On the use of phase semantics: to make sense out of nonsense

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What is phase semantics?

- A semantics complete for provability in Linear Logic (LL)
- Both for classical LL (Girard 87) and intuitionistic LL (Abrusci 90, Troelstra 92, Okada 96, etc.)
- Can be accommodated for all FL-systems (Ono 94)
- Sometimes considered as “abstract nonsense” (Girard) in the LL community
- Today we overview some uses of phase semantics.

Time Table

- Motivation (17.10 -)
- LL vs. Intuitionistic LL (17.17 -)
- Decidability and Finite Model Property (17.27 -)
- Cut Elimination (17.35 -)
- Criteria for Cut-Elimination (17.45 -)
- Interpolation and Amalgamation (17.52 -)
- Polarity and Focalization (18.00 -)
- Conclusion (- 18.10)

Why study phase semantics? (1)

- Phase semantics is
  - A useful tool to show various properties.
  - Source of inspiration.
  - Simple accounts for various phenomena.

Why study phase semantics? (1)

- Semantic cut-elimination (Okada 96)
- Undecidability of MALL2 (Lafont 96)
- Decidability of LL/ILL with weakening/contraction via finite model property (Lafont 96, Okada-Terui 99)
- Denotational completeness (Girard 98, Streicher ??, Ehrhard ??)
- Verification of concurrent constraint programs (Fages, Ruet, Soliman 98)
- Criteria for Cut-Elimination (Terui 07, Ciabattoni-Terui 06, 07)
- Interpolation/Amalgamation (Terui)
- Focalization

Why study phase semantics? (2)

- It models not only provability, but also counter-proofs.
- Counter-proofs: possibly infinite trees, defined dually to proofs, not reaching axioms. E.g.,

$$\dfrac{\dfrac{\dfrac{\vdash \alpha, \beta^\perp}{\vdash \alpha \,\&\, \beta, \beta^\perp} \quad \vdash \alpha, \beta^\perp}{\vdash (\alpha \,\&\, \beta) \oplus \alpha, \beta^\perp}}{\vdash (\alpha \,\&\, \beta) \oplus \alpha, \alpha^\perp \,\&\, \beta^\perp}$$

Why study phase semantics? (3)

- Fact: Any formula has either a proof or a counter-proof.
- Theorem (Terui 98): To each counter-proof π of a formula A, one can associate a phase model $\pi^\bullet$ such that $\pi^\bullet \not\models A$.

$$
\begin{array}{ccc}
\text{Proofs} & \stackrel{dual}{\Longleftrightarrow} & \text{Counter-proofs} \\
\Downarrow & & \Downarrow \\
\text{Denotations} & \stackrel{?}{\Longleftrightarrow} & \text{Phase models}
\end{array}
$$

Why study phase semantics? (4)

- Similar to classical logic proofs, from the viewpoint of computational complexity.
- Classical logic provability: coNP-complete (Cook 71).
- MLL provability: NP-complete (Kanovitch 92).
- Syntax-semantics twist between CL and MLL:

Classical logic		MLL
Proofs	$\approx$	Phase models
Boolean valuations	$\approx$	Proofs

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Intuitionistic LL is almost classical

- Intuitionistic connectives: $\mathbf{1}, \bot, \top, \mathbf{0}, \otimes, \multimap, \oplus, \&, !$
- Conservativity Theorem (Schellinx 91): A propositional formula A of ILL without $\mathbf{0}$ and $\bot$ is provable in ILL iff it is provable in LL.
- Should be contrasted with the CL/IL case. CL and IL are different even without negation and absurdity:

$$\text{CL} \vdash \quad ((\alpha \to \beta) \to \alpha) \to \alpha \quad \not\dashv \text{ IL}$$

- Syntactically, LL and ILL are almost equivalent. However, semantically, they look so different...

Phase semantics for LL

- Classical phase space: $(M, \perp)$ such that
  - M: a commutative monoid
  - $\perp \subseteq M$.
- $X^{\perp} = \{y \in M \mid \forall x \in X (xy \in \perp)\}$
- $X \subseteq M$ is closed if $X^{\perp\perp} = X$.
- Properties of $(\bullet)^{\perp\perp}$:

($\perp$ 1) $X \subseteq X^{\perp\perp}$,

($\perp$ 2) $X^{\perp\perp\perp\perp} \subseteq X^{\perp\perp}$,

($\perp$ 3) $X \subseteq Y \Longrightarrow X^{\perp\perp} \subseteq Y^{\perp\perp}$,

($\perp$ 4) $X^{\perp\perp} \bullet Y^{\perp\perp} \subseteq (X \bullet Y)^{\perp\perp}$.

ILL

- What is ILL?
- Syntax suggests it is a restriction of LL.
- Phase semantics seems to suggest it is a generalization of LL.

Phase semantics for ILL (1)

- Intuitionistic phase space: (M, Cl) such that
  - M: a commutative monoid
  - $Cl : \mathscr{P}(M) \longrightarrow \mathscr{P}(M)$ (closure operator):

(Cl1) $X \subseteq Cl(X)$,

(Cl2) $Cl(Cl(X)) \subseteq Cl(X)$,

(Cl3) $X \subseteq Y \Longrightarrow Cl(X) \subseteq Cl(Y)$,

(Cl4) $Cl(X) \bullet Cl(Y) \subseteq Cl(X \bullet Y)$.

- $X \subseteq M$ is closed if $Cl(X) = X$.

Phase semantics for ILL (2)

- Intuitionistic phase model: intuitionistic phase space (M, Cl) with a valuation of atoms and $\perp$ into the set of closed sets.
- Formulas interpreted by closed sets:

$$\begin{array}{rclrcl} \mathbf{1}^\bullet & = & Cl(\{1\}) & \mathbf{0}^\bullet & = & Cl(\emptyset) \\ \top^\bullet & = & M & \perp^\bullet & = & \text{prescribed by valuation} \end{array}$$

$$\begin{array}{rcl} (A \otimes B)^\bullet & = & Cl(A^\bullet \cdot B^\bullet) \\ (A \oplus B)^\bullet & = & Cl(A^\bullet \cup B^\bullet) \\ (A \& B)^\bullet & = & A^\bullet \cap B^\bullet \\ (A \multimap B)^\bullet & = & \{y \in M \mid \forall x \in A^\bullet (xy \in B^\bullet)\} \\ (!A)^\bullet & = & Cl(A \cap \{x \in \mathbf{1} \mid xx = x\}) \end{array}$$

Phase semantics for ILL (3)

- A formula A is satisfied in $(M, Cl, \bullet)$ if $1 \in A^{\bullet}$.
- Completeness Theorem: A formula of ILL is provable iff it is satisfied in every intuitionistic phase model.
- Trouble: Intuitionistic phase semantics requires of a second-order closure operator even for propositional ILL!

Concrete closure operators

- (Abrusci 90): For some presupposed set $\mathscr{B} \subseteq \mathscr{P}(M)$,

$$Cl(X) = \bigcap_{Y \in \mathscr{B}} (X \multimap Y) \multimap Y.$$

- (Okada 96): For some set $\mathscr{C} \subseteq \mathscr{P}(M)$ closed under intersection and implication,

$$Cl(X) = \bigcap_{Y \in \mathscr{C}, X \subseteq Y} Y.$$

- Again second-order!
- Question: Is it possible to give a first-order definition to intuitionistic phase semantics?

Subspaces

- (M_I, Cl) is a subspace of a classical phase space $(M_C, \perp)$ if

$$\begin{aligned} M_I &\subseteq M_C \\ Cl(X) &= X^{\perp\perp} \cap M_I, \text{ for } X \subseteq M_I. \end{aligned}$$

- Theorem: Every subspace of a classical phase space is an intuitionistic phase space.
- Proof:

$$\begin{aligned} X^{\perp\perp} \cdot Y^{\perp\perp} &\subseteq (X \cdot Y)^{\perp\perp} \\ M_I \bullet M_I &\subseteq M_I \\ (X^{\perp\perp} \cap M_I) \bullet (Y^{\perp\perp} \cap M) &\subseteq (X \bullet Y)^{\perp\perp} \cap M \end{aligned}$$

- Question: What about the converse?

Subspaces

- Theorem: Every intuitionistic phase space is a subspace of a classical phase space.
- Proof: Given (M, Cl), define $(M_C, \perp)$ by:

$$
\begin{aligned}
M_C &= \{(x, \Phi) \mid x \in M,\ \Phi : \text{ a multiset of } Cl\text{-closed sets}\} \\
(x, \Phi) \cdot (y, \Psi) &= (x \cdot y, \Phi \uplus \Psi) \\
\mathbf{0}_C &= \{(x, \Phi) \mid x \in \mathbf{0}, \Phi : \text{ arbitrary}\} \\
\perp &= \{(x, \{X\}) \mid X : \text{closed set in } (M, Cl),\ x \in X\} \cup \mathbf{0}_C
\end{aligned}
$$

- Original (M, Cl) is identified with $\{(x, \emptyset) \mid x \in M\} \subseteq M_C$.
- Remark: In general, M_C is uncountable. However, it can be made countable when the original (M, Cl) has a countable *basis*.

Syntactic embedding of ILL into LL (1)

- Recall Schellinx' Theorem: A propositional formula A of ILL without $\mathbf{0}$ and $\perp$ is provable in ILL iff it is provable in LL.
- It fails in the presence of $\mathbf{0}$ or $\perp$:

$$\begin{array}{llr} \mathsf{LL} \vdash & \alpha^{\perp\perp} \multimap \alpha & \not\dashv \mathsf{ILL} \\ \mathsf{LL} \vdash & (\top \multimap \mathbf{1}) \multimap \alpha^{\mathbf{00}} \multimap \alpha & \not\dashv \mathsf{ILL} \end{array}$$

- We need to translate ILL formulas into LL formulas.
- Idea: ILL is a "submonoid-restriction" of LL.

Syntactic embedding of ILL into LL (2)

- Fix a propositional variable M. Define

$$Monoid(M) \equiv !(\mathbf{1} \multimap M) \otimes !(M \otimes M \multimap M)$$

- Lemma: $(\;\bullet\;) \& M$ is an S4-modality for LL.

Syntactic embedding of ILL into LL (2)

- Define an embedding $^\circ : ILL \longrightarrow LL$:

$$
\begin{array}{rcll}
q^\circ & := & q\&M, & \text{for a propositional variable } q \\
c^\circ & := & c\&M, & \text{for } c \in \{\top, \bot\} \\
d^\circ & := & d, & \text{for } d \in \{\mathbf{1}, \mathbf{0}\} \\
(A \multimap B)^\circ & := & (A^\circ \multimap B^\circ) \,\&\, M & \\
(A \star B)^\circ & := & A^\circ \star B^\circ, & \text{for } \star \in \{\otimes, \&, \oplus\} \\
(!A)^\circ & := & !A^\circ. &
\end{array}
$$

- Theorem: A is provable in ILL iff $Monoid(M) \multimap A^\circ$ is provable in LL.

- Remark: Linear analogue of Gödel's translation of IL into S4. However, our modality $(\ \bullet\ ) \,\&\, M$ is definable in terms of LL connectives.

Morals

- Intuitionistic phase spaces = subspaces of classical phase spaces
- Reduction of second-order to first-order
- ILL is a submonoid restriction of LL, both semantically and syntactically.

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Decidability and FMP

- MALL is decidable, since proof search trees are always finite.
- MALL also satisfies the finite model property (FMP).
- Idea: To each finite counter-proof π of a formula A, one can associate a finite phase model $\pi^\bullet$ such that $\pi^\bullet \not\models A$.
- LL is undecidable, since proof search encodes machine computation (Kanovitch, Lafont).

Quiz

- What about the decidability/FMP of:
  1. LL + W?
  2. LL + C?
  3. LL + W + C?
  4. LL – E?
  5. LL – E + W?
  6. LL – E + C?
  7. LL – E + W + C?

Answers

- What about the decidability/FMP of:
  1. LL + W? yes (Kopylov)
  2. LL + C? yes (Okada-Terui)
  3. LL + W + C? yes (=IS4)
  4. LL – E? yes (Lafont)
  5. LL – E + W? yes (Lafont)
  6. LL – E + C? open
  7. LL – E + W + C? yes (=IS4)
- Except 6, phase semantics provides a simple, unified proof.

Decidability of LL +C

- (Kripke's argument) Define a partial order $\leq$ on sequents:

$$\vdash \Gamma, \Sigma \quad \leq \quad \vdash \Gamma, \Sigma, \Sigma$$

- Design a sequent calculus for LL+C such that:

  If $S_1 \leq S_2$ and S_2 has a proof, then S_1 has a shorter (or equivalent length) proof.

$$\frac{\vdash \Gamma, \Sigma, (A \otimes B), A \quad \vdash \Delta, \Sigma, (A \otimes B), B}{\vdash \Gamma, \Delta, \Sigma, A \otimes B}$$

- Observe: $\leq$ admits no infinite anti-chain

$$S_1 \not\leq S_2 \not\leq S_3 \not\leq \ldots$$

- Proof search tree is finite!

Phase semantics for decidability

- Phase spaces admit quotientation by logical congruence (Lafont)

Syntactic model	Quotientation	Finite model
(validity = provability)	$\Longrightarrow$	(validity = provability)

- Decidability/FMP of MALL –E + C is still open.

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Phase semantics for cut-elimination

- (Okada 92,96) builds an (intuitionistic) phase space in which validity implies cut-free provability.
- $\mathscr{F}^*$: free monoid generated by the formulas $\mathscr{F}$ of ILL
- The closure operator $Cl : \wp(\mathscr{F}^*) \longrightarrow \wp(\mathscr{F}^*)$ defined by:

$$\begin{array}{rcl} [\![\Gamma \Rightarrow C]\!] & = & \{\Sigma \mid \Sigma, \Gamma \Rightarrow C \text{ is cut-free provable in ILL}\}, \\ Y \text{ is closed} & \Leftrightarrow & Y = \bigcap_{i \in \Lambda} [\![\Gamma_i \Rightarrow C_i]\!] \\ Cl(X) & = & \text{the minimal closed set that includes } X \end{array}$$

Phase semantics for cut-elimination

- Meaning of closure operator:

  $\Sigma \in Cl(\{\Lambda\}) \iff$

  Whenever $\Gamma, \Lambda, \Delta \Rightarrow C$ is cut-free derivable, so is $\Gamma, \Sigma, \Delta \Rightarrow C$.

Phase semantics for cut-elimination

- Define $\alpha^\bullet = [\![\alpha]\!]$.
- Okada's Lemma: For every formula A,

$$A \in A^\bullet \subseteq [\![A]\!].$$

  In particular, if A is satisfied, then

$$1 \in A^\bullet \subseteq [\![A]\!].$$

  I.e., $\Rightarrow A$ is cut-free provable.
- Use left logical rules to show $A \in A^\bullet$.
- Use right logical rules to show $A^\bullet \subseteq [\![A]\!]$.

Phase semantics for cut-elimination

- Cut-elimination theorem: obtained by composition with soundness:

$$\frac{\vdash A \xrightarrow{\text{sound}} \models A \qquad \models A \xrightarrow{\text{complete}} \vdash_{cut-free} A}{\vdash A \longrightarrow \vdash_{cut-free} A}$$

- Object-CUTs are replaced with META-CUTs.
- Question: what happens if we eliminate META-CUTs?

Algebraic perspective on cut-elimination

- Residuated lattices: algebras associated to (noncommutative) IMALL.
- One can consider its intransitive variant.
- Algebraic understanding of cut-elimination: For any intransitive residuated lattice $\mathbf{A}$, there exists a (transitive) residuated lattice $\mathbf{A}^+$ and a surjective homomorphism

$$f : \mathbf{A}^+ \longrightarrow \mathbf{A}$$

- Intuition:

$$\begin{array}{rcl} \mathbf{A} & = & \text{cut-free proof system} \\ \mathbf{A}^+ & = & \text{Okada's phase space} \end{array}$$

- Cut-Elimination = Algebraic Completion

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When do you have cut-elimination?

- Cut-elimination holds when ILL is entended with a natural structural rule:
  - Contraction: $A \multimap A \otimes A$
- It fails when extended with an unnatural one:
  - Broccoli: $A \otimes A \multimap A$ $\qquad \dfrac{A, \Gamma \Rightarrow C}{A, A, \Gamma \Rightarrow C}$
- Broccoli is equivalent to Mingle: $A \otimes B \multimap A \oplus B$, which admits cut-elimination:

$$\frac{\Gamma, \Sigma_1, \Delta \Rightarrow C \quad \Gamma, \Sigma_2, \Delta \Rightarrow C}{\Gamma, \Sigma_1, \Sigma_2, \Delta \Rightarrow C}$$

- What is the general principle?

Broccoli and Mingle

- Broccoli does not admit cut elimination:

$$\dfrac{\dfrac{\overline{\beta \Rightarrow \beta}}{\beta \Rightarrow \alpha \vee \beta} \qquad \dfrac{\dfrac{\overline{\alpha \Rightarrow \alpha}}{\alpha \Rightarrow \alpha \vee \beta} \qquad \dfrac{\overline{\alpha \vee \beta \Rightarrow \alpha \vee \beta}}{\alpha \vee \beta, \alpha \vee \beta \Rightarrow \alpha \vee \beta}\,Exp}{\alpha, \alpha \vee \beta \Rightarrow \alpha \vee \beta}\,Cut}{\alpha, \beta \Rightarrow \alpha \vee \beta}\,Cut$$

- When Broccoli is replaced with Mingle, the above cut can be eliminated:

$$\dfrac{\dfrac{\overline{\alpha \Rightarrow \alpha}}{\alpha \Rightarrow \alpha \vee \beta} \qquad \dfrac{\overline{\beta \Rightarrow \beta}}{\beta \Rightarrow \alpha \vee \beta}}{\alpha, \beta \Rightarrow \alpha \vee \beta}\,Min$$

Girard's test

- Girard's test for naturality of structural rules (Meaning I, 1999).
  - A logical principle (structural rule) passes Girard's test if, in every phase space $(M, \perp)$, it propagates from atomic facts $\{x\}^{\perp\perp}$ to all facts $X^{\perp\perp}$.
- Contraction and Mingle pass it:

$$\forall x \in M.\ \{x\}^{\perp\perp} \multimap \{x \cdot x\}^{\perp\perp} \implies \forall X:\ \text{fact } (X \multimap X \otimes X).$$

- Broccoli fails:

$$\forall x \in M.\ \{x \cdot x\}^{\perp\perp} \multimap \{x\}^{\perp\perp} \not\Longrightarrow \forall X:\ \text{fact } (X \otimes X \multimap X).$$

- Relationship with cut elimination is hinted, but not proved.

Structural rules in general

- (Additive) structural rules:

$$\frac{\Gamma, \vec{X}_1, \Delta \Rightarrow C \quad \cdots \quad \Gamma, \vec{X}_n, \Delta \Rightarrow C}{\Gamma, \vec{X}_0, \Delta \Rightarrow C}\ R$$

such that $\{\vec{X}_1, \ldots, \vec{X}_n\} \subseteq \{\vec{X}_0\}$.

Correctness of Girard's test

- Cut elimination implies Girard's test.
- Build a phase space $\mathbf{P}(R) = (\mathscr{F}^*, Cl)$ based on:

$$[\![\Gamma \Rightarrow C]\!] = \{\Sigma \mid \Sigma, \Gamma \Rightarrow C \text{ is } \textcolor{red}{\text{cut-free}} \text{ provable in ILL+R}\}$$

- R holds on atomic facts $Cl(\{A\})$.
- We want to argue

$$\frac{\vdash A \overset{\textcolor{red}{\text{sound}}}{\longrightarrow} MODEL(ILL+R) \models A \qquad \mathbf{P}(R) \models A \overset{\textcolor{red}{\text{complete}}}{\longrightarrow} \vdash_{cut-free}}{\vdash A \longrightarrow \vdash_{cut-free} A}$$

- It works only when $\mathbf{P}(R) \in MODEL(ILL+R)$, i.e. R passes Girard's test.

Correctness of Girard's test

- Theorem: For any structural rule R, IMALL+R admits cut-elimination iff R passes Girard's test.
- See (Terui 07) for more general results.
- (Ciabattoni-Terui 06) gives CE-criteria for logics with arbitrary (strange) connectives.

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Intermezzo: 短歌

- 瀬をはやみ　岩にせかるる　滝川の

  われてもすえに　あはんとぞおもふ

  崇徳院 (1119 – 1164)

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  崇徳院 (1119 – 1164)

The stream goes down a mountain rapidly
Even if it runs into a rock
And is forced to divide into two
They will join together in the end.

Sutoku-in (1119 – 1164)

Interporation and Amalgamation

- $\mathscr{F}(X)$: formulas over variables $\alpha, \beta, \dots \in X$.
- Craig interpolation property: Suppose $A \in \mathscr{F}(X)$ and $B \in \mathscr{F}(Y)$. If $A \multimap B$ is provable, then there is $I \in \mathscr{F}(X \cap Y)$ such that

$$A \multimap I \qquad I \multimap B$$

  are provable.
- True for most natural logics including LL and ILL.
- Open for NCILL.

Residuated lattices

- A residuated lattice is an algebra

$$\mathbf{P} = \langle P, \wedge, \vee, \otimes, \rightarrow, \leftarrow, 1 \rangle$$

  where

  1. $\langle P, \wedge, \vee \rangle$ is a lattice.
  2. $\langle P, \otimes, 1 \rangle$ is a monoid.
  3. For any $x, y, z \in P$,

$$x \otimes y \leq z \iff x \leq z \rightarrow y \iff y \leq x \leftarrow z.$$

- NCIMALL : Residuated lattices = $\mathbf{IL}$: Heyting algebras
- For any substructural logic L over NCIMALL, $\mathscr{V}(L)$ is the class of residuated lattices validating all theorems of L.

Interporation and Amalgamation

- A class $\mathscr{V}$ of algebras has the amalgamation property if for any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathscr{V}$ with embeddings

$$f_1 : \mathbf{A} \longrightarrow \mathbf{B} \qquad f_2 : \mathbf{A} \longrightarrow \mathbf{C},$$

there are $\mathbf{D} \in \mathscr{V}$ and embeddings

$$g_1 : \mathbf{B} \longrightarrow \mathbf{D} \qquad g_2 : \mathbf{C} \longrightarrow \mathbf{D}$$

such that $g_1 \circ f_1 = g_2 \circ f_2$.

Interporation and Amalgamation

- A class $\mathscr{V}$ of algebras has the amalgamation property if for any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathscr{V}$ with embeddings

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such that $g_1 \circ f_1 = g_2 \circ f_2$.

- I would call it Sutoku-in property.

Interporation and Amalgamation

- Theorem (Maximova): For any superintuitionstic logic L,

$$L \text{ admits interpolation} \iff \mathscr{V}(L) \text{ admits amalgamation}$$

- Extended for logics over IMALL by Wroński, Kowalski, Galatos-Ono, etc.

Interporation and Amalgamation

- Usually proved by a chain of arguments ...
- Why are they equivalent?
- Phase semantics gives a uniform account (for some special cases)

Phase space for interpolation

- $\Gamma \Rightarrow_M C$ (Γ maeharaly implies C) iff for any partition $\Gamma_1; \Gamma_2 = \Gamma$ such that $\Gamma_1 \in \mathscr{F}^*(X)$ and $\Gamma_2, A \in \mathscr{F}^*(Y)$, there is $I \in \mathscr{F}(X \cap Y)$ such that

$$\Gamma_1 \Rightarrow I \quad \text{and} \quad I, \Gamma_2 \Rightarrow A.$$

  The same holds with X and Y exchanged.

- Build a phase space $(\mathscr{F}^*, Cl)$ based on:

$$[\![\Gamma \Rightarrow C]\!] = \{\Sigma \mid \Sigma, \Gamma \Rightarrow_M C \text{ holds in ILL}\}$$

Phase space for interpolation

- Define $\alpha^\bullet = [\![\alpha]\!]$
- Main Lemma: $A^\bullet = [\![A]\!]$ for all formula A
- Corollary: If $A \Rightarrow B$ is provable, then there is an interpolant I such that $A \Rightarrow I$ and $I \Rightarrow B$.

$$\frac{A \Rightarrow B \quad \xrightarrow{\text{sound}} \quad \models A \Rightarrow B \qquad \models A \Rightarrow B \quad \xrightarrow{\text{complete}} \quad A \Rightarrow_M B}{A \Rightarrow B \quad \longrightarrow \quad A \Rightarrow_M B}$$

Phase space for amalgamation

- We assume $\mathbf{A} = \mathbf{B} \cap \mathbf{C}$
- Consider $((B \cup C)^*, \circ, \varepsilon)$, the free commutative monoid generated by $B \cup C$.
- Given $d \in B \cup C$, define $[\![d]\!] \subseteq (B \cup C)^*$ by: $t \in [\![d]\!]$ holds $\iff$
  1. if $d \in B$, then for any partition $t_1 \circ t_2 = t$ with $t_1 \in C^*$ and $t_2 \in B^*$, there is $i \in A$ such that
  $$t_1 \leq_{\mathbf{C}} i \qquad i \cdot t_2 \leq_{\mathbf{B}} d.$$
  2. The same holds with B and C exchanged.

Phase space for amalgamation

- We have

$$\begin{aligned} [\![\ \ ]\!] &: \mathbf{B} \longrightarrow \mathbf{D} = ((B \cup C)^*, Cl) \\ [\![\ \ ]\!] &: \mathbf{C} \longrightarrow \mathbf{D} \end{aligned}$$

- The $g_1 \circ f_1 = g_2 \circ f_2$ requirement trivially holds.
- Main Lemma: $[\![\ \ ]\!]$ is an embedding.

Polarity

- Positive and negative connectives have nice characterizations in terms of phase semantics.
- Positive connectives $\mathbf{1}, \mathbf{0}, \otimes, \oplus$ propagate closure operator:

$$
\begin{array}{rcll}
X^{\perp\perp} \bullet Y^{\perp\perp} & \subseteq & (X \bullet Y)^{\perp\perp} & = & X \otimes Y \\
X^{\perp\perp} \cup Y^{\perp\perp} & \subseteq & (X \cup Y)^{\perp\perp} & = & X \oplus Y \\
\{1\} & \subseteq & \{1\}^{\perp\perp} & = & \mathbf{1} \\
\emptyset & \subseteq & \emptyset^{\perp\perp} & = & \mathbf{0}
\end{array}
$$

- Negative connectives $\top, \perp, \&, ⅋$ distribute closure operator:

$$
\begin{array}{rcl}
(X \cap Y)^{\perp\perp} & \subseteq & X^{\perp\perp} \cap Y^{\perp\perp} \\
\top^{\perp\perp} & \subseteq & \top
\end{array}
$$

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- LL vs. Intuitionistic LL (17.17 -)
- Decidability and Finite Model Property (17.27 -)
- Cut Elimination (17.35 -)
- Criteria for Cut-Elimination (17.45 -)
- Interpolation and Amalgamation (17.52 -)
- Polarity and Focalization (18.00 -)
- Conclusion (- 18.10)

Invertibility and Focalization

- Invertibility and Focalization can be stated as phase semantic propertiies.
- Invertibility: negative connectives do not need closure operator:

$$(X \cap Y)^{\perp\perp} = X \cap Y$$

  if X, Y are closed.
- Focalization: closure operator between two positive connectives can be omitted:

$$X \otimes (Y \oplus Z) = (X \bullet (Y \cup Z)^{\perp\perp})^{\perp\perp} = (X \bullet (Y \cup Z))^{\perp\perp}$$

- Question: Do they have applications to syntax?

Phase space for focalization

- Build a phase space $(\mathscr{F}^*_+, Cl)$ based on:

$$[\![\Gamma]\!] = \{\Sigma \mid \vdash \Sigma, \Gamma \text{ has a cut-free focusing proof in MALL}\}$$

- Define $\alpha^\bullet = [\![\alpha]\!]$.
- Main Lemma: $A^\bullet \subseteq [\![A]\!]$ for any formula A.
- Crucial use of phase-semantic focalization property!
- Corollary: Every provable formula in MALL has a focusing proof.

$$\frac{\vdash A \overset{\text{sound}}{\longrightarrow} \models A \qquad \models A \overset{\text{complete}}{\longrightarrow} \vdash_{focusing} A}{\vdash A \longrightarrow \vdash_{focusing} A}$$

Conclusion

- Phase semantics gives simplified accounts for:
  LL vs ILL, Decidability, Cut Elimination, Positive vs Negative
- By considering various condition sets $[\![\ \ ]\!]$, one can uniformly prove various properties.
  - $[\![A]\!] = \{\Sigma \mid \Sigma \Rightarrow A$ is cut-free provable in ILL$\}$ for C.-E.
  - $[\![A]\!] = \{\Sigma \mid \Sigma \Rightarrow_M A$ holds in ILL$\}$ for Interpolation
  - $[\![d]\!] = \{t \mid t \geq_M d$ holds for $\mathbf{B}, \mathbf{C}\}$ for Amalgamation
  - $[\![A]\!] = \{\Sigma \mid\ \vdash \Sigma, A$ has a cut-free focusing proof$\}$ for Foc.
- Export to universal algebra
- By extending the above argument to denotational phase spaces (Girard, Streicher), one would be able to prove similar properties for denotational semantics.