: a commutative monoid
  - $Cl : \mathscr{P}(M) \longrightarrow \mathscr{P}(M)$.

(Cl1) $X \subseteq Cl(X)$,

(Cl2) $Cl(Cl(X)) \subseteq Cl(X)$,

(Cl3) $X \subseteq Y \Longrightarrow Cl(X) \subseteq Cl(Y)$,

(Cl4) $Cl(X) \cdot Cl(Y) \subseteq Cl(X \cdot Y)$.

- $X \subseteq M$ is closed if $Cl(X) = X$.

Phase semantics for ILL (2)

- Intuitionistic phase model: intuitionistic phase space (M, Cl) with a valuation of atoms and $\perp$ into the set of closed sets.
- Formulas interpreted by closed sets:

$$
\begin{array}{rclrcl}
\mathbf{1}^\bullet & = & Cl(\{1\}) & \mathbf{0}^\bullet & = & Cl(\emptyset) \\
\top^\bullet & = & M & \perp^\bullet & = & \text{prescribed by valuation} \\
(A \otimes B)^\bullet & = & Cl(A^\bullet \cdot B^\bullet) & (A \oplus B)^\bullet & = & Cl(A^\bullet \cup B^\bullet) \\
(A \& B)^\bullet & = & A^\bullet \cap B^\bullet & (A \multimap B)^\bullet & = & \{y \in M \mid \forall x \in A^\bullet (xy \in B^\bullet)\}
\end{array}
$$

$$(!A)^\bullet = Cl(A \cap \{x \in \mathbf{1} \mid xx = x\})$$

Phase semantics for ILL (3)

- A formula A is satisfied in $(M, Cl, \bullet)$ if $1 \in A^\bullet$.
- Theorem: A formula of ILL is provable iff it is satisfied in every intuitionistic phase model.
- Problem: Second-order even for propositional ILL!

Concrete closure operators

- (Abrusci 90): For some presupposed set $\mathscr{B} \subseteq \mathscr{P}(M)$,

$$Cl(X) = \bigcap_{Y \in \mathscr{B}} (X \multimap Y) \multimap Y.$$

- (Okada 96): For some set $\mathscr{C} \subseteq \mathscr{P}(M)$ closed under intersection and implication,

$$Cl(X) = \bigcap_{Y \in \mathscr{C}, X \subseteq Y} Y.$$

- Impredicative, whereas phase semantics for LL is entirely first-order and predicative.
- Is there a first-order, predicative characterization of intuitionistic phase semantics?

Subspaces

- A subspace of a classical phase space $(M_C, \perp)$ is (M_I, Cl) such that

$$
\begin{array}{rcl}
M_I & \subseteq & M_C \\
Cl(X) & \subseteq & X^{\perp\perp} \cap M_I, \text{ for } X \subseteq M_I.
\end{array}
$$

- Theorem: Every subspace of a classical phase space is an intuitionistic phase space.
- Q1: Is every intuitionistic phase space a subspace of a classical phase space?

Quasi-classical phase spaces

- A quasi-classical phase space (M_Q, Cl_Q) is a subspace of a classical phase space $(M_C, \perp)$ such that

$$Cl_Q(X) = X^{\perp\perp}, \quad \text{for } X \subseteq M_Q.$$

- Q2: Is every intuitionistic phase space quasi-classical?
- A phase-isomorphism from (M_1, Cl_1) to (M_2, Cl_2) is a bijection $\mathscr{F} : ClosedSets(M_1, Cl_1) \longrightarrow ClosedSets(M_2, Cl_2)$ that preserves $\otimes, \oplus, \&, \multimap, \mathbf{1}, \mathbf{0}, \top, !$.
- Remark: Phase-isomorphic spaces are identical as FL-algebra (Ono 94).
- Q2': Is every intuitionistic phase space phase-isomorphic to a quasi-classical one?

Answer to Q1 (1)

- Theorem: Every intuitionistic phase space is a subspace of a classical phase space.
- Proof: Given (M, Cl), define $(M_C, \perp)$ by:

$$
\begin{array}{rcl}
M_C & = & \{(x, \Phi) \mid x \in M,\ \Phi : \text{ a multiset of } Cl\text{-closed sets}\} \\
(x, \Phi) \cdot (y, \Psi) & = & (x \cdot y, \Phi \uplus \Psi) \\
\mathbf{0}_C & = & \{(x, \Phi) \mid x \in \mathbf{0}, \Phi : \text{ arbitrary}\} \\
\perp & = & \{(x, \{X\}) \mid X : \text{closed set in } (M, Cl),\ x \in X\} \cup \mathbf{0}_C
\end{array}
$$

- Original (M, Cl) is identified with $\{(x, \emptyset) \mid x \in M\} \subseteq M_C$.

Answer to Q1 (2)

- Lemma: For any $X \subseteq M$, $(1, \{Cl(X)\}) \in X^{\perp}$.
- Proof: If $(x, \emptyset) \in X$, then $(x, \emptyset) \cdot (1, \{Cl(X)\}) = (x, \{Cl(X)\}) \in \perp$.
- Lemma: For any $X \subseteq M$, $X^{\perp\perp} \subseteq Cl(X) \cup \mathbf{0}_C$.
- Proof: Two cases to be considered:
  - For any $(x, \emptyset) \in X^{\perp\perp}$, $(x, \{Cl(X)\}) = (x, \emptyset) \cdot (1, \{Cl(X)\}) \in \perp$. Hence $x \in Cl(X)$.
  - For any $(x, \Phi) \in X^{\perp\perp}$ with Φ non empty, $(x, \Phi \uplus \{Cl(X)\}) = (x, \Phi) \cdot (1, \{Cl(X)\}) \in \perp$. This means $x \in \mathbf{0}$. Hence $(x, \Phi) \in \mathbf{0}_C$.

Answer to Q1 (3)

- Lemma: For any $X \subseteq M$, $Cl(X) \cup \mathbf{0}_C \subseteq X^{\perp\perp}$.
- Proof: Omitted.
- Corollary: For any $X \subseteq M$, $Cl(X) = X^{\perp\perp} \cap M$.
- Proof: $Cl(X) = (Cl(X) \cup \mathbf{0}_C) \cap M = X^{\perp\perp} \cap M$.
- Remark: In general, M_C is uncountable. However, it can be made countable when the original (M, Cl) has a countable *basis*.

Answer to Q2'

- Theorem: Every intuitionistic phase space is phase-isomorphic to a quasi-classical one.
- Proof: Define (M_Q, Cl_Q) by

$$\begin{aligned} M_Q &= M \cup \mathbf{0}_C \\ Cl_Q(X) &= X^{\perp\perp} \cap M_Q \end{aligned}$$

- Then $Cl_Q(X) = X^{\perp\perp}$ for any $X \subseteq M_Q$.
- Projection $M_Q \longrightarrow M$ gives a phase-isomorphism.
- Corollary: ILL is complete with respect to the quasi-classical phase models.

Summary

- Theorem 1: Every intuitionistic phase space is a subspace of a classical phase space.
- First order, predicative semantics for propositional ILL.
- Theorem 2: Every intuitionistic phase space is phase-isomorphic to a quasi-classical one.
- Intuitionistic phase spaces are almost classical.
- Theorem 3: A syntactic embedding of full propositional ILL into LL (based on these semantic ideas).
- Open problems:
  1. Is every intuitionistic phase space quasi-classical?
  2. Second-order case?