# A flaw in R.B. White's article "The consistency of the axiom of comprehension in the infinite-valued predicate logic of Łukasiewicz."

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There seems to be a flaw in the paper "The consistency of the axiom of comprehension in the infinite-valued predicate logic of Łukasiewicz" (White 1979). Although the flaw lies in a seemingly innocent statement "clearly every proof can be converted to a pure proof" (page 518), it seems fatal to me, and indeed all my attempts to correct it have failed. Below, I give some background, recall the concept of purity, give a counterexample to White's argument, and finally explain how a naive attempt to correct the flaw fails.

**1. Background.** White's proof of consistency rests on the normalization argument for a peculiar natural deduction system $G$. It involves a $\tau$-term $\tau_x^n(A(x))$ for every one-variable formula $A(a)$ and a natural number $n$. $\tau$-terms are used to manipulate quantifiers, as typically seen in the inference rule ($ui$) below; we also mention the dual rule ($-ui$):

$$\frac{A(\tau)}{(x)A(x)}\ ui \qquad \frac{-B(t)}{-(x)B(x)}\ -ui$$

In the rule ($ui$), $A(a)$ contains at most one free variable and $\tau = \tau_x^n A(x)$ is a $\tau$-term associated to $A(x)$. One says that $\tau$ is *used* in the rule ($ui$). There is no other constraint on application of the rule ($ui$) such as the eigenvariable condition in the ordinary natural deduction system. Lack of any "eigenvariable condition" (or rather "eigen-$\tau$-term condition") seems inevitable; we will see later that imposing an eigenvariable-like condition makes Theorem 5 problematic.

[0]This note was written in the autumn of 2010. Following some suggestions by colleagues, I plan to submit this as a journal article after minor revision.

In the rule $(-ui)$, the only constraint on $t$ is that it is closed; in particular, $t$ may contain any $\tau$-terms.

He then considers the following *u-reduction* rule:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\Pi\\ A(\tau)\end{array}}{(x)A(x)}\,ui \qquad \dfrac{\begin{array}{c}\vdots\,\Pi_1\\ -A(t)\end{array}}{-(x)A(x)}\,-ui}{f} \qquad \longmapsto \qquad \dfrac{\begin{array}{c}\vdots\,\Pi_t^\tau\\ A(t)\end{array} \qquad \begin{array}{c}\vdots\,\Pi_1\\ -A(t)\end{array}}{f}$$

where $\Pi_t^\tau$ is obtained from $\Pi$ by replacing all occurrences of $\tau$ by $t$. Such a replacement cannot always be done; it presupposes that $\tau$ is not used in $\Pi$ (Theorem 3, page 520), namely, the whole proof is *pure* (see below). However, I claim that one cannot always assume that a given proof is pure, despite that White assumes so. Hence the normalization argument breaks down.

**2. Purity.** Let us have a closer look at his argument. On page 518, White calls a proof *pure* if each $\tau$-term is used at most once on any branch of the proof, namely the rule $(ui)$

$$\frac{A(\tau_x^n A(x))}{(x)A(x)}\,ui$$

with the identical $\tau$-term does not appear more than once on any branch.

He then claims that "clearly every proof can be converted to a pure proof" by applying the following purification procedure (see page 519):

$$\begin{array}{c}\vdots\,\Pi_1\\ \dfrac{A(\tau_x^n A(x))}{(x)A(x)}\,ui\\ \vdots\,\Pi_2\\ \dfrac{A(\tau_x^n A(x))}{(x)A(x)}\,ui\end{array} \qquad \longmapsto \qquad \begin{array}{c}\vdots\,\Pi_3\\ \dfrac{A(\tau_x^k A(x))}{(x)A(x)}\,ui\\ \vdots\,\Pi_2\\ \dfrac{A(\tau_x^n A(x))}{(x)A(x)}\,ui\end{array}$$

where $k$ is a fresh natural number and $\Pi_3$ is obtained from $\Pi_1$ by changing all occurrences of $\tau_x^n(A(x))$ to $\tau_x^k(A(x))$. Henceforth he assumes that *all the proofs are pure.*

However, things are not so easy because the transformation may affect the $\tau$-terms occurring in the *assumptions* of $\Pi_1$, which may be *discharged* in $\Pi_2$. Indeed, it seems that the proof below provides a counterexample to his claim (and to the validity of the u-reduction rule).

**3. A counterexample.** Let $s(x)$ be a term of Cantor-Łukasiewicz naive set theory that contains one free variable $x$. One can for instance take $s(x) = \{x\} = \{y|x = y\}$. Let $N$ be a closed term that satisfies

(*) $\forall x.\ x \in N \leftrightarrow (x \in N \to s(x) \in N)$.

For instance $N = \{x|\top\}$ will do. A more nontrivial term can be obtained by applying the fixed point theorem to the formula $x \in y \to s(x) \in y$.

Since (*) is closed, Theorem 1 (page 516) ensures that it is provable in system $G$. Let $\tau = \tau^0_x(x \in N)$, that is the $\tau$-term of index 0 associated to formula $x \in N$. We then have the following proof in system $G$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\tau \in N]^1}{(x)x \in N}\,ui}{s(\tau) \in N}\,ue}{\tau \in N \to s(\tau) \in N}\,1 \qquad \dfrac{(*)}{(\tau \in N \to s(\tau) \in N) \to \tau \in N}}{\tau \in N}}{(x)x \in N}\,ui$$

where $(ue)$ is a rule easily derived from $(-ui)$, and $[\tau \in N]^1$ indicates that the assumption $\tau \in N$ is discharged at the implication introduction rule marked by 1. By lack of any "eigenvariable condition," the occurrence of $\tau$ in the discharged assumption is not problematic. (At this point, we can already see how weird his proof system is, since it does not seem true that $(x)x \in N$ holds for any $N$ satisfying (*). But let us continue.)

The above proof is not pure since the identical $\tau$ is used twice on the left branch. It cannot however be converted to a pure proof. Indeed, if we apply White's purification procedure, we obtain an ill-formed proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\tau' \in N]^1}{(x)x \in N}\,ui}{s(\tau) \in N}}{\tau' \in N \to s(\tau) \in N}\,1 \qquad \dfrac{(*)}{(\tau' \in N \to s(\tau) \in N) \to \tau \in N}\,???}{\tau \in N}\,???}{(x)x \in N}\,ui$$

where $\tau = \tau^0_x(x \in N)$ as before and $\tau' = \tau^1_x(x \in N)$. Namely, the formula $(\tau' \in N \to s(\tau) \in N) \to \tau \in N$ cannot be derived from (*) due to the mismatch between $\tau$ and $\tau'$.

As a result, we cannot directly apply the u-reduction rule to a proof of the form (if there is any):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\tau \in N]^1}{(x)x \in N}\,ui}{s(\tau) \in N}\,ue}{\tau \in N \to s(\tau) \in N}\,1 \qquad \dfrac{(*)}{(\tau \in N \to s(\tau) \in N) \to \tau \in N}}{\tau \in N}}{(x)x \in N}\,ui \qquad \dfrac{\begin{matrix}\vdots\\ -(t \in N)\end{matrix}}{-(x)x \in N}}{f} \qquad \longmapsto \qquad ???$$

Hence his normalization argument gets stuck.

**4. A (failed) attempt of correction.** An obvious way to fix this problem would be to impose the "eigen-$\tau$ condition" on the rule $(ui)$: one can form a proof

$$\dfrac{\begin{matrix}\vdots\;\Pi\\ A(\tau)\end{matrix}}{(x)A(x)}\; ui$$

*only when* $\tau$ does not appear in any of the open assumptions of $\Pi$.

It looks fine at a first glance since it is a natural condition, preserves all closed theorems of CŁ (the proofs of axioms H5, H7, H8, H9 on pages 517 and 518 can be suitably modified), and circumvents the above difficulty. However, it causes another difficulty in Theorem 5, that is the most crucial part of the paper.

In the proof of Theorem 5, White claims that an instance of the rule

$$\dfrac{\begin{matrix}\vdots\;\Pi\\ A\vee B\end{matrix}}{B\vee A}\;\vee$$

can be removed from any weakly-normal proof of contradiction, provided that $\Pi$ is *categorical*, namely without open assumptions. The core of his argument lies in the following (quite ingenious!) proof transformation:

$$\begin{matrix}\vdots\;\Pi_1\\ \dfrac{A\vee B}{B\vee A}\;\vee\\ \vdots\;\Pi_2\\ f\end{matrix}
\qquad\longmapsto\qquad
\begin{matrix}
\dfrac{\dfrac{\dfrac{\begin{matrix}\dfrac{[A]^1}{B\vee A}\\ \vdots\;\Pi_2\\ \bot\end{matrix}}{B}}{A\to B}\;1 \qquad \begin{matrix}\vdots\;\Pi_1\\ A\vee B\end{matrix}}{\dfrac{B}{B\vee A}}\\ \vdots\;\Pi_2\\ f\end{matrix}$$

where $\Pi_1$ is categorical.

However, this reduction does not preserve the "eigen-$\tau$ condition" above. For suppose that $A = A(\tau)$ contains a $\tau$-term which is used by an instance of the rule $(ui)$ in $\Pi_2$. Then after the above reduction, we obtain an instance of $(ui)$ which comes from an open assumption $A(\tau)$ (note that it is closed at the $\to$-introduction rule marked by 1). Hence imposing the eigen-$\tau$ condition is not a good idea...

**Remark.** In 1987, White published a proof of the consistency of naive set theory over the logic BCK. Since BCK is subsumed by Łukasiewicz logic, the result is merely a corollary of the consistency of CŁ. Nevertheless, he did not even cite the earlier 1979 paper in the 1987 one. This suggests a possibility that White himself was not satisfied by his proof in the 1979 paper.