On space efficiency of Krivine's abstract machine and Hyland-Ong games

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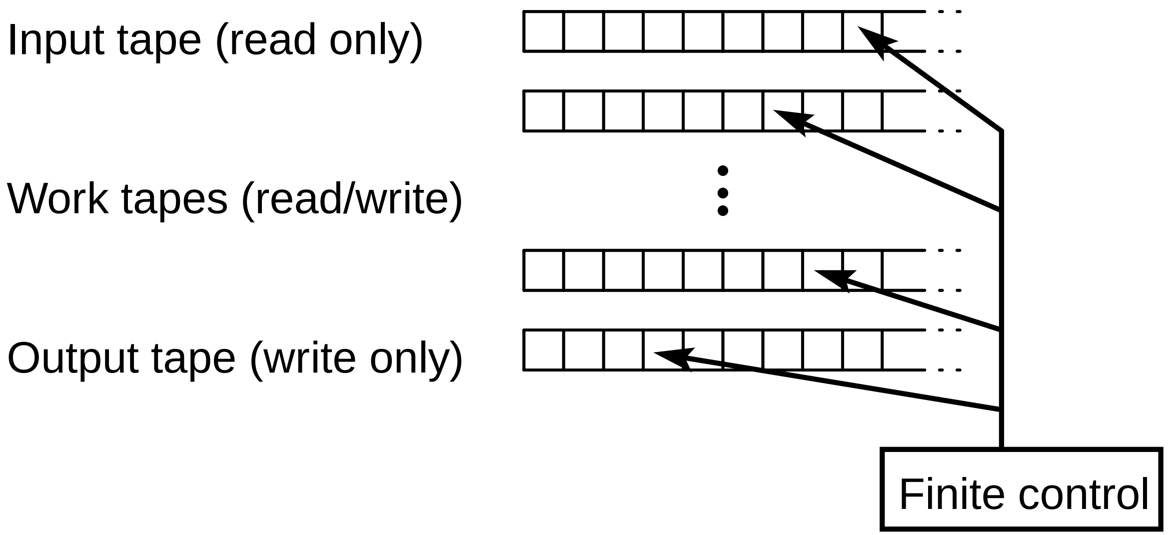
ICC

- Intensional Computational Complexity (ICC)
  - algorithms (eg. λ terms)
  - evaluation mechanisms (eg. β-reduction)

Space sensitive Turing machines



- Space = the num of cells used on worktapes.
- The input and output can be significantly larger than the space: If M works in space $f(n)$ then the output is of size $O(2^{f(n)})$.
- Given two Turing machines M, N, we want to compose them. How?

Space sensitive Turing machines

- Functional composition is not good:
  If M, N work in space $f(n), g(n)$, then $M; N$ works in space $O(f(n) + g(2^{f(n)}) + 2^{f(n)})$.
- In particular, logspace functions do not compose.
- Composition must be interactive: $M \rightleftarrows N$ works in $O(f(n) + g(2^{f(n)}))$.
- From functional to interactive computation!

Towards a logical theory of computational complexity

- Shortcomings of Turing Machines:
  - Poor data structures (only tapes)
  - Poor control mechanisms (only transitions)
  - Ad hoc constructions: no canonical way of composition
  - Results in lack of beautiful mathematical theory
- Logic and lambda calculus
  - Rich data structures and controls
  - Canonical composition $T \circ U = \lambda x.T(U(x))$
  - Hope to make complexity theory more mathematical

Space in lambda calculus

- Composition does not preserve good space bounds when using β-reduction, because β-reduction is not interactive!
- Interactive evaluation mechanisms
  - Geometry of Interaction
  - Krivine's abstract machine (KAM)
  - Game semantics

KAM

- KAM implements weak head linear reduction:

$$(\lambda \cdots \lambda x \cdots .x\vec{U}) \cdots T \cdots \Longrightarrow (\lambda \cdots \lambda x \cdots .T\vec{U}) \cdots T \cdots$$

- GoI with "jumps" under $A \Rightarrow B = !(A \multimap B)$ (Danos-Regnier 94)
- Syntax:

Terms	t	$::=$	$x \mid \lambda x.t_1 \mid t_1 t_2$
Environments	ρ	$::=$	$[x_1 \mapsto t_1\rho_1,\ \ldots,\ x_n \mapsto t_n\rho_n]$
Closures	$t\rho$		
Stacks	π	$::=$	$t_1\rho_1 : \cdots : t_n\rho_n$
States	S	$::=$	$(t\rho,\ \pi)$

KAM

- Initial state: $(t[], \mathsf{nil})$
- Transitions

$$
\begin{array}{rcll}
(x\rho,\ \pi) & \longrightarrow & (\rho(x),\ \pi) & \text{if } x \in Dom(\rho) \\
((tu)\rho,\ \pi) & \longrightarrow & (t\rho,\ u\rho : \pi) & \\
((\lambda x.t)\rho,\ u\rho' : \pi) & \longrightarrow & (t\rho[x \mapsto u\rho'],\ \pi) &
\end{array}
$$

- Terminations

$$
\begin{array}{lcl}
(x\rho,\ \pi) \text{ with } x \notin Dom(\rho) & — & \text{Success with output } x \\
((\lambda x.t)\rho,\ \mathsf{nil}) & — & \text{Failure}
\end{array}
$$

- Not space efficient: ρ contains a lot of redundant bindings.

Optimized KAM

- Optimized transitions

$$
\begin{array}{rcll}
(x\rho,\ \pi) & \longrightarrow & (\rho(x),\ \pi) & \text{if } x \in Dom(\rho) \\
((tx)\rho,\ \pi) & \longrightarrow & (t(\rho|_t),\ \rho(x) : \pi) & \text{if } x \in Dom(\rho) \\
((tu)\rho,\ \pi) & \longrightarrow & (t{\color{red}(\rho|_t)},\ u{\color{red}(\rho|_u)} : \pi) & \text{otherwise} \\
((\lambda x.t)\rho,\ u\rho' : \pi) & \longrightarrow & (t\rho[x \mapsto u\rho'],\ \pi) &
\end{array}
$$

- $\rho|_t$ is a restriction of ρ to the free variables of t.

Example

Suppose $(K[],\ \mathsf{nil}) \longrightarrow^* (\lambda z.z,\ \mathsf{nil})$, $\quad \overline{2} = \lambda f x.f(fx)$.

$(\overline{2} K c,\ \mathsf{nil})$

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$(f[f \mapsto K],\ c)$

$(K,\ c)$

Size of a state

- Subterm property: if $(t[], \mathsf{nil}) \longrightarrow^* S$, all terms occurring in S are subterms of t.
- Allows us to think of each state as made of pointers rather than actual terms.
- $\sharp S$ = the number of pointers in S:

$$\begin{array}{rcll} \sharp\rho & = & \sharp\rho_1 + \cdots + \sharp\rho_n + n & \text{if } \rho = [x_1 \mapsto t_1\rho_1, \ldots, x_n \mapsto t_n\rho_n \\ \sharp\pi & = & \sharp\rho_1 + \cdots + \sharp\rho_n + n & \text{if } t_1\rho_1 : \cdots : t_n\rho_n \\ \sharp(t\rho,\ \pi) & = & \sharp\rho + \sharp\pi + 1 & \end{array}$$

- In particular, $\sharp(t[], \mathsf{nil}) = 1$.
- KAM evaluates t with n pointers if for any intermediate state $(t[],\ \mathsf{nil}) \longrightarrow^* S, \quad \sharp S \leq n$.

TM space-efficiently simulates KAM

- Naively, TM requires of $\log n$ overhead to simulate KAM:

  if KAM evaluates t with n pointers, then $M(t)$ works in space $O(n \cdot \log |t|)$.

- We are interested in terms td, where t (program) is fixed while d (input) is varying.

- Consider a huge alphabet

$$\Sigma = \{u \mid u \text{ is a subterm occ. of } t\} \cup \{0, 1\}$$

TM space-efficiently simulates KAM

- t satisfies the input condition if in any intermediate state $(td[], \mathsf{nil}) \longrightarrow^* S$, the number of pointers pointing subterms of d is constant (independent of d).
- Analogy: the num of heads on input tape is fixed.
- Theorem: There is a TM M which simulates KAM linearly: if KAM evaluates td with $f(|d|)$ pointers, t satisfies the input condition and $f(x) \geq \log x$, then $M(t)$ works in space $O(f(|d|))$.

Data structures

- Booleans: $\mathbf{t} = \lambda xy.x$, $\mathbf{f} = \lambda xy.y$
- Scott numerals: $\mathbf{w}$ for $w \in \{0,1\}^*$

$$
\begin{aligned}
\varepsilon &= \lambda x_\varepsilon x_0 x_1 . x_\varepsilon \\
\mathbf{0} \cdot \mathbf{w} &= \lambda x_\varepsilon x_0 x_1 . x_0 \mathbf{w} \\
\mathbf{1} \cdot \mathbf{w} &= \lambda x_\varepsilon x_0 x_1 . x_1 \mathbf{w}
\end{aligned}
$$

- Purely linear. Successor, predecessor and conditional are also linear. Useful for worktapes.
- What are read-only/write-only tapes?
- They should be interactive entities (analogy: Book vs Dialogue)

Data structures

- Databases: For $d \in \{0,1\}^{2^n}$, define a term $\mathbf{d}$: Word $\to$ Bool s.t. Given $w \in \{0,1\}^n$, $\mathbf{d w}$ returns the wth bit of d.

$$
\begin{array}{rcll}
\mathbf{d} & = & \lambda w.w\mathbf{i}\Omega\Omega & \text{if } d = i \in \{0,1\} \\
& = & \lambda w.w\Omega\mathbf{d}_0\mathbf{d}_1 & \text{if } d_1 = i_{2^n} \cdots i_{2^{n-1}+1}, \\
& & & d_0 = i_{2^{n-1}} \cdots i_1,\ d = d_1 d_0
\end{array}
$$

KAM space-efficiently simulates TM

- Theorem: For any TM M, there is a term t_M s.t.
  - Given $d \in \{0,1\}^*$, $t_M \mathbf{d} *_0 *_1$ returns $*_0$ or $*_1$ according to the output of $M(d)$.
  - When M works in space $f(n) \geq \log n$, KAM evaluates $t_M \mathbf{d} *_0 *_1$ with $O(f(n))$ pointers.
- Now the space hierarchy theorem implies

  There is no TM M that evaluates $t_M \mathbf{d}$ in space $O(f(n)^\epsilon)$ with $\epsilon < 1$.
- Corollary: There is no evaluator that is uniformly (i.e. for all terms) and significantly (i.e. super-linearly) more space-efficient than KAM.

Composition

- Let M, N be transducer TMs which work in space $f(n), g(n)$ and produce outputs of size $2^{f(n)}$ and $2^{g(n)}$.

$$\frac{\begin{array}{rcl} t_M & : & (\mathsf{Word} \to \mathsf{Bool}) \longrightarrow (\mathsf{Word} \to \mathsf{Bool}) \\ t_N & : & (\mathsf{Word} \to \mathsf{Bool}) \longrightarrow (\mathsf{Word} \to \mathsf{Bool}) \end{array}}{t_M \circ t_N \; : \; (\mathsf{Word} \to \mathsf{Bool}) \longrightarrow (\mathsf{Word} \to \mathsf{Bool})}$$

- $(t_N \circ t_M)\mathbf{dw} *_0 *_1$ returns the wth bit of $N \circ M(d)$.
- KAM evaluates it with $O(f(|d|) + g(|d'|))$ pointers, where d' is the output of $M(d)$.
- KAM + canonical composition simulates the interactive composition of TMs!

Moral

	Turing Machines	λ-calculus
Composition	non canonical	canonical
Evaluation	canonical	non canonical

- In λ-calculus, TIME-SPACE tradeoff shows up at the stage of evaluation:
  - KAM is space-efficient
  - CBV seems to be time-efficient (cf. Dal Lago-Martini).
- Work to be done: identify a subclass of λ-terms (via a type system) on which space bounds are well preserved by KAM composition.

Warning

- There is no uniformly more efficient evaluator than KAM.
- KAM is like a student Bob with score

Math	Science	Grammer	Latin	Music
1	1	1	1	10

- Although there is no uniformly better student than Bob, this does not mean Bob is the best student.

From KAM to HO games

- Empirically, KAM is good at "tall" terms:

$$t_n = (\lambda f x. f^n(x))(\lambda y.y)*$$

KAM evaluates t_n with $O(1)$ pointers.

- KAM is not good at "wide" terms:

$$u_n = (\lambda x_1 \cdots \lambda x_n . x_n) *_1 \cdots *_n$$

KAM evaluates u_n with $O(n)$ pointers.

- For simply typed terms with unbounded width and fixed rank, HO games seem to be more efficient.

HO games

- A fully abstract model of PCF (Hyland-Ong 00). Successfully applied to various programming languages.
- Useful for compositional and higher-order model checking (Ghica-Mckusker 00, Ong 04, etc.)

$$
\begin{array}{ccc}
\text{Types} & = & \text{Arenas} \\
\text{Terms} & = & \text{Strategies} \\
\text{Computation} & = & \text{play}
\end{array}
$$

- Play = interactive composition of two strategies
- Composition is effective (cf. Ong 04). Corresponds to Pointer Abstract Machine (Danos-Herbelin-Regnier 96).

Ranks and complexity

- The rank of a type: $rank(\iota) = 0$,
  $rank(A \to B) = max(rank(A) + 1, rank(B))$
- A term is at rank n if all of its subterms have rank n types.

Rank	Quantative	Qualitative
1	boolean formulas $\neg b = \lambda xy.b^{\iota\to\iota\to\iota}yx$	(open) addition
2	boolean circuits $(\lambda x.F(x,\ldots,x))^{bool\to bool}G(y)$	multiplication $\overline{n} \circ \overline{m}$
3	QBFs $\forall x.F(x) =$ $(\lambda X.X(\mathbf{0}) \wedge X(\mathbf{1}))^{(bool\to bool)\to bool}\lambda x.F(x)$	exponentiation $(\overline{n})^{int\to int}\overline{m}$

Ranks and complexity

- Normalization problem:

  Given two terms t, u where u is normal,
  does t reduce to u?

- (Schubert 2001) shows that this problem is
  1. in $DTIME(n \log n)$ for rank 1 terms
  2. P-complete for rank 2 terms
  3. PSPACE-complete for rank 3 terms

- Hardness is easy. Membership is difficult, since β-reduction does not work.

- Schubert uses graph rewriting. We use HO games.

HO games for simple types

- Arena for type A:

Moves	=	occurrences of atomic types in A
P-moves	=	negative occurrences
O-moves	=	positive occurrences
Initial move	=	$tail(A)$
$tail(B) \vdash tail(C)$	where	C is an immediate subformula of B

- Legal play: a PO-alternating pointing sequence like

$$s = m_0 m_1 \cdots m_j \cdots m_i \cdots$$

  - m_0 is the initial O-move
  - each m_i ($1 \leq i$) is justified by a preceding move m_j ($m_j \vdash m_i$, $0 \leq j < i$).

Let's play the game!

- P-view: a legal play in which any O-move is justfied by the move immediately before.
- Given an arbitrary legal play s, a P-view $\lceil s \rceil$ can be extracted.
- Innocent strategy (as view function): a P-deterministic tree made of P-ending P-views.
- Every closed normal term $t : A$ can be interpreted by an innocent strategy.
- HO-dialogue: Given innocent strategies $\sigma : A \to B$ and $\tau : A$,
  1. Start with the initial move $m_0 = tail(B)$
  2. Expand an odd length play s to $s \cdot m$ such that $\lceil s \rceil \cdot m \in \sigma$
  3. Expand an even length play $m_0 \cdot s$ to $m_0 \cdot s \cdot m$ such that $\lceil s \rceil \cdot m \in \tau$

A lemma in game semantics

- Difficulty: HO game is history sensitive. But recording the whole history leads to TIME = SPACE.
- Lemma (taught by P. Boude): For any legal play satisfying P- and O-visibility

$$s = \cdots m_1 \cdots\cdots m_2 \cdots\cdots$$

if m_2 justifies no moves, it never happens that

$$s = \cdots m_1 \cdots\cdots m_2 \cdots\cdots$$

- So one can safely contract s to

$$s' = \cdots m_1 m_2 \cdots$$

and continue the play.

Complexity of Rank 3 games

- Theorem: Given a term t at rank 3 (which is an applicative combination of closed normal terms), the length of play can be kepth within $O(|t|)$.
- Corollary: One can determine the head variable of $nf(t)$ in polynomial space via HO games.
- Work to be done: What about rank ≥ 4? Do we have a hierarchy result as in Kristiansen's talk?

Conclusion

- Lambda calculus admits a canonical composition $M \circ N$
- KAM is optimal for simulating TM.
- Composition works nicely with KAM.
- (KAM allows natural programming eg. boolean matrix closure)
- HO games seem to be good at least for rank bounded terms.
- Good understanding of game semantics leads to clever space management. Game semantics may implicitly explain complexity.