On the asymptotic expansion of the Kashaev invariant of the 5_2 knot

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Abstract

We give a presentation of the asymptotic expansion of the Kashaev invariant of the 5_2 knot. As the volume conjecture states, the leading term of the expansion presents the hyperbolic volume and the Chern-Simons invariant of the complement of the 5_2 knot. Further, we obtain a method to compute the full Poincare asymptotics to all orders of the Kashaev invariant of the 5_2 knot.

1 Introduction

In [16, 17] Kashaev defined the Kashaev invariant $\langle L \rangle_N \in \mathbb{C}$ of a link L for $N = 2, 3, \cdots$ by using the quantum dilogarithm. In [18] he conjectured that, for any hyperbolic link L,

$$2\pi \cdot \lim_{N\to\infty} \frac{\log \langle L \rangle_N}{N} \;=\; \mathrm{vol}(S^3 - L),$$

where "vol" denotes the hyperbolic volume, and gave evidence for the conjecture for the figure-eight knot, the 5_2 knot and the 6_1 knot, by formal calculations. In 1999, H. Murakami and J. Murakami [24] proved that the Kashaev invariant $\langle L \rangle_N$ of any link L is equal to the N-colored Jones polynomial $J_N(L; e^{2\pi\sqrt{-1}/N})$ of L evaluated at $e^{2\pi\sqrt{-1}/N}$, where $J_N(L; q)$ denotes the invariant obtained as the quantum invariant of links associated with the N-dimensional irreducible representation of the quantum group $U_q(sl_2)$. Further, as an extension of Kashaev's conjecture, they conjectured that, for any knot K,

$$2\pi \cdot \lim_{N\to\infty} \frac{\log |J_N(K; e^{2\pi\sqrt{-1}/N})|}{N} \;=\; \mathrm{vol}(S^3 - K),$$

where "vol" in this formula denotes the simplicial volume (normalized by multiplying by the hyperbolic volume of the regular ideal tetrahedron). This is called *the volume conjecture*. As a complexification of the volume conjecture, it is conjectured in [25] that, for a hyperbolic link L,

$$2\pi\sqrt{-1} \cdot \lim_{N\to\infty} \frac{\log J_N(L; e^{2\pi\sqrt{-1}/N})}{N} \;=\; \mathrm{cs}(S^3 - L) + \sqrt{-1}\,\mathrm{vol}(S^3 - L) \qquad (1)$$

for an appropriate choice of a branch of the logarithm, where "cs" denotes the Chern-Simons invariant. Furthermore, it is conjectured [11] (see also [3, 12, 46]) from the viewpoint of the $\mathrm{SL}(2, \mathbb{C})$ Chern-Simons theory that the asymptotic expansion of $J_N(K; e^{2\pi\sqrt{-1}/k})$

of a hyperbolic knot K as $N, k \to \infty$ fixing $u = N/k$ is given by the following form,

$$J_N(K; e^{2\pi\sqrt{-1}/k}) \underset{\substack{N,k\to\infty \\ u=N/k:\ \text{fixed}}}{\sim} e^{N\varsigma} N^{3/2}\, \omega \cdot \Big(1 + \sum_{i=1}^{\infty} \kappa_i \cdot \big(\frac{2\pi\sqrt{-1}}{N}\big)^i\Big) \tag{2}$$

for some scalars $\varsigma, \omega, \kappa_i$ depending on K and u, though they do not discuss the Jones polynomial in the Chern-Simons theory in the case of vanishing quantum dimension, which is discussed in [38]. These conjectures look interesting in the sense that they make a bridge between quantum topology and hyperbolic geometry, and it suggests the existence of a future theory between them.

An approach toward a proof of the volume conjecture has been known, which is due to Kashaev [18], Thurston [35] and Yokota [42]. We briefly review this approach, as follows. By definition, the Kashaev invariant $\langle K \rangle_N$ (which we review in Section 2.1) of a knot K is given by a sum of fractions whose denominators are product of copies of $(q)_n$, where $q = \exp(2\pi\sqrt{-1}/N)$ and $(x)_n = (1-x)(1-x^2)\cdots(1-x^n)$; for example, the Kashaev invariant $\langle 5_2 \rangle_N$ of the 5_2 knot is given by

$$\langle 5_2 \rangle_N = \sum_{\substack{0 \le i,j \\ i+j<N}} \frac{N^3 q}{(q)_{i+j}(q)_{N-i-j-1}(q)_j(\overline{q})_j(\overline{q})_i},$$

as shown in (7). By the approximation $(q)_n \sim \exp\big(\frac{N}{2\pi\sqrt{-1}}\big(\mathrm{Li}_2(1) - \mathrm{Li}_2(e^{2\pi n\sqrt{-1}/N})\big)\big)$, we expect the following approximation,

$$\langle 5_2 \rangle_N \underset{?}{\sim} N^3 q \sum_{\substack{0 \le i,j \\ i+j<N}} \exp\Big(\frac{N}{2\pi\sqrt{-1}} \check{V}(e^{2\pi\sqrt{-1}\,i/N}, e^{2\pi\sqrt{-1}\,j/N})\Big),$$

where we put

$$\check{V}(x,y) = \mathrm{Li}_2(xy) + \mathrm{Li}_2\big(\frac{1}{xy}\big) + \mathrm{Li}_2(y) - \mathrm{Li}_2\big(\frac{1}{y}\big) - \mathrm{Li}_2\big(\frac{1}{x}\big) - \mathrm{Li}_2(1).$$

Further, by formally replacing the sum with an integral putting $t = i/N$ and $s = j/N$, we expect that

$$\langle 5_2 \rangle_N \underset{??}{\sim} N^5 q \int_{\substack{0 \le t,s \\ t+s \le 1}} \exp\Big(\frac{N}{2\pi\sqrt{-1}} \check{V}(e^{2\pi\sqrt{-1}\,t}, e^{2\pi\sqrt{-1}\,s})\Big)\, dt\, ds.$$

Furthermore, by applying the saddle point method, we expect that the asymptotic behavior might be described by a critical value of $\check{V}$. Yokota [42] showed that a critical value of such a function $\check{V}$ is given by the hyperbolic volume of the knot complement. There are problems justifying this series of arguments.

The volume conjecture has been rigorously proved for some particular knots and links such as torus knots [19] (see also [5][1]), the figure-eight knot (by Ekholm, see also [1][2]),

[1] A detailed asymptotic expansion of the colored Jones polynomial for torus knots is given in [5].

[2] A detailed proof of the volume conjecture for the figure-eight knot was given in [1] and the term $N^{3/2}$ in (2) was also verified there.

Whitehead doubles of $(2,p)$-torus knots [47], positive iterated torus knots [37], the 5_2 knot [20], and some links [9, 15, 36, 37, 41, 47]; for details see *e.g.* [22]. They (except for the 5_2 knot) have particular properties; for example, the simplicial volumes of the complements of torus knots are 0, and a critical point of $\check{V}$ for the figure-eight knot is on the original contour of the integral. The volume conjecture for them has been proved by using such particular properties. In particular, the 5_2 knot is of a general case; the volume conjecture for the 5_2 knot has been proved by Kashaev and Yokota [20] by presenting the above mentioned sum by the residue of a certain integral.

The aim of this paper is to give a presentation of the asymptotic expansion of the Kashaev invariant $\langle 5_2 \rangle_N$ of the 5_2 knot rigorously (Theorem 1.1 below). Let y_0 be the unique solution with positive imaginary part of $(y-1)^3 = y$,

$$y_0 = 0.3376410213... + \sqrt{-1}\cdot 0.5622795120... \ .$$

We put

$$\begin{aligned} x_0 &= 1 - \frac{1}{y_0}\,, \\ \varsigma &= \frac{1}{2\pi\sqrt{-1}}\Big(\mathrm{Li}_2(x_0 y_0) + \mathrm{Li}_2\big(\frac{1}{x_0 y_0}\big) + \mathrm{Li}_2(y_0) - \mathrm{Li}_2\big(\frac{1}{y_0}\big) - \mathrm{Li}_2\big(\frac{1}{x_0}\big) - \frac{\pi^2}{6}\Big) \\ &= 0.4501096100... + \sqrt{-1}\cdot 0.4813049796...\ , \\ \omega &= e^{\pi\sqrt{-1}/4}\big(\frac{y_0-1}{2y_0+1}\big)^{1/2} = 0.0901905774... - \sqrt{-1}\cdot 0.6499757866... \ . \end{aligned}$$

Then, we have

Theorem 1.1. *The asymptotic expansion of the Kashaev invariant $\langle 5_2 \rangle_N$ of the 5_2 knot is given by the following form,*

$$\langle 5_2 \rangle_N \;=\; e^{N\varsigma} N^{3/2}\,\omega\cdot\Big(1 + \sum_{i=1}^{d} \kappa_i \cdot \big(\frac{2\pi\sqrt{-1}}{N}\big)^i \;+ O\big(\frac{1}{N^{d+1}}\big)\Big),$$

for any d, where κ_i is some constant given by a polynomial in y_0 with rational coefficients; in particular, κ_1 is given by

$$\begin{aligned} \kappa_1 \;&=\; \frac{1}{(2y_0+1)^3}\Big(\frac{37}{4}y_0^2 - \frac{31}{6}y_0 + \frac{35}{8}\Big) + 1 \;=\; \frac{1}{12696}(1650y_0^2 - 3498y_0 + 2197) + 1 \\ &=\; 1.0537470859... - \sqrt{-1}\cdot 0.1055728779... \ . \end{aligned}$$

We can numerically observe that the limit of $q^{-1}\langle 5_2 \rangle_N\, e^{-N\varsigma}N^{-3/2}$ tends to the above mentioned value of ω.

N	$q^{-1}\langle 5_2 \rangle_N\, e^{-N\varsigma}\, N^{-3/2}$
200	$0.0915851738... - \sqrt{-1}\cdot 0.6519891312...$
500	$0.0907489101... - \sqrt{-1}\cdot 0.6507787459...$
1000	$0.0904698237... - \sqrt{-1}\cdot 0.6503768725...$

Further, we can numerically observe that the limit of $(\langle 5_2 \rangle_N (e^{N\varsigma} N^{3/2}\omega)^{-1} - 1)N/(2\pi\sqrt{-1})$ tends to the above mentioned value of κ_1.

N	$(\langle 5_2 \rangle_N (e^{N\varsigma} N^{3/2}\omega)^{-1} - 1)N/(2\pi\sqrt{-1})$
200	$1.0567234007... - \sqrt{-1} \cdot 0.0885918466...$
500	$1.0549811019... - \sqrt{-1} \cdot 0.0987904427...$
1000	$1.0543713307... - \sqrt{-1} \cdot 0.1021833710...$

As the conjecture (2) suggests, ω and κ_i's of (2) are expected to be invariants of K for any hyperbolic knot K. We have computed ω and κ_1 for the 5_2 knot in this paper. It is conjectured that $2\sqrt{-1}\,\omega^2$ of a hyperbolic knot is equal to the twisted Reidemeister torsion associated with the action on $\mathfrak{sl}_2$ of the holonomy representation of the hyperbolic structure; see Remark 1.4 below. We discuss about it for some knots in [28].

We give a proof of the theorem in Section 5 by justifying the above mentioned approach. An outline of the proof is as follows. We rewrite the sum (7) by an integral by the Poisson summation formula. When we apply the Poisson summation formula, the right-hand side of the Poisson summation formula consists of infinitely many summands, and we show that we can ignore them all except for the one at 0 in the sense that they are of sufficiently small order at $N \to \infty$ (Proposition 4.6 and Lemma 5.8). Further, by the saddle point method (Proposition 3.5), we calculate the asymptotic expansion of the integral, and obtain the presentation of the theorem.

By the method of this paper, the asymptotic behavior of the Kashaev invariant is discussed for the hyperbolic knots with up to 7 crossings in [27, 26] and for some hyperbolic knots with 8 crossings in [34].

Remark 1.2. The author has written the first version of this paper in August 2011. In February 2012, [2] was uploaded in the arXiv, in which Dimofte and Garoufalidis define a formal power series from an ideal tetrahedral decomposition of a knot complement, which is expected to be equal to the asymptotic expansion of the Kashaev invariant of the knot.

Remark 1.3. The right-hand side of (1) is equal to $2\pi\sqrt{-1}\,\varsigma$, and it is called the *complex volume*. It is known, see *e.g.* [45], that the complex volume can be expressed by a critical value of the potential function. It is also known, see *e.g.* [10], that the complex volume can be regarded as the $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons invariant.

Remark 1.4. The normalization of the above mentioned Reidemeister torsion is the cohomological Reidemeister torsion associated with the meridian used in [23]. We note that the twisted Reidemeister torsion in [4] is the twisted Reidemeister torsion associated with the longitude, and it can be changed to the twisted Reidemeister torsion associated with the meridian by [29, Théorème 4.1] as mentioned in [23].

The paper is organized as follows. In Section 2, we review definitions and basic properties of the notation used in this paper. In Section 3, we calculate the asymptotic expansion of Gaussian integrals by the saddle point method, and show Proposition 3.5. In Section 4, we calculate the sum corresponding to the integrals of Section 3 by the Poisson summation formula, and show Proposition 4.6. In Section 5, we give a proof of Theorem 1.1

by using the Poisson summation formula (Proposition 4.6) and the saddle point method (Proposition 3.5).

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2 Calculation of the Kashaev invariant of the 5_2 knot

In this section, we review the definition of the Kashaev invariant and the calculation of the Kashaev invariant $\langle 5_2 \rangle_N$ of the 5_2 knot in Section 2.1. Further, we continue to calculate the value of $\langle 5_2 \rangle_N$ toward its asymptotic expansion in Section 2.2.

2.1 The Kashaev invariant of the 5_2 knot

In this section, we review the definition of the Kashaev invariant of oriented knots, and review the calculation of the Kashaev invariant $\langle 5_2 \rangle_N$ of the 5_2 knot, which are due to Yokota [43]. The aim of this section is to show (7) which gives the value of $\langle 5_2 \rangle_N$.

Let N be an integer ≥ 2. We put $q = \exp(2\pi\sqrt{-1}/N)$, and put

$$(x)_n \;=\; (1-x)(1-x^2)\cdots(1-x^n)$$

for $n \geq 0$. It is known [24] that, for any n, m with $n \leq m$,

$$(q)_n(\overline{q})_{N-n-1} = N, \tag{3}$$

$$\sum_{n\leq k\leq m} \frac{1}{(q)_{m-k}(\overline{q})_{k-n}} = 1. \tag{4}$$

Following Yokota [43],[3] we review the definition of the Kashaev invariant. We put

$$\mathcal{N} \;=\; \{0, 1, \cdots, N-1\}.$$

For $i, j, k, l \in \mathcal{N}$, we put

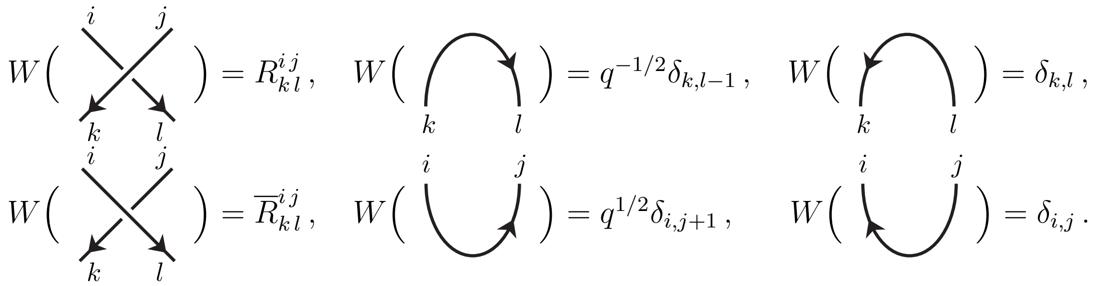
$$R^{ij}_{kl} \;=\; \frac{N\, q^{-\frac{1}{2}+i-k}\, \theta^{ij}_{kl}}{(q)_{[i-j]}(\overline{q})_{[j-l]}(q)_{[l-k-1]}(\overline{q})_{[k-i]}}, \qquad \overline{R}^{ij}_{kl} \;=\; \frac{N\, q^{\frac{1}{2}+j-l}\, \theta^{ij}_{kl}}{(\overline{q})_{[i-j]}(q)_{[j-l]}(\overline{q})_{[l-k-1]}(q)_{[k-i]}},$$

where $[m] \in \mathcal{N}$ denotes the residue of m modulo N, and we put

$$\theta^{ij}_{kl} \;=\; \begin{cases} 1 & \text{if } \; [i-j]+[j-l]+[l-k-1]+[k-i] = N-1, \\ 0 & \text{otherwise.} \end{cases}$$

[3]We make a minor modification of the definition of weights of critical points from the definition in [43], in order to make $\langle K \rangle_N$ invariant under Reidemeister moves.

Let K be an oriented knot. We consider a 1-tangle whose closure is isotopic to K such that its string is oriented downward at its end points. Let D be a diagram of the 1-tangle. We present D by a union of elementary tangle diagrams shown in (5). We decompose the string of D into edges by cutting it at crossings and critical points with respect to the height function of $\mathbb{R}^2$. A *labeling* is an assignment of an element of $\mathcal{N}$ to each edge. Here, we assign 0 to the two edges adjacent to the end points of D. For example, see (6). We define the *weights* of labeled elementary tangle diagrams by

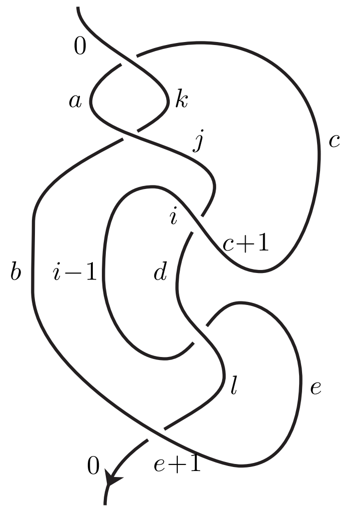


(5)

Then, *the Kashaev invariant* $\langle K \rangle_N$ of K is defined by

$$\langle K \rangle_N \;=\; \sum_{\text{labelings}} \; \prod_{\substack{\text{crossings}\\ \text{of } D}} W(\text{crossings}) \prod_{\substack{\text{critical}\\ \text{points of } D}} W(\text{critical points}) \;\in \mathbb{C}.$$

Following Yokota [43],[4] we review the calculation of the Kashaev invariant $\langle 5_2 \rangle_N$ of the 5_2 knot, where *the* 5_2 *knot* is the closure of the following 1-tangle.



(6)

We consider the above labeling. Then, it is shown by arguments in [43] that the labelings of edges adjacent to the unbounded regions vanish, *i.e.*, $a = b = c = d = e = 0$. Hence, the Kashaev invariant of the 5_2 knot is given by

$$\langle 5_2 \rangle_N \;=\; \sum_{i,j,k,l} q^{1/2}\, \overline{R}^{00}_{0k}\, \overline{R}^{0k}_{0j}\, \overline{R}^{ij}_{01}\, \overline{R}^{00}_{i-1\,l}\, \overline{R}^{0l}_{01}$$

[4] Our resulting formula (7) in this section is q times the corresponding formula in [43]. This difference is because of the difference of the definitions of $\langle K \rangle_N$ between ours and [43].

$$
\begin{aligned}
&= \sum_{i,j,k,l} q^{1/2} \cdot \frac{N q^{\frac{1}{2}-k}}{(q)_{[-k]}(\overline{q})_{[k-1]}} \cdot \frac{N q^{\frac{1}{2}+k-j}}{(\overline{q})_{[-k]}(q)_{[k-j]}(\overline{q})_{[j-1]}} \cdot \frac{N q^{-\frac{1}{2}+j}}{(\overline{q})_{[i-j]}(q)_{[j-1]}(q)_{[-i]}} \\
&\qquad \times \frac{N q^{\frac{1}{2}-l}}{(q)_{[-l]}(\overline{q})_{[l-i]}(q)_{[i-1]}} \cdot \frac{N q^{-\frac{1}{2}+l}}{(\overline{q})_{[-l]}(q)_{[l-1]}} \\
&= \sum_{i,j,k,l} \frac{N^3 q}{(\overline{q})_{[-k]}(q)_{[k-j]}(\overline{q})_{[j-1]}(\overline{q})_{[i-j]}(q)_{[j-1]}(q)_{[-i]}(q)_{[-l]}(\overline{q})_{[l-i]}(q)_{[i-1]}} ,
\end{aligned}
$$

where we obtain the last equality from (3). Further, by (4), the above formula is rewritten,

$$
\begin{aligned}
\langle\, 5_2 \,\rangle_N &= \sum_{1 \le j \le i \le N} \frac{N^3 q}{(q)_{i-1}(q)_{N-i}(q)_{j-1}(\overline{q})_{j-1}(\overline{q})_{i-j}} \\
&= \sum_{0 \le j \le i < N} \frac{N^3 q}{(q)_{i}(q)_{N-i-1}(q)_{j}(\overline{q})_{j}(\overline{q})_{i-j}} \\
&= \sum_{\substack{0 \le i,j \\ i+j < N}} \frac{N^3 q}{(q)_{i+j}(q)_{N-i-j-1}(q)_{j}(\overline{q})_{j}(\overline{q})_{i}} , \qquad (7)
\end{aligned}
$$

where we obtain the second equality by replacing i and j with $i+1$ and $j+1$, and obtain the last equality by replacing i with $i+j$. Hence, we obtain the presentation (7) of $\langle\, 5_2 \,\rangle_N$.

2.2 Calculation of $\langle\, 5_2 \,\rangle_N$ toward its asymptotic expansion

In this section, we continue to calculate the value of the Kashaev invariant $\langle\, 5_2 \,\rangle_N$ of the 5_2 knot toward its asymptotic expansion.

To calculate the asymptotic expansion of $\langle\, 5_2 \,\rangle_N$, we review an integral expression of $(q)_n$. It is known [8, 40] that

$$
\begin{aligned}
(q)_n &= \exp\Big(\varphi\big(\frac{1}{2N}\big) - \varphi\big(\frac{2n+1}{2N}\big)\Big), \\
(\overline{q})_n &= \exp\Big(\varphi\big(1-\frac{2n+1}{2N}\big) - \varphi\big(1-\frac{1}{2N}\big)\Big).
\end{aligned} \qquad (8)
$$

Here, following Faddeev [6], we define a holomorphic function $\varphi(t)$ on $\{t \in \mathbb{C} \mid 0 < \operatorname{Re} t < 1\}$ by

$$
\varphi(t) \;=\; \int_{-\infty}^{\infty} \frac{e^{(2t-1)x} dx}{4x \,\sinh x \,\sinh(x/N)} ,
$$

noting that this integrand has poles at $n\pi\sqrt{-1}$ $(n \in \mathbb{Z})$, where, to avoid the pole at 0, we choose the following contour of the integral,

$$
\gamma \;=\; (-\infty, -1\,] \;\cup\; \big\{ z \in \mathbb{C} \;\big|\; |z| = 1,\ \operatorname{Im} z \ge 0 \big\} \;\cup\; [\,1, \infty).
$$

We review some properties of $\varphi(t)$ in Appendix A.

By using $\varphi(t)$, we rewrite the presentation (7) of $\langle\, 5_2 \,\rangle_{N}$ by (8) as

$$\langle\, 5_2 \,\rangle_{N} \;=\; N^3 q \sum_{\substack{0 \le i,j \\ i+j<N}} \exp\Big(N \cdot \tilde{V}\big(\frac{2i+1}{2N},\, \frac{2j+1}{2N}\big)\Big),$$

where we put

$$\begin{aligned}
\tilde{V}(t,s) \;&=\; \frac{1}{N}\Big(\varphi\big(t+s-\frac{1}{2N}\big)+\varphi\big(1-t-s+\frac{1}{2N}\big)+\varphi(s)-\varphi(1-s) \\
&\qquad -\varphi(1-t)-3\,\varphi\big(\frac{1}{2N}\big)+2\,\varphi\big(1-\frac{1}{2N}\big)\Big) \\
&=\; \frac{1}{N}\big(-\varphi(1-t)-2\,\varphi(1-s)\big)-\frac{1}{2\pi\sqrt{-1}}\frac{\pi^2}{6} \\
&\quad -2\pi\sqrt{-1}\Big(\frac{1}{2}\big(t+s-\frac{1}{2N}\big)^2+\frac{1}{2}s^2-\frac{1}{2}t-s+\frac{1}{6}\Big) \\
&\quad -\frac{5}{2N}\log N-\frac{3\pi\sqrt{-1}}{4N}+\frac{\pi\sqrt{-1}}{4N^2}\,.
\end{aligned}$$

Here, we obtain the second equality by Lemmas A.2 and A.3. Hence, by putting

$$\begin{aligned}
V(t,s) \;&=\; \tilde{V}(t,s)+\frac{5}{2N}\log N \\
&=\; \frac{1}{N}\big(-\varphi(1-t)-2\,\varphi(1-s)\big)-\frac{1}{2\pi\sqrt{-1}}\frac{\pi^2}{6} \\
&\quad -2\pi\sqrt{-1}\Big(\frac{1}{2}\big(t+s-\frac{1}{2N}\big)^2+\frac{1}{2}s^2-\frac{1}{2}t-s+\frac{1}{6}\Big)-\frac{3\pi\sqrt{-1}}{4N}+\frac{\pi\sqrt{-1}}{4N^2}\,,
\end{aligned}$$

the presentation of $\langle\, 5_2 \,\rangle_{N}$ is rewritten

$$\langle\, 5_2 \,\rangle_{N} \;=\; N^{1/2}\, q \sum_{\substack{0 \le i,j \\ i+j<N}} \exp\Big(N \cdot V\big(\frac{2i+1}{2N},\, \frac{2j+1}{2N}\big)\Big). \tag{9}$$

The range of $(i/N, j/N)$ in this sum is given by the following domain,

$$\Delta \;=\; \big\{(t,s)\in\mathbb{R}^2 \;\big|\; 0\le t,\quad 0\le s,\quad t+s\le 1\big\}.$$

Further, it follows from Proposition A.1 that $V(t,s)$ converges to the following function as $N\to\infty$,

$$\begin{aligned}
\hat{V}(t,s) \;&=\; \frac{1}{2\pi\sqrt{-1}}\Big(-\operatorname{Li}_2(e^{-2\pi\sqrt{-1}\,t})-2\operatorname{Li}_2(e^{-2\pi\sqrt{-1}\,s})-\frac{\pi^2}{6}\Big) \\
&\quad -2\pi\sqrt{-1}\,\Big(\frac{1}{2}t^2+s^2+ts-\frac{1}{2}t-s+\frac{1}{6}\Big).
\end{aligned} \tag{10}$$

We will show that the asymptotic expansion of (9) is of order $e^{N\varsigma_R}$ times polynomial order in N, where ς_R is the real part of ς which is given in the introduction,

$$\varsigma_R \;=\; \operatorname{Re}\varsigma \;=\; 0.4501096100...\;.$$

Hence, we can ignore summands of (9) in the domain where $\operatorname{Re}\hat{V}(t,s) \leq \varsigma_R - \varepsilon$, since they do not contribute to the resulting expansion. Further, as we can see in Figure 1, $\operatorname{Re}\hat{V}(t,s) - \varsigma_R$ is positive only for (t,s) in a particular subdomain of Δ. In the following lemma, we consider to restrict Δ to a smaller domain Δ' which includes this subdomain; this restriction will be used to verify the assumptions of the Poisson summation formula and the saddle point method later.

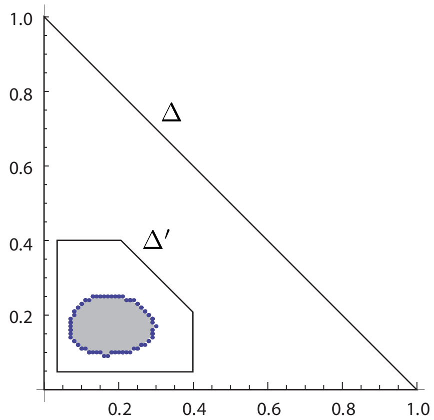


Figure 1: The domain $\{(t,s) \mid \operatorname{Re}\hat{V}(t,s) \geq \varsigma_R\}$

Lemma 2.1. *We put*

$$\Delta' = \{(t,s) \in \Delta \mid 0.04 \leq t \leq 0.4,\ 0.05 \leq s \leq 0.4,\ t+s \leq 0.6\}.$$

Then, the following domain

$$\{(t,s) \in \mathbb{R}^2 \mid \operatorname{Re}\hat{V}(t,s) \geq \varsigma_R - \varepsilon\} \tag{11}$$

is included in Δ' for some sufficiently small $\varepsilon > 0$.

We give a proof of the lemma in Appendix D. we can graphically observe the inclusion of the lemma in Figure 1.

We put

$$\Delta'' = \{(t,s) \in \Delta \mid 0.04 \leq t \leq 0.9,\ 0.05 \leq s \leq 0.9\}. \tag{12}$$

By Lemma B.1, $V(t,s)$ uniformly converges to $\hat{V}(t,s)$ as $N \to \infty$ in this domain. By this uniform convergence, we can restrict Δ'' to Δ' later. Before this restriction, we consider to restrict Δ to Δ'' in the following calculation of $\langle 5_2 \rangle_N$.

We consider the value of $\langle 5_2 \rangle_N$ given in (7). The summand of the sum (7) is estimated by

$$\Big| \frac{N^3 q}{(q)_{i+j}(q)_{N-i-j-1}(q)_j(\overline{q})_j(\overline{q})_i} \Big| = \Big| \frac{N^2}{(q)_i (q)_j^2} \Big|, \tag{13}$$

where we obtain the equality by (3). Since $(q)_n = (1-q^n)(q)_{n-1}$ from the definition of $(q)_n$, the value of $1/|(q)_n|$ is monotonically increasing with respect to n for $0 \leq n \leq \frac{N}{6}$. Hence, for $0 \leq i \leq 0.04 \cdot N$ and $0 \leq j \leq 0.05 \cdot N$, the value of (13) is monotonically increasing with respect to i and j. Further, we can similarly show that, for $0.9 \cdot N \leq i < N$ and $0.9 \cdot N \leq j < N$, the value of (13) is monotonically decreasing with respect to i and j. Furthermore, since $V(t,s)$ uniformly converges to $\hat{V}(t,s)$ as $N \to \infty$ on Δ'' by Lemma B.1, $\operatorname{Re} V(t,s)$ is bounded by $e^{N(\varsigma_R - \varepsilon_1)}$ for some $\varepsilon_1 > 0$ on $\partial\Delta''$ by Lemma 2.1. Hence, (13) is bounded by the order $e^{N(\varsigma_R - \varepsilon_2)}$ for some $\varepsilon_2 > 0$. Therefore, by (7),

$$\langle\, 5_2 \,\rangle_N \;=\; \sum_{\substack{i,j \in \mathbb{Z} \\ (i/N, j/N) \in \Delta''}} \frac{N^3 q}{(q)_{i+j}(q)_{N-i-j-1}(q)_j(\overline{q})_j(\overline{q})_i} \;+ O(N^2 e^{N(\varsigma_R - \varepsilon_2)}).$$

Hence, similarly as (9), we have that

$$\begin{aligned} \langle\, 5_2 \,\rangle_N \;&=\; N^{1/2} q \sum_{\substack{i,j \in \mathbb{Z} \\ (i/N, j/N) \in \Delta''}} \exp\Big(N \cdot V\big(\frac{2i+1}{2N}, \frac{2j+1}{2N}\big)\Big) \;+ O\big(N^2 e^{N(\varsigma_R - \varepsilon_2)}\big) \\ &=\; e^{N\varsigma} N^{1/2} q \sum_{\substack{i,j \in \mathbb{Z} \\ (i/N, j/N) \in \Delta''}} \exp\Big(N \cdot V\big(\frac{2i+1}{2N}, \frac{2j+1}{2N}\big) - N\varsigma\Big) \;+ O\big(N^2 e^{N(\varsigma_R - \varepsilon_2)}\big). \end{aligned}$$

By Lemma 2.1, we can restrict Δ'' to Δ', where the error term of the sum is estimated by the order $N^2 e^{-N\varepsilon}$. Hence,

$$\langle\, 5_2 \,\rangle_N \;=\; e^{N\varsigma} N^{1/2} q \left(\sum_{\substack{i,j \in \mathbb{Z} \\ (i/N, j/N) \in \Delta'}} \exp\Big(N \cdot V\big(\frac{2i+1}{2N}, \frac{2j+1}{2N}\big) - N\varsigma\Big) \;+ O(e^{-N\varepsilon_3}) \right), \tag{14}$$

for some $\varepsilon_3 > 0$. Further, by the Poisson summation formula, we will see that the above sum is expressed by an integral,

$$\langle\, 5_2 \,\rangle_N \;=\; e^{N\varsigma} N^{5/2} q \left(\int_{\Delta'} \exp\Big(N \cdot V(t,s) - N\varsigma\Big) dt\, ds + O(e^{-N\varepsilon_4}) \right)$$

for some $\varepsilon_4 > 0$.

We will analyse this integral in Section 5 using the saddle point method and give the proof our main Theorem 1.1 there. We will therefore devote the following two sections to recalling the saddle point method and the Poisson summation formula.

3 Calculation by the saddle point method

In this section, we calculate Gaussian integrals by the saddle point method. We calculate a Gaussian integral in Proposition 3.1, a Gaussian integral with perturbative terms in Proposition 3.2, and multi-variable cases in Propositions 3.4 and 3.5. We use Proposition 3.5 in the proof of Theorem 1.1 in Section 5. For the saddle point method, see *e.g.* [39].

Proposition 3.1. *For a non-zero $a \in \mathbb{C}$, the domain $\{z \in \mathbb{C} \mid \operatorname{Re} az^2 < 0\}$ has two connected components. We choose z_0, z_1 from the two components respectively. Let C be a path from z_0 to z_1 in $\mathbb{C}$. (See Figure 2.) Then, there exists $\varepsilon > 0$ such that*

$$\int_C e^{N\cdot az^2} dz \;=\; \frac{\sqrt{\pi}}{\sqrt{-a}\cdot\sqrt{N}} \;+ O\big(e^{-\varepsilon N}\big),$$

where we choose the sign of $\sqrt{-a}$ such that $\operatorname{Re} z_1\sqrt{-a} > 0$.

We note that ε depends on z_0, z_1 and a.

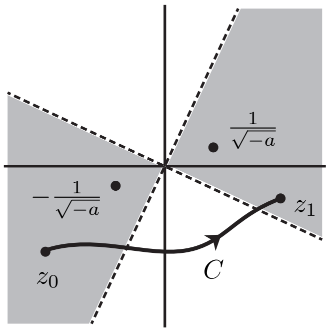


Figure 2: The domain $\{z \in \mathbb{C} \mid \operatorname{Re} az^2 < 0\}$ is shaded.

Proof. By replacing z with $z/\sqrt{-a}$, we can reduce the proof to the case where $a = -1$. Putting $a = -1$, we show that

$$\int_{z_0}^{z_1} e^{-Nz^2} dz \;=\; \frac{\sqrt{\pi}}{\sqrt{N}} \;+ O(e^{-\varepsilon N}).$$

Since $\int_{-\infty}^{\infty} e^{-Nz^2} dz \;=\; \sqrt{\pi}/\sqrt{N}$, it is sufficient to show that

$$\int_{-\infty}^{z_0} e^{-Nz^2} dz = O(e^{-\varepsilon N}), \qquad \int_{z_1}^{\infty} e^{-Nz^2} dz = O(e^{-\varepsilon N}).$$

We show the latter formula. (The former formula can be shown similarly.) The latter formula is calculated,

$$\int_{z_1}^{\infty} e^{-Nz^2} dz \;=\; \int_{z_1}^{\operatorname{Re} z_1} e^{-Nz^2} dz + \int_{\operatorname{Re} z_1}^{\infty} e^{-Nz^2} dz,$$

and the two terms of the right-hand side are estimated,

$$\begin{aligned}\Big|\int_{z_1}^{\operatorname{Re} z_1} e^{-Nz^2} dz\Big| \;&\le\; \int_{z_1}^{\operatorname{Re} z_1} |e^{-Nz^2}|\cdot|dz| \;\le\; \int_{z_1}^{\operatorname{Re} z_1} e^{-N\operatorname{Re} z^2}|dz| \\ &\le\; \int_{z_1}^{\operatorname{Re} z_1} e^{-N\operatorname{Re} z_1^2}|dz| \;=\; |\operatorname{Im} z_1|\cdot e^{-N\operatorname{Re} z_1^2} \;=\; O(e^{-\varepsilon N}),\end{aligned}$$

$$0 \;\le\; \int_{\mathrm{Re}\, z_1}^{\infty} e^{-Nz^2} dz \;\le\; \int_{\mathrm{Re}\, z_1}^{\infty} e^{-N(\mathrm{Re}\, z_1) z} dz \;=\; \frac{e^{-N(\mathrm{Re}\, z_1)^2}}{N \cdot \mathrm{Re}\, z_1} \;=\; O(e^{-\varepsilon N})$$

for some $\varepsilon > 0$. Hence, we obtain the required formula. □

We generalize Proposition 3.1 to the case where there are perturbative terms in the exponential of the integrand.

Proposition 3.2. *Let a be a non-zero complex number, and let $\psi(z)$ and $r(z)$ be holomorphic functions of the forms,*

$$\psi(z) \;=\; az^2 + r(z), \qquad r(z) \;=\; b_3 z^3 + b_4 z^4 + \cdots ,$$

defined in a neighborhood of 0*. The domain*

$$\big\{ z \in \mathbb{C} \;\big|\; \mathrm{Re}\; \psi(z) < 0 \big\} \tag{15}$$

has two connected components in a neighborhood of 0*. We choose z_0, z_1 from these two components respectively. Let C be a path from z_0 to z_1 in $\mathbb{C}$. (See Figure 3.) Then,*

$$\int_C e^{N\psi(z)} dz \;=\; \frac{\sqrt{\pi}}{\sqrt{-a} \cdot \sqrt{N}} \Big(1 + \sum_{k=1}^{d} \frac{\lambda_k}{N^k} \; + O\big(\frac{1}{N^{d+1}}\big) \Big),$$

for any d, where we choose the sign of $\sqrt{-a}$ similarly as in Proposition 3.1, and λ_k's are constants given by using coefficients of the expansion of $\psi(z)$; such presentations are obtained by formally expanding the following formula,

$$1 + \sum_{k=1}^{\infty} \frac{\lambda_k}{N^k} \;=\; \exp\Big(N\, r\big(\frac{\partial}{\partial u}\big)\Big) \; \exp\Big(- \frac{u^2}{4Na} \Big)\Big|_{u=0} . \tag{16}$$

In particular, λ_1 is given by

$$\lambda_1 \;=\; -\frac{15 b_3^2}{16 a^3} + \frac{3 b_4}{4 a^2} .$$

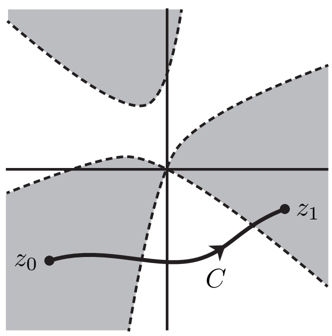


Figure 3: The domain $\{z \in \mathbb{C} \mid \mathrm{Re}\, \psi(z) < 0\}$ is shaded.

Proof. We show the proposition modifying a proof of the saddle point method written in [39].[5] We show the proposition, for simplicity, putting $a = -1$ as in the proof of Proposition 3.1. We put $\hat{r}(z) = r(z)/z^2$. Since $\hat{r}(z)$ is analytic in a neighborhood of 0, there exists a sufficiently small $\delta_1 > 0$ such that

$$\hat{r}(z) \;=\; \sum_{i=1}^{\infty} b_{k+2}\, z^k$$

for $|z| < \delta_1$. Let w be a non-negative real parameter. For each fixed w, we have that

$$e^{w\,\hat{r}(z)} \;=\; \sum_{k=0}^{\infty} P_k(w)\, z^k$$

for $|z| < \delta_1$, where $P_k(w)$ is a polynomial in w of degree $\le k$. Since $\hat{r}(0) = 0$, there exist small $\delta_2 > 0$ and $\varepsilon_1 > 0$ such that $|\hat{r}(x)| \le \varepsilon_1$ for $-\delta_2 \le x \le \delta_2$; we can further assume that $\varepsilon_1 < 1$ and $\delta_2 \le \delta_1$. For each fixed integer $m > 0$, we can put

$$e^{w\,\hat{r}(x)} \;=\; \sum_{k=0}^{m} P_k(w)\, x^k + R_m\, x^{m+1} \tag{17}$$

for $-\delta_2 \le x \le \delta_2$ and any $w \ge 0$, where $R_m x^{m+1}$ is the error term, which is estimated by

$$|\mathrm{Re}\, R_m| \;\le\; \max_{|x| \le \delta_2} \Big| \mathrm{Re}\, \frac{d^{m+1}}{dx^{m+1}}\, e^{w\,\hat{r}(x)} \Big|, \qquad |\mathrm{Im}\, R_m| \;\le\; \max_{|x| \le \delta_2} \Big| \mathrm{Im}\, \frac{d^{m+1}}{dx^{m+1}}\, e^{w\,\hat{r}(x)} \Big|.$$

Further, since

$$\frac{d^{m+1}}{dx^{m+1}}\, e^{w\,\hat{r}(x)} \;=\; e^{w\,\hat{r}(x)} \cdot \big(\text{polynomial in } w \text{ and differentials of } \hat{r}(x) \text{ of degree} \le m{+}1\big),$$

we have that

$$|R_m| \;\le\; e^{\varepsilon_1 w} K_1 (w^{m+1} + 1) \;\le\; K_2\, e^{\varepsilon_2 w} \tag{18}$$

for some $K_1, K_2 > 0$ and $\varepsilon_2 < 1$, which are independent of x and w (noting that m is bounded by using d later). Further, we replace the path C with the union of a path C_1 from z_0 to $-\delta$, a path C_2 from $-\delta$ to δ along the real axis, and a path C_3 from δ to z_1. We can assume that there exist sufficiently small δ, $\varepsilon_3 > 0$ such that $\delta \le \delta_2$ and C_1 and C_3 are in the domain

$$\big\{ z \in \mathbb{C} \;\big|\; \mathrm{Re}\, \psi(z) \le -\varepsilon_3 \big\}. \tag{19}$$

Then, the integral of the proposition is given by

$$\int_C e^{N\,\psi(z)} dz \;=\; \int_{C_1} e^{N\,\psi(z)} dz + \int_{C_2} e^{N\,\psi(z)} dz + \int_{C_3} e^{N\,\psi(z)} dz. \tag{20}$$

[5] As for the 1-variable case, a proof of a more general statement of the saddle point method is written in [39], though the multi-variable case is not written in [39]. We review (a simpler modification of) the proof of [39], in order to generalize it to the multi-variable case later (Proposition 3.5).

Since C_1 and C_3 are in the domain (19), we have that

$$\int_{C_1} e^{N\,\psi(z)} dz \;=\; O(e^{-N\varepsilon_3}) \quad \text{and} \quad \int_{C_3} e^{N\,\psi(z)} dz \;=\; O(e^{-N\varepsilon_3}). \tag{21}$$

Hence, it is sufficient to show that the integral along C_2 gives the required formula.

The integral along C_2 is calculated as

$$\begin{aligned}
\int_{C_2} e^{N\,\psi(z)} dz \;&=\; \int_{-\delta}^{\delta} e^{N\,\psi(x)} dx \;=\; \int_{-\delta}^{\delta} e^{-Nx^2} e^{Nx^2 \hat{r}(x)} dx \\
&=\; \sum_{k=0}^{2d+1} \int_{-\delta}^{\delta} P_k(Nx^2)\, x^k e^{-Nx^2} dx \;+\; \int_{-\delta}^{\delta} R_{2d+1}\, x^{2d+2} e^{-Nx^2} dx,
\end{aligned} \tag{22}$$

by (17), putting $w = Nx^2$ and $m = 2d+1$. When k is odd, the summand of the first term of (22) is equal to 0, since the integrand is an odd function. When k is even, the summand of the first term of (22) is given by a sum of integrals of the following form,

$$\begin{aligned}
\int_{-\delta}^{\delta} (Nx^2)^l\, x^k e^{-Nx^2} dx \;&=\; \int_{-\infty}^{\infty} N^l x^{2l+k} e^{-Nx^2} dx \;+ O(e^{-N\varepsilon_4}) \\
&=\; \frac{1}{N^{k/2}\sqrt{N}} \int_{-\infty}^{\infty} y^{2l+k} e^{-y^2} dx \;+ O(e^{-N\varepsilon_4}) \\
&=\; \frac{(2l+k-1)!! \cdot \sqrt{\pi}}{2^{l+k/2} N^{k/2} \sqrt{N}} \;+ O(e^{-N\varepsilon_4}),
\end{aligned}$$

for some $\varepsilon_4 > 0$, where we obtain the first equality in a similar way as in the proof of Proposition 3.1, and obtain the second equality putting $y = \sqrt{N}x$. Hence, the first term of (22) is given by the following form,

$$\frac{\sqrt{\pi}}{\sqrt{N}} \Big(1 + \sum_{k=1}^{d} \frac{\lambda_k}{N^k} \Big) \;+ O(e^{-N\varepsilon_4}).$$

Further, by (18) putting $w = Nx^2$, the second term of (22) is estimated by

$$\begin{aligned}
\Big| \int_{-\delta}^{\delta} R_{2d+1}\, x^{2d+2} e^{-Nx^2} dx \Big| \;&\le\; \int_{-\delta}^{\delta} |R_{2d+1}|\, x^{2d+2} e^{-Nx^2} dx \\
&\le\; K_2 \int_{-\delta}^{\delta} x^{2d+2} e^{-(1-\varepsilon_2)Nx^2} dx \;=\; O\big(\frac{1}{N^{d+\frac{3}{2}}}\big),
\end{aligned}$$

where we obtain the last equality in a similar way as the above calculation. Hence, by (22),

$$\int_{C_2} e^{N\,\psi(z)} dz \;=\; \frac{\sqrt{\pi}}{\sqrt{N}} \Big(1 + \sum_{k=1}^{d} \frac{\lambda_k}{N^k} \;+ O\big(\frac{1}{N^{d+1}}\big) \Big).$$

Therefore, by (20) and (21), the integral of the proposition is given by the following form,

$$\int_{C} e^{N\,\psi(z)} dz \;=\; \frac{\sqrt{\pi}}{\sqrt{N}} \Big(1 + \sum_{k=1}^{d} \frac{\lambda_k}{N^k} \;+ O\big(\frac{1}{N^{d+1}}\big) \Big).$$

In particular, λ_1 is concretely calculated by refining the above calculation, as follows. Since

$$\hat{r}(z) \;=\; b_3 z + b_4 z^2 + \cdots\,,$$

we have that

$$e^{w\,\hat{r}(z)} \;=\; 1 + w\hat{r}(z) + \frac{1}{2}w^2\hat{r}(z)^2 + \cdots \;=\; 1 + b_3 wz + \big(\frac{b_3}{2}w^2 + b_4 w\big)z^2 + \cdots\,.$$

Hence,

$$\begin{aligned} \int_C e^{N\,\psi(z)}dz \;&=\; \int_{-\infty}^{\infty} e^{-Nx^2}\Big(1 + b_3 Nx^2\cdot x + \big(\frac{b_3^2}{2}(Nx^2)^2 + b_4 Nx^2\big)x^2 + \cdots\Big)dx \\ &=\; \frac{\sqrt{\pi}}{\sqrt{N}}\Big(1 + \big(\frac{15}{16}b_3^2 + \frac{3}{4}b_4\big)\frac{1}{N} \;+ O\big(\frac{1}{N^2}\big)\Big). \end{aligned}$$

This is the required formula for $d=1$ when $a=-1$.

We obtain a concrete presentation of any λ_k by the following formal calculation. Noting that

$$z^m \;=\; \big(\frac{\partial}{\partial u}\big)^m e^{uz}\big|_{u=0}\,,$$

we have that

$$\begin{aligned} \int e^{Naz^2} z^m dz \;&=\; \int \big(\frac{\partial}{\partial u}\big)^m e^{Naz^2+uz}\big|_{u=0}\, dz \\ &=\; \int \big(\frac{\partial}{\partial u}\big)^m \exp\Big(Naw^2 - \frac{u^2}{4Na}\Big)\Big|_{u=0}\, dw \\ &=\; \frac{\sqrt{\pi}}{\sqrt{-a}\cdot\sqrt{N}}\cdot\big(\frac{\partial}{\partial u}\big)^m \exp\Big(-\frac{u^2}{4Na}\Big)\Big|_{u=0}\,, \end{aligned}$$

which can be justified by a similar calculation as above, to be precise. Hence, putting $\exp\big(N\,r(z)\big) = \sum_m \tilde{b}_m z^m$,

$$\begin{aligned} \int e^{N\,\psi(z)}dz \;&=\; \int e^{Naz^2}\exp\big(N\,r(z)\big)\,dz \\ &=\; \int e^{Naz^2}\big(\sum_m \tilde{b}_m z^m\big)\,dz \\ &=\; \frac{\sqrt{\pi}}{\sqrt{-a}\cdot\sqrt{N}}\cdot\sum_m \tilde{b}_m\big(\frac{\partial}{\partial u}\big)^m \exp\Big(-\frac{u^2}{4Na}\Big)\Big|_{u=0} \\ &=\; \frac{\sqrt{\pi}}{\sqrt{-a}\cdot\sqrt{N}}\cdot\exp\Big(N\,r\big(\frac{\partial}{\partial u}\big)\Big)\,\exp\Big(-\frac{u^2}{4Na}\Big)\Big|_{u=0}\,, \end{aligned}$$

and this gives (16). By expanding this formula formally, we obtain concrete presentations of λ_k's in terms of coefficients of the expansion of $\psi(z)$. □

Remark 3.3. We can extend Proposition 3.2 to the case where $\psi(z)$ depends on N in such a way that $\psi(z)$ is of the form

$$\psi(z) \;=\; \psi_0(z) + \psi_1(z)\frac{1}{N} + \psi_2(z)\frac{1}{N^2} + \cdots + \psi_m(z)\frac{1}{N^m} + r_m(z)\frac{1}{N^{m+1}}\,, \tag{23}$$

where $\psi_i(z)$'s are holomorphic functions independent of N, and we assume that $\psi_0(z)$ satisfies the assumption of the proposition and $|r_m(z)|$ is bounded by a constant which is independent of N. Then, $e^{N\psi(z)} = \phi(z)e^{N\psi_0(z)}$, where we put

$$\begin{aligned} \phi(z) \;&=\; \exp\big(\psi_1(z) + \psi_2(z)\frac{1}{N} + \cdots + \psi_m(z)\frac{1}{N^{m-1}} + r_m(z)\frac{1}{N^m}\,\big) \\ &=\; \phi_0(z) + \phi_1(z)\frac{1}{N} + \phi_2(z)\frac{1}{N^2} + \cdots + \phi_{m-1}(z)\frac{1}{N^{m-1}} + \tilde{r}_m(z)\frac{1}{N^m}. \end{aligned}$$

Here, $\tilde{r}_m(z)$ is the error term such that $|\tilde{r}_m(z)|$ is bounded by a constant which is independent of N. As written in [39], in the same way as in the proof of Proposition 3.2, we can show the asymptotic expansion of $\int \phi_i(z)e^{N\psi_0(z)}dz$ by expanding $\phi_i(z)$ at $z=0$. Further, we can estimate $\int \tilde{r}_m(z)e^{N\psi_0(z)}dz$ similarly as in the proof of Proposition 3.2, noting that only values of $\tilde{r}_m(z)$ in a sufficiently small neighborhood of 0 contribute to the resulting expansion. In this way, we can justify the statement of Proposition 3.2 in the case where $\psi(z)$ depends on N as in the form (23).

We generalize Proposition 3.1 to the case of n variables. Let A be a non-singular symmetric complex $n\times n$ matrix, and let $\mathbf{z}$ be a column vector $(z_1,\cdots,z_n)^T \in \mathbb{C}^n$. The domain

$$\big\{\mathbf{z}\in\mathbb{C}^n \;\big|\; \mathrm{Re}\,\mathbf{z}^T A\,\mathbf{z} < 0\big\} \tag{24}$$

is homotopy equivalent to S^{n-1}. Let D be an oriented n-ball embedded in $\mathbb{C}^n$ such that ∂D is included in the domain (24), whose inclusion is homotopy equivalent. There exists a matrix P such that $-A = P^T P$; we note that $\det P = \pm\sqrt{\det(-A)}$. We choose an n-ball as a neighborhood of the origin in $\mathbb{R}^n \subset \mathbb{C}^n$. The matrix P^{-1} takes this n-ball to an n-ball satisfying the above assumption of D. We choose the sign of $\sqrt{\det(-A)}$ by setting it to be $\det P$ if P takes the orientation of D to the standard orientation of $\mathbb{R}^n$, and $-\det P$ otherwise.

Proposition 3.4. *Let A, D be as above. Then, there exists $\varepsilon > 0$ such that*

$$\int_D e^{N\cdot\mathbf{z}^T A\,\mathbf{z}} d\mathbf{z} \;=\; \frac{\pi^{n/2}}{N^{n/2}\sqrt{\det(-A)}} \;+ O\big(e^{-\varepsilon N}\big),$$

where we put $d\mathbf{z} = dz_1\cdots dz_n$, and we choose the sign of $\sqrt{\det(-A)}$ as above.

Proof. By changing the coordinate of $\mathbf{z}$ linearly, we can reduce the proof to the case where A is a diagonal matrix. Further, since $d(e^{N\cdot\mathbf{z}^T A\,\mathbf{z}}d\mathbf{z}) = 0$, we can move the domain D in $\mathbb{C}^n$ by Stokes' theorem; we can also move ∂D in the domain (24) ignoring error terms of order $e^{-N\varepsilon}$ for some $\varepsilon > 0$ as in the proof of Proposition 3.1. In this way, we can move D into $\mathbb{R}^n \subset \mathbb{C}^n$, which means that we can reduce the proof to the product of copies of the formula of Proposition 3.1. □

We generalize Proposition 3.4 to the case where there are perturbative terms in the exponential of the integrand.

Proposition 3.5. *Let A be a non-singular symmetric complex $n\times n$ matrix, and let $\psi(\mathbf{z})$ and $r(\mathbf{z})$ be holomorphic functions of the forms,*

$$\begin{aligned} \psi(\mathbf{z}) &= \mathbf{z}^T A\, \mathbf{z} + r(\mathbf{z}), \\ r(\mathbf{z}) &= r(z_1, \cdots, z_n) \;=\; \textstyle\sum_{i,j,k} b_{ijk} z_i z_j z_k + \sum_{i,j,k,l} c_{ijkl} z_i z_j z_k z_l + \cdots, \end{aligned} \tag{25}$$

defined in a neighborhood of $\mathbf{0} \in \mathbb{C}^n$. The restriction of the domain

$$\big\{ \mathbf{z} \in \mathbb{C}^n \;\big|\; \mathrm{Re}\, \psi(\mathbf{z}) < 0 \big\} \tag{26}$$

to a neighborhood of $\mathbf{0} \in \mathbb{C}^n$ is homotopy equivalent to S^{n-1}. Let D be an oriented n-ball embedded in $\mathbb{C}^n$ such that ∂D is included in the domain (26) whose inclusion is homotopic to a homotopy equivalence to the above S^{n-1} in the domain (26). Then,

$$\int_D e^{N\,\psi(\mathbf{z})} d\mathbf{z} \;=\; \frac{\pi^{n/2}}{N^{n/2}\sqrt{\det(-A)}} \Big(1 + \sum_{i=1}^{d} \frac{\lambda_i}{N^i} \; + O\big(\frac{1}{N^{d+1}}\big) \Big),$$

for any d, where we choose the sign of $\sqrt{\det(-A)}$ as in Proposition 3.4, and λ_i's are constants given by using coefficients of the expansion of $\psi(\mathbf{z})$; such presentations are obtained by formally expanding the following formula,

$$1 + \sum_{i=1}^{\infty} \frac{\lambda_i}{N^i} \;=\; \exp\Big(N\, r\big(\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_n}\big) \Big) \exp\Big(-\frac{1}{4N} \mathbf{u}^T A^{-1} \mathbf{u} \Big)\Big|_{\mathbf{u}=\mathbf{0}}. \tag{27}$$

In particular, λ_1 is given by

$$\lambda_1 \;=\; \frac{-1}{2^8\cdot 3} \sum_{\substack{i_1, i_2, \cdots, i_6 \\ \sigma\in\mathfrak{S}_6}} b_{i_1 i_2 i_3} b_{i_4 i_5 i_6} \overline{a}_{i_{\sigma(1)} i_{\sigma(2)}} \overline{a}_{i_{\sigma(3)} i_{\sigma(4)}} \overline{a}_{i_{\sigma(5)} i_{\sigma(6)}} + \frac{1}{32} \sum_{\substack{i_1, i_2, i_3, i_4 \\ \tau\in\mathfrak{S}_4}} c_{i_1 i_2 i_3 i_4} \overline{a}_{i_{\tau(1)} i_{\tau(2)}} \overline{a}_{i_{\tau(3)} i_{\tau(4)}},$$

where $\mathfrak{S}_n$ is the nth symmetric group, and we put $\big(\overline{a}_{ij}\big)_{i,j} = A^{-1}$.

Proof. Similarly as in the proof of Proposition 3.4, we can reduce the proof to the case where $A = -E$, where E denotes the identity matrix of size n. Then, the integral of the problem is rewritten,

$$\int_D e^{N\cdot \mathbf{z}^T \mathbf{z}} \exp\big(N\, r(\mathbf{z})\big) d\mathbf{z}, \tag{28}$$

where we can let D be a sufficiently small neighborhood of $\mathbf{0}$ in $\mathbb{R}^n$ as in the proof of Proposition 3.4. The second exponential in the integrand of (28) is calculated as

$$\begin{aligned} \exp\big(N\, r(\mathbf{z})\big) &= 1 + N\, r(\mathbf{z}) + \frac{1}{2} N^2 r(\mathbf{z})^2 + \cdots \\ &= 1 + N \sum_{i,j,k} b_{ijk} z_i z_j z_k + N \sum_{i,j,k,l} c_{ijkl} z_i z_j z_k z_l \\ &\quad + \frac{1}{2} N^2 \sum_{i_1, i_2, \cdots, i_6} b_{i_1 i_2 i_3} b_{i_4 i_5 i_6} z_{i_1} z_{i_2} z_{i_3} z_{i_4} z_{i_5} z_{i_6} + \cdots. \end{aligned}$$

Hence, we can calculate the asymptotic expansion of the integral (28) similarly as the calculation of λ_1 in the proof of Proposition 3.2, and we can show that the asymptotic expansion of (28) is given by the following form,

$$\int_D e^{N\,\psi(\mathbf{z})} d\mathbf{z} \;=\; \frac{\pi^{n/2}}{N^{n/2}} \Big(1 + \frac{\lambda_1}{N} + \frac{\lambda_2}{N^2} + \cdots \Big).$$

We estimate the remaining part "$\cdots$" of the above formula as in the proof of Proposition 3.2, as follows. We can put

$$r(\mathbf{z}) \;=\; \sum_{i,j} z_i z_j \hat{r}_{ij}(\mathbf{z})$$

for some $\hat{r}_{ij}(\mathbf{z})$ which satisfies that $\hat{r}_{ij}(\mathbf{z}) = \hat{r}_{ji}(\mathbf{z})$. Let w_{ij} be non-negative real parameters, and put $\mathbf{w} = (w_{ij})$. For each fixed $\mathbf{w}$, we have that

$$\exp\Big(\sum_{i,j} w_{ij} \hat{r}_{ij}(\mathbf{z}) \Big) \;=\; \sum_{k=0}^{\infty} \sum_{i_1,\cdots,i_k} P_{i_1\cdots i_k}(\mathbf{w})\, z_{i_1} \cdots z_{i_k}$$

for sufficiently small $\mathbf{z}$, where $P_{i_1\cdots i_k}(\mathbf{w})$ is a polynomial in w_{ij}'s of degree $\leq k$. Further, for each fixed integer $m > 0$, we can put

$$\exp\Big(\sum_{i,j} w_{ij} \hat{r}_{ij}(\mathbf{x}) \Big) \;=\; \sum_{k=0}^{m} \sum_{i_1,\cdots,i_k} P_{i_1\cdots i_k}(\mathbf{x})\, x_{i_1} \cdots x_{i_k} \;+ \sum_{j_1,\cdots,j_{m+1}} R_{j_1\cdots j_{m+1}} x_{j_1} \cdots x_{j_{m+1}}$$

where $R_{j_1\cdots j_{m+1}} x_{j_1} \cdots x_{j_{m+1}}$ is the error term. Similarly as in the proof of Proposition 3.2, we can estimate it by

$$|R_{j_1\cdots j_{m+1}}| \;\leq\; K \exp\Big(\sum_{i,j} \varepsilon_{ij} w_{ij} \Big)$$

for some $K > 0$ and $\varepsilon_{ij} > 0$ which satisfies that $\varepsilon_{ij} = \varepsilon_{ji}$ and $E - (\varepsilon_{ij})$ is positive definite. Further, similarly as in the proof of Proposition 3.2, putting $m = 2d+1$, we can show that the error term is of order $O(1/N^{n/2+d+1})$. Hence, we can show that the integral of the problem has the asymptotic expansion of the required formula.

We obtain concrete presentations of λ_k's by the following formal calculation. Noting that

$$z_{i_1} \cdots z_{i_m} \;=\; \frac{\partial}{\partial u_{i_1}} \cdots \frac{\partial}{\partial u_{i_m}} e^{\mathbf{u}^T \mathbf{z}} \Big|_{\mathbf{u}=0}$$

we have that

$$\begin{aligned}
\int e^{N\cdot \mathbf{z}^T A\, \mathbf{z}} z_{i_1} \cdots z_{i_m} d\mathbf{z} \;&=\; \int \frac{\partial}{\partial u_{i_1}} \cdots \frac{\partial}{\partial u_{i_m}} e^{N\cdot \mathbf{z}^T A\, \mathbf{z} + \mathbf{u}^T \mathbf{z}} \Big|_{\mathbf{u}=0} d\mathbf{z} \\
&=\; \int \frac{\partial}{\partial u_{i_1}} \cdots \frac{\partial}{\partial u_{i_m}} \exp\Big(N \cdot \mathbf{w}^T A\, \mathbf{w} \;-\; \frac{1}{4N} \mathbf{u}^T A^{-1} \mathbf{u} \Big) \Big|_{\mathbf{u}=0} d\mathbf{w} \\
&=\; \frac{\pi^{n/2}}{N^{n/2} \sqrt{\det(-A)}} \cdot \frac{\partial}{\partial u_{i_1}} \cdots \frac{\partial}{\partial u_{i_m}} \exp\Big(-\frac{1}{4N} \mathbf{u}^T A^{-1} \mathbf{u} \Big) \Big|_{\mathbf{u}=0}.
\end{aligned}$$

Hence, we obtain (27) from this formula similarly as in the proof of Proposition 3.2.

The above expansion is concretely calculated as

$$\frac{\partial}{\partial u_{i_1}}\cdots\frac{\partial}{\partial u_{i_{2k}}}\exp\Big(-\frac{1}{4N}\mathbf{u}^TA^{-1}\mathbf{u}\Big)\Big|_{\mathbf{u}=0} \;=\; \frac{\partial}{\partial u_{i_1}}\cdots\frac{\partial}{\partial u_{i_{2k}}}\frac{1}{k!}\big(-\frac{1}{4N}\mathbf{u}^TA^{-1}\mathbf{u}\big)^k$$
$$= \frac{(-1)^k}{k!\,4^kN^k}\sum_{\sigma\in\mathfrak{S}_{2k}}\overline{a}_{i_{\sigma(1)}i_{\sigma(2)}}\cdots\overline{a}_{i_{\sigma(2k-1)}i_{\sigma(2k)}}.$$

In particular, we obtain the presentation of λ_1 from such calculation. □

Remark 3.6. Similarly as Remark 3.3, we can extend Proposition 3.5 to the case where $\psi(\mathbf{z})$ depends on N in such a way that $\psi(\mathbf{z})$ is of the form

$$\psi(\mathbf{z}) \;=\; \psi_0(\mathbf{z})+\psi_1(\mathbf{z})\frac{1}{N}+\psi_2(\mathbf{z})\frac{1}{N^2}+\cdots+\psi_m(\mathbf{z})\frac{1}{N^m}+r_m(\mathbf{z})\frac{1}{N^{m+1}}\,, \tag{29}$$

where $\psi_i(\mathbf{z})$'s are holomorphic functions independent of N, and we assume that $\psi_0(\mathbf{z})$ satisfies the assumption of the proposition and $|r_m(\mathbf{z})|$ is bounded by a constant which is independent of N.

4 Calculation by the Poisson summation formula

In this section, we calculate the sums corresponding to the integrals of the propositions in the previous section by the Poisson summation formula. Corresponding to Propositions 3.1, 3.2, 3.4 and 3.5 in the previous section, we show Propositions 4.1, 4.2, 4.5 and 4.6 in this section. We use Proposition 4.6 in the proof of Theorem 1.1 in Section 5. We remark that the Poisson summation formula has also been used in the study of large level asymptotics of quantum invariants of Seifert 3-manifolds [13, 14, 30, 31, 32].

Recall (see *e.g.* [33]) that the Poisson summation formula states that

$$\sum_{\mathbf{m}\in\mathbb{Z}^n} f(\mathbf{m}) \;=\; \sum_{\mathbf{m}\in\mathbb{Z}^n}\hat{f}(\mathbf{m}) \tag{30}$$

for a continuous integrable function f on $\mathbb{R}^n$ which satisfies that

$$|f(\mathbf{z})| \;\le\; C\big(1+|\mathbf{z}|\big)^{-n-\delta}, \qquad |\hat{f}(\mathbf{z})| \;\le\; C\big(1+|\mathbf{z}|\big)^{-n-\delta} \tag{31}$$

for some $C,\delta>0$, where $\hat{f}$ is the Fourier transform of f defined by

$$\hat{f}(\mathbf{w}) \;=\; \int_{\mathbb{R}^n} f(\mathbf{z})\,e^{-2\pi\sqrt{-1}\,\mathbf{w}^T\mathbf{z}}d\mathbf{z}.$$

Proposition 4.1. *Let a, c be complex numbers satisfying that*

$$\mathrm{Re}\,a \;<\; 0, \quad \big|\mathrm{Im}\,c\big| \;<\; -\frac{\pi}{2}\,\mathrm{Re}\,\frac{1}{a}\,.$$

We put

$$\Lambda \;=\; \Big\{\frac{k}{N}+c\;\in\;\mathbb{C}\;\Big|\;k\in\mathbb{Z},\;b_0\le\frac{k}{N}\le b_1\Big\},$$

$$C \;=\; \big\{ t+c \,\in\, \mathbb{C} \;\big|\; t\in\mathbb{R},\ b_0 \le t \le b_1 \big\}$$

for some $b_0, b_1 \in \mathbb{R}$ *and* $\varepsilon_0 > 0$ *satisfying that* $\operatorname{Re}(b_0+c) < 0 < \operatorname{Re}(b_1+c)$ *and* $\operatorname{Re} a(b_i+c)^2 < -\varepsilon_0$ $(i=0,1)$*; see Figure 4. Then, there exists* $\varepsilon > 0$ *such that*

$$\frac{1}{N}\sum_{z\in\Lambda} e^{N\cdot az^2} \;=\; \int_C e^{N\cdot az^2} dz \;+O\big(e^{-\varepsilon N}\big).$$

We note that ε depends on a, c and ε_0.

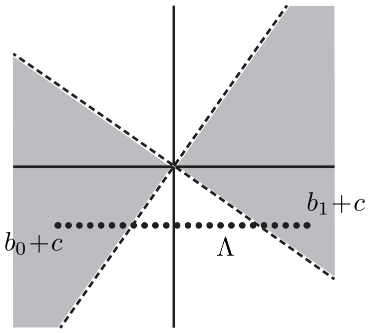


Figure 4: The domain $\{z\in\mathbb{C} \mid \operatorname{Re} az^2 < 0\}$ is shaded.

Proof. By Proposition 3.1, the required formula is rewritten,

$$\sum_{\substack{k\in\mathbb{Z}\\ b_0\le k/N\le b_1}} e^{N\cdot a(k/N+c)^2} \;=\; \frac{\sqrt{\pi}\cdot\sqrt{N}}{\sqrt{-a}} \;+O\big(e^{-\varepsilon N}\big).$$

Since

$$\sum_{\substack{k\in\mathbb{Z}\\ k/N<b_0}} e^{N\cdot a(k/N+c)^2} \;=\; O\big(e^{-\varepsilon N}\big), \qquad \sum_{\substack{k\in\mathbb{Z}\\ b_1<k/N}} e^{N\cdot a(k/N+c)^2} \;=\; O\big(e^{-\varepsilon N}\big),$$

for some $\varepsilon > 0$, it is sufficient to show that

$$\sum_{k\in\mathbb{Z}} e^{N\cdot a(k/N+c)^2} \;=\; \frac{\sqrt{\pi}\cdot\sqrt{N}}{\sqrt{-a}} \;+O\big(e^{-\varepsilon N}\big). \tag{32}$$

In order to apply the Poisson summation formula (30), we put $f(z) = e^{N\cdot a(z/N+c)^2}$. Then, its Fourier transform is

$$\begin{aligned}
\hat f(\zeta) \;&=\; \int_{-\infty}^{\infty} e^{N\cdot a(z/N+c)^2 - 2\pi\sqrt{-1}\,\zeta z} dz \\
&=\; \int_{-\infty}^{\infty} e^{(a/N)(z+cN-\pi\sqrt{-1}\zeta N/a)^2} dz \cdot e^{N\pi^2\zeta^2/a\,+\,2\pi\sqrt{-1}\,cN\zeta} \\
&=\; \frac{\sqrt{\pi}\cdot\sqrt{N}}{\sqrt{-a}}\, e^{N\pi^2\zeta^2/a\,+\,2\pi\sqrt{-1}\,cN\zeta},
\end{aligned}$$

where we obtain the last equality by Proposition 3.1. These $f(z)$ and $\hat{f}(\zeta)$ satisfy the assumption (31) of the Poisson summation formula. Hence, by the Poisson summation formula (30),

$$\sum_{k\in\mathbb{Z}} e^{N\cdot a(k/N+c)^2} \;=\; \frac{\sqrt{\pi}\cdot\sqrt{N}}{\sqrt{-a}} \sum_{m\in\mathbb{Z}} e^{N\pi^2 m^2/a + 2\pi\sqrt{-1}\,cNm}.$$

The summand at $m=0$ gives the right-hand side of (32). When $m\neq 0$, the power of the summand is

$$N\cdot \pi m^2\big(\frac{\pi}{a}+\frac{2\sqrt{-1}\,c}{m}\big),$$

and its real part is negative by the assumption of the proposition. Hence, the summands at $m\neq 0$ are of order $e^{-\varepsilon N}$ for some $\varepsilon>0$. Therefore, we obtain (32). □

We generalize Proposition 4.1 to the case where there are perturbative terms in the exponential of the summand.

Proposition 4.2. *We put*

$$\Lambda \;=\; \Big\{\frac{k}{N}+c\,\in\,\mathbb{C}\;\Big|\; k\in\mathbb{Z},\; b_0\le\frac{k}{N}\le b_1\Big\},$$
$$C \;=\; \big\{t+c\,\in\,\mathbb{C}\;\big|\; t\in\mathbb{R},\; b_0\le t\le b_1\big\}$$

for some $b_0,b_1\in\mathbb{R}$ and $c\in\mathbb{C}$. Let a be a complex number whose real part is negative, let δ_0 be a real positive number, and let $\psi(z)$ be a holomorphic function of the form,

$$\psi(z) \;=\; az^2+b_3z^3+b_4z^4+\cdots,$$

defined in a neighborhood of 0 including the δ_0-neighborhood of C. The domain (15) has two connected components in a neighborhood of 0. We assume that b_0+c and b_1+c are in these two components respectively, and $\operatorname{Re}\psi(b_i+c)<-\varepsilon_0$ $(i=0,1)$ for some $\varepsilon_0>0$. (See Figure 5.) Further, we assume that

$$\begin{gathered}b_0+c \text{ and } b_1+c \text{ are in the same connected component of}\\ \big\{w+\delta\sqrt{-1}\in\mathbb{C}\;\big|\;w\in C,\;\delta\in[0,\delta_0],\;\operatorname{Re}\big(\psi(w+\delta\sqrt{-1})-2\pi\delta\big)<0\big\},\end{gathered}\tag{33}$$

and

$$\begin{gathered}\text{they are in the same connected component of}\\ \big\{w-\delta\sqrt{-1}\in\mathbb{C}\;\big|\;w\in C,\;\delta\in[0,\delta_0],\;\operatorname{Re}\big(\psi(w-\delta\sqrt{-1})-2\pi\delta\big)<0\big\}.\end{gathered}\tag{34}$$

Then,

$$\frac{1}{N}\sum_{z\in\Lambda} e^{N\psi(z)} \;=\; \int_C e^{N\psi(z)}dz \;+O(e^{-N\varepsilon})$$

for some $\varepsilon>0$.

We note that ε depends on $\psi(z)$, c, δ_0 and ε_0; in particular, ε directly depends on ε_2' of (39) in the proof of the proposition.

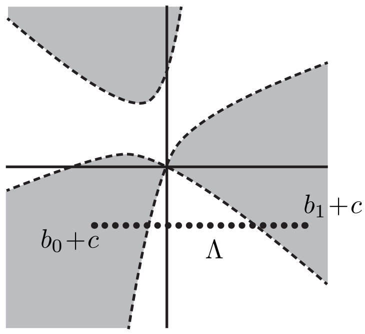


Figure 5: The domain $\{z \in \mathbb{C} \mid \operatorname{Re}\psi(z) < 0\}$ is shaded.

Proof. The sum of the left-hand side of the required formula is rewritten,

$$\sum_{\substack{k \in \mathbb{Z} \\ b_0 \le k/N \le b_1}} \exp\Big(N \cdot \psi\big(\frac{k}{N} + c\big)\Big). \tag{35}$$

In order to apply the Poisson summation formula (30), we put

$$f(z) \;=\; g\big(\frac{z}{N} + c\big)\,\exp\Big(N \cdot \psi\big(\frac{z}{N} + c\big)\Big),$$

where g is a differentiable function on $\mathbb{R} + c$ satisfying that

$$g(w) \;=\; \begin{cases} 1 & \text{if } w \in C, \\ 0 & \text{if } w \notin N(C), \end{cases}$$
$$0 \le g(w) \le 1 \quad \text{if } w \in N(C) - C.$$

Here, $N(C)$ is a neighborhood of C in $\mathbb{R}+c$ such that $N(C)-C$ is included in the domain $\big\{z \in \mathbb{C} \;\big|\; \operatorname{Re}\psi(z) < -\varepsilon_0/2\big\}$. Then, the Fourier transform of f is given by

$$\begin{aligned} \hat{f}(\zeta) \;&=\; \int_{\mathbb{R}} g\big(\frac{z}{N} + c\big)\,\exp\Big(N \cdot \psi\big(\frac{z}{N} + c\big)\Big)\,e^{-2\pi\sqrt{-1}\,\zeta z}dz \\ &=\; N \int_{\mathbb{R}+c} g(w)\,e^{N\left(\psi(w) - 2\pi\sqrt{-1}\,\zeta(w-c)\right)}dw, \end{aligned}$$

where we put $w = z/N + c$. Further,

$$\begin{aligned} \zeta^2 \hat{f}(\zeta) \;&=\; -\frac{1}{4\pi^2 N}\int_{\mathbb{R}+c} g(w)\,e^{N\,\psi(w)}\Big(\big(\frac{d}{dw}\big)^2 e^{-2\pi\sqrt{-1}\,N\zeta(w-c)}\Big)dw \\ &=\; -\frac{1}{4\pi^2 N}\int_{\mathbb{R}+c} \Big(\big(\frac{d}{dw}\big)^2 g(w)\,e^{N\,\psi(w)}\Big)e^{-2\pi\sqrt{-1}\,N\zeta(w-c)}dw \\ &=\; -\frac{1}{4\pi^2 N}\int_{\mathbb{R}+c} h(w)\,e^{N\,\psi(w)}\,e^{-2\pi\sqrt{-1}\,N\zeta(w-c)}dw, \end{aligned}$$

where we put

$$h(w) \;=\; g''(w) + 2\,g'(w)\,\psi'(w) + g(w)\big(\psi''(w) + \psi'(w)^2\big).$$

Since the above integral is bounded independently of ζ, $\hat{f}(\zeta)$ satisfies the assumption (31) of the Poisson summation formula. Further, $f(z)$ also satisfies (31). Therefore, by the Poisson summation formula (30),

$$(35) \;=\; \sum_{m\in\mathbb{Z}} \hat{f}(m).$$

When $m \neq 0$, we have that

$$\begin{aligned} \hat{f}(m) \;&=\; -\frac{1}{4\pi^2 N}\cdot\frac{1}{m^2}\int_{\mathbb{R}+c} h(w)\, e^{N\left(\psi(w)-2\pi\sqrt{-1}\,m(w-c)\right)}dw \\ &=\; -\frac{1}{4\pi^2 N}\cdot\frac{1}{m^2}\int_{C} \left(\psi''(w)+\psi'(w)^2\right) e^{N\left(\psi(w)-2\pi\sqrt{-1}\,m(w-c)\right)}dw \qquad (36) \\ &\quad\; -\frac{1}{4\pi^2 N}\cdot\frac{1}{m^2}\int_{N(C)-C} h(w)\, e^{N\left(\psi(w)-2\pi\sqrt{-1}\,m(w-c)\right)}dw, \qquad (37) \end{aligned}$$

since $h(w) = \psi''(w)+\psi'(w)^2$ for $w\in C$ and $h(w)=0$ for $w\in(\mathbb{R}+c)-N(C)$. Further, since $\operatorname{Re}\psi(w) < -\varepsilon_0/2$ for $w\in N(C)-C$,

$$\sum_{m\neq 0}(37) \;=\; O(e^{-N\varepsilon_1})$$

for some $\varepsilon_1>0$. Furthermore, when $m>0$, by pushing the contour C to a contour C' in the domain of (34), we can show that

$$\begin{aligned} &\int_C \left(\psi''(w)+\psi'(w)^2\right) e^{N\left(\psi(w)-2\pi\sqrt{-1}\,m(w-c)\right)}dw \\ &=\; \int_{C'} \left(\psi''(w)+\psi'(w)^2\right) e^{N\left(\psi(w)-2\pi\sqrt{-1}\,m(w-c)\right)}dw \;=\; O(e^{-N\varepsilon_2}) \qquad (38) \end{aligned}$$

for some $\varepsilon_2>0$, which we can choose independently of m, since there exists $\varepsilon_2'>0$ such that

$$\operatorname{Re}\left(\psi(w)-2\pi\sqrt{-1}\,(w-c)\right) \;<\; -\varepsilon_2' \qquad (39)$$

for any $w\in C'$. Hence,

$$\sum_{m>0}(36) \;=\; O(e^{-N\varepsilon_2}).$$

When $m<0$, by pushing the contour C into the domain of (33), we similarly obtain

$$\sum_{m<0}(36) \;=\; O(e^{-N\varepsilon_3})$$

for some $\varepsilon_3>0$. Therefore,

$$\sum_{m\neq 0}\hat{f}(m) \;=\; O(e^{-N\varepsilon_4})$$

for some $\varepsilon_4>0$. Hence,

$$(35) \;=\; \hat{f}(0) + O(e^{-N\varepsilon_4}) \;=\; N\int_C e^{N\psi(w)}dw \;+O(e^{-N\varepsilon_5})$$

for some $\varepsilon_5>0$, and this implies the required formula. □

Remark 4.3. The assumptions (33) and (34) of Proposition 4.2 can be replaced with the condition that there exist positive integers m_0, m_1 and positive real numbers δ_0, δ_1 such that for any integer m satisfying that $-m_0 < m < 0$ or $0 < m < m_1$,

$$\int_C e^{N\left(\psi(w)-2\pi\sqrt{-1}\,m(w-c)\right)}dw \;=\; O(e^{-N\varepsilon})$$

for some $\varepsilon > 0$, and

$b_0 + c$ and $b_1 + c$ are in the same connected component of
$\big\{w + \delta\sqrt{-1} \in \mathbb{C} \;\big|\; w \in C,\ \delta \in [0, \delta_1],\ \mathrm{Re}\,\big(\psi(w + \delta\sqrt{-1}) - 2\pi\delta m_1\big) < 0\big\},$

and

they are in the same connected component of
$\big\{w - \delta\sqrt{-1} \in \mathbb{C} \;\big|\; w \in C,\ \delta \in [0, \delta_0],\ \mathrm{Re}\,\big(\psi(w - \delta\sqrt{-1}) - 2\pi\delta m_0\big) < 0\big\}.$

Proposition 4.2 for this assumption can be proved by modifying the above proof of Proposition 4.2.

Remark 4.4. Proposition 4.2 can naturally be extended to the case where the holomorphic function $\psi(z)$ depends on N, if $\psi(z)$ uniformly converges to $\psi_0(z)$ as $N \to \infty$, and $\psi_0(z)$ satisfies the assumption of the proposition, and $|\psi''(z)+\psi'(z)^2|$ is bounded by a constant which is independent of N. In this case, (39) holds for sufficiently large N, and hence, (38) holds. Therefore, Proposition 4.2 also holds in this case, where we note that we can choose ε independently of N.

We generalize Proposition 4.1 to the case of n variables.

Proposition 4.5. *For* $\mathbf{c} \in \mathbb{C}^n$ *and an oriented* n*-ball* D' *in* $\mathbb{R}^n$*, we put*

$$\begin{aligned} \Lambda &= \Big\{\frac{1}{N}\mathbf{k} + \mathbf{c} \in \mathbb{C}^n \;\Big|\; \mathbf{k} \in \mathbb{Z}^n,\ \frac{1}{N}\mathbf{k} \in D'\Big\}, \\ D &= \big\{\mathbf{z} + \mathbf{c} \in \mathbb{C}^n \;\big|\; \mathbf{z} \in D' \subset \mathbb{R}^n\big\}. \end{aligned}$$

Let A *be a non-singular symmetric complex* $n\times n$ *matrix such that* $\mathrm{Re}\,(A^{-1})$ *is negative definite. We assume that* ∂D *is included in the domain (24). Further, we assume that*

$$(\mathrm{Im}\,\mathbf{c})^T\,\mathbf{m} \;<\; -\frac{\pi}{2}\,\mathbf{m}^T\cdot \mathrm{Re}\,(A^{-1})\cdot \mathbf{m}$$

for any $\mathbf{m} \in \mathbb{Z}^n - \{\mathbf{0}\}$*. Then,*

$$\frac{1}{N^n}\sum_{\mathbf{z}\,\in\,\Lambda} e^{N\cdot \mathbf{z}^T A\,\mathbf{z}} \;=\; \int_D e^{N\cdot \mathbf{z}^T A\,\mathbf{z}} d\mathbf{z} \;+ O(e^{-N\varepsilon}),$$

for some $\varepsilon > 0$.

Proof. By Proposition 3.4, the required formula is rewritten,

$$\sum_{\mathbf{z}\,\in\,\Lambda} e^{N\cdot \mathbf{z}^T A\,\mathbf{z}} \;=\; \frac{\pi^{n/2}N^{n/2}}{\sqrt{\det(-A)}} \;+ O\big(e^{-\varepsilon N}\big). \tag{40}$$

Further, in a similar way as in the proof of Proposition 4.1, putting $f(\mathbf{z}) = e^{N\cdot(\mathbf{z}/N+\mathbf{c})^T A\,(\mathbf{z}/N+\mathbf{c})}$, we can show that

$$\hat{f}(\zeta) \;=\; \frac{\pi^{n/2}N^{n/2}}{\sqrt{\det(-A)}}\, e^{N\pi^2\,\zeta^T A^{-1}\zeta \,+\, 2\pi\sqrt{-1}N\mathbf{c}^T\zeta}$$

by using Proposition 3.4. Hence, we obtain

$$\sum_{\mathbf{k}\in\mathbb{Z}^n} e^{N\cdot(\mathbf{k}/N+\mathbf{c})^T A\,(\mathbf{k}/N+\mathbf{c})} \;=\; \frac{\pi^{n/2}N^{n/2}}{\sqrt{\det(-A)}} \sum_{\mathbf{m}\in\mathbb{Z}^n} e^{N\pi^2\,\mathbf{m}^T A^{-1}\mathbf{m} \,+\, 2\pi\sqrt{-1}N\mathbf{c}^T\mathbf{m}}$$

by the Poisson summation formula (30). By the assumption of the proposition, the real part of

$$N\pi^2\,\mathbf{m}^T A^{-1}\mathbf{m} \;+\; 2\pi\sqrt{-1}N\mathbf{c}^T\mathbf{m}$$

is negative for any $\mathbf{m}\in\mathbb{Z}^n-\{\mathbf{0}\}$. Therefore, only the summand at $\mathbf{m}=\mathbf{0}$ survives, and we obtain (40) from it. □

We generalize Proposition 4.5 to the case where there are perturbative terms in the exponential of the summand. Let $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ be the standard basis vectors of $\mathbb{C}^n$,

$$\mathbf{e}_1 = (1,0,\cdots,0)^T, \quad \mathbf{e}_2 = (0,1,0,\cdots,0)^T, \quad \cdots, \quad \mathbf{e}_n = (0,\cdots,0,1)^T.$$

Proposition 4.6. *For $\mathbf{c}\in\mathbb{C}^n$ and an oriented n-ball D' in $\mathbb{R}^n$, we put*

$$\begin{aligned} \Lambda \;&=\; \Big\{\frac{1}{N}\mathbf{k}+\mathbf{c}\,\in\mathbb{C}^n \;\Big|\; \mathbf{k}\in\mathbb{Z}^n,\ \frac{1}{N}\mathbf{k}\,\in D'\Big\},\\ D \;&=\; \big\{\mathbf{z}+\mathbf{c}\in\mathbb{C}^n \;\big|\; \mathbf{z}\in D'\subset\mathbb{R}^n\big\}. \end{aligned}$$

Let A be a non-singular symmetric complex $n\times n$ matrix, and let $\psi(\mathbf{z})$ be a holomorphic function of the form (25) defined in a neighborhood of $\mathbf{0}\in\mathbb{C}^n$ including D. We assume that ∂D is included in the domain $\big\{\mathbf{z}\in\mathbb{C}^n \mid \operatorname{Re}\psi(\mathbf{z}) < -\varepsilon_0\big\}$ for some $\varepsilon_0>0$. Further, we assume that there exist $\delta_i, \delta_i' > 0$ $(i=1,\cdots,n)$ such that

$$\begin{aligned} &\partial D \text{ is null-homotopic in}\\ &\big\{\mathbf{w}+\delta\sqrt{-1}\,\mathbf{e}_i\,\in\mathbb{C}^n \;\big|\; \mathbf{w}\in D,\ \delta\in[0,\delta_i],\ \operatorname{Re}\big(\psi(\mathbf{w}+\delta\sqrt{-1}\,\mathbf{e}_i)-2\pi\delta\big)<0\big\}, \end{aligned} \tag{41}$$

and

$$\begin{aligned} &\partial D \text{ is null-homotopic in}\\ &\big\{\mathbf{w}-\delta\sqrt{-1}\,\mathbf{e}_i\,\in\mathbb{C}^n \;\big|\; \mathbf{w}\in D,\ \delta\in[0,\delta_i'],\ \operatorname{Re}\big(\psi(\mathbf{w}-\delta\sqrt{-1}\,\mathbf{e}_i)-2\pi\delta\big)<0\big\}, \end{aligned} \tag{42}$$

for $i=1,\cdots,n$, assuming that ψ is holomorphic in these domains. Then,

$$\frac{1}{N^n}\sum_{\mathbf{z}\in\Lambda} e^{N\,\psi(\mathbf{z})} \;=\; \int_D e^{N\,\psi(\mathbf{z})}d\mathbf{z} \;+O(e^{-N\varepsilon}),$$

for some $\varepsilon>0$.

We note that ε depends on $\psi(\mathbf{z})$, $\mathbf{c}$, δ_i, δ_i' and ε_0.

Proof. The sum of the left-hand side of the required formula is rewritten,

$$\sum_{\substack{\mathbf{k}\in\mathbb{Z}^n\\ \mathbf{k}/N\in D'}} \exp\Big(N\cdot\psi\big(\frac{1}{N}\mathbf{k}+\mathbf{c}\big)\Big). \tag{43}$$

In order to apply the Poisson summation formula (30), we put

$$f(\mathbf{z}) \;=\; g\big(\frac{1}{N}\mathbf{z}+\mathbf{c}\big)\,\exp\Big(N\cdot\psi\big(\frac{1}{N}\mathbf{z}+\mathbf{c}\big)\Big),$$

where g is a differentiable function on $\mathbb{R}^n+\mathbf{c}$ satisfying that

$$g(\mathbf{w}) \;=\; \begin{cases} 1 & \text{if } \mathbf{w}\in D,\\ 0 & \text{if } \mathbf{w}\notin N(D),\end{cases}$$
$$0\le g(\mathbf{w})\le 1 \quad \text{if } \mathbf{w}\in N(D)-D.$$

Here, $N(D)$ is a neighborhood of D in $\mathbb{R}^n+\mathbf{c}$ such that $N(D)-D$ is included in the domain $\big\{\mathbf{z}\in\mathbb{C}^n \mid \operatorname{Re}\psi(\mathbf{z})<-\varepsilon_0/2\big\}$. Then, the Fourier transform of f is given by

$$\begin{aligned}\hat{f}(\zeta) \;&=\; \int_{\mathbb{R}^n} g\big(\frac{1}{N}\mathbf{z}+\mathbf{c}\big)\,\exp\Big(N\cdot\psi\big(\frac{1}{N}\mathbf{z}+\mathbf{c}\big)\Big)\,e^{-2\pi\sqrt{-1}\,\zeta^T\mathbf{z}}d\mathbf{z}\\ &=\; N^n\int_{\mathbb{R}^n+\mathbf{c}} g(\mathbf{w})\,e^{N\big(\psi(\mathbf{w})-2\pi\sqrt{-1}\,\zeta^T(\mathbf{w}-\mathbf{c})\big)}d\mathbf{w},\end{aligned}$$

where we put $\mathbf{w}=\mathbf{z}/N+\mathbf{c}$. Further, for an integer $l>n/2$,

$$\begin{aligned}|\zeta|^{2l}\hat{f}(\zeta) \;&=\; \big(\sum_{i=1}^n\zeta_i^2\big)^l\hat{f}(\zeta)\\ &=\; N^n\big(\frac{-1}{4\pi^2N}\big)^l\int_{\mathbb{R}^n+\mathbf{c}} g(\mathbf{w})\,e^{N\,\psi(\mathbf{w})}\Big(\big(\sum_{i=1}^n\frac{\partial^2}{\partial w_i^2}\big)^l e^{-2\pi\sqrt{-1}\,N\zeta^T(\mathbf{w}-\mathbf{c})}\Big)d\mathbf{w}\\ &=\; N^n\big(\frac{-1}{4\pi^2N}\big)^l\int_{\mathbb{R}^n+\mathbf{c}}\Big(\big(\sum_{i=1}^n\frac{\partial^2}{\partial w_i^2}\big)^l g(\mathbf{w})\,e^{N\,\psi(\mathbf{w})}\Big)e^{-2\pi\sqrt{-1}\,N\zeta^T(\mathbf{w}-\mathbf{c})}d\mathbf{w}\\ &=\; N^n\big(\frac{-1}{4\pi^2N}\big)^l\int_{\mathbb{R}^n+\mathbf{c}} h(\mathbf{w})\,e^{N\,\psi(\mathbf{w})}\,e^{-2\pi\sqrt{-1}\,N\zeta^T(\mathbf{w}-\mathbf{c})}d\mathbf{w},\end{aligned}$$

where $h(\mathbf{w})$ is some polynomial in derivatives of $g(\mathbf{w})$ and $\psi(\mathbf{w})$. Since the above integral is bounded independently of ζ, $\hat{f}(\zeta)$ satisfies the assumption (31) of the Poisson summation formula. Further, $f(\mathbf{z})$ also satisfies (31). Therefore, by the Poisson summation formula (30),

$$(43) \;=\; \sum_{\mathbf{m}\in\mathbb{Z}^n}\hat{f}(\mathbf{m}).$$

When $\mathbf{m}\neq\mathbf{0}$, we have that

$$\hat{f}(\mathbf{m}) \;=\; N^n\big(\frac{-1}{4\pi^2N}\big)^l\cdot\frac{1}{|\mathbf{m}|^{2l}}\int_{\mathbb{R}^n+\mathbf{c}} h(\mathbf{w})\,e^{N\big(\psi(\mathbf{w})-2\pi\sqrt{-1}\,\mathbf{m}^T(\mathbf{w}-\mathbf{c})\big)}d\mathbf{w}$$

$$= \; N^n\Big(\frac{-1}{4\pi^2 N}\Big)^l \cdot \frac{1}{|\mathbf{m}|^{2l}} \int_D \Psi(\mathbf{w})\, e^{N\big(\psi(\mathbf{w})-2\pi\sqrt{-1}\,\mathbf{m}^T(\mathbf{w}-\mathbf{c})\big)} d\mathbf{w} \tag{44}$$

$$+ N^n\Big(\frac{-1}{4\pi^2 N}\Big)^l \cdot \frac{1}{|\mathbf{m}|^{2l}} \int_{N(D)-D} h(\mathbf{w})\, e^{N\big(\psi(\mathbf{w})-2\pi\sqrt{-1}\,\mathbf{m}^T(\mathbf{w}-\mathbf{c})\big)} d\mathbf{w}, \tag{45}$$

where $\Psi(\mathbf{w})$ is some polynomial in (at most the $2l$th) derivatives of $\psi(\mathbf{w})$. Further, since $\operatorname{Re}\psi(\mathbf{w}) < 0$ for $\mathbf{w} \in N(D) - D$,

$$\sum_{\mathbf{m}\neq\mathbf{0}} (45) \;=\; O(e^{-N\varepsilon_1})$$

for some $\varepsilon_1 > 0$. Furthermore, when $m_1 > 0$, pushing the contour D into the domain of (42), we obtain

$$\sum_{\substack{\mathbf{m}\in\mathbb{Z}^n \\ m_1>0}} (44) \;=\; O(e^{-N\varepsilon_2})$$

for some $\varepsilon_2 > 0$, similarly as in the proof of Proposition 4.2. Similarly we obtain

$$\sum_{\substack{\mathbf{m}\in\mathbb{Z}^n \\ m_1<0}} (44) \;=\; O(e^{-N\varepsilon_3})$$

for some $\varepsilon_3 > 0$, from the assumption (41). Hence,

$$\sum_{\mathbf{m}\neq\mathbf{0}} (44) \;=\; \sum_{\substack{\mathbf{m}\neq\mathbf{0} \\ m_1=0}} (44) \;+ O(e^{-N\varepsilon_4})$$

for some $\varepsilon_4 > 0$. By repeating this argument for $m_2, \cdots, m_n$, we obtain

$$\sum_{\mathbf{m}\neq\mathbf{0}} (44) \;=\; O(e^{-N\varepsilon_5})$$

for some $\varepsilon_5 > 0$. Therefore,

$$(43) \;=\; \hat{f}(\mathbf{0}) \;+ O(e^{-N\varepsilon_6}) \;=\; N^n \int_D e^{N\,\psi(\mathbf{w})} d\mathbf{w} \;+ O(e^{-N\varepsilon_6})$$

for some $\varepsilon_6 > 0$, and this implies the required formula. □

Remark 4.7. The assumptions (41) and (42) of Proposition 4.6 can be modified similarly as in Remark 4.3.

Remark 4.8. Similarly as in Remark 4.4, Proposition 4.6 can naturally be extended to the case where the holomorphic function $\psi(\mathbf{z})$ depends on N, if $\psi(\mathbf{z})$ uniformly converges to $\psi_0(\mathbf{z})$ as $N \to \infty$, and $\psi_0(\mathbf{z})$ satisfies the assumption of the proposition, and $|\Psi(\mathbf{z})|$ is bounded by a constant which is independent of N. Similarly as in Remark 4.4, we can also choose ε independently of N in this case.

5 Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1. In Section 5.1, we give a proof of the theorem by using lemmas proved in Sections 5.2, 5.3 and 5.4. In Section 5.2, we show lemmas which calculate the asymptotic expansion. In Section 5.3, we show a lemma which verifies the assumption of the saddle point method. In Section 5.4, we show a lemma which verifies the assumption of the Poisson summation formula.

5.1 Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1, which presents the asymptotic expansion of the Kashaev invariant $\langle 5_2 \rangle_N$ of the 5_2 knot. To obtain the asymptotic expansion, as mentioned in Section 2.2, we rewrite the sum (14) of $\langle 5_2 \rangle_N$ by using an integral by the Poisson summation formula (Proposition 4.6) and calculate the asymptotic expansion of the integral by the saddle point method (Proposition 3.5).

Proof of Theorem 1.1. We recall that $\langle 5_2 \rangle_N$ is presented by the sum (14). By Proposition 4.6 (Poisson summation formula) (see also Remark 4.8 and Appendix B), this sum is expressed by an integral,

$$\langle 5_2 \rangle_N \;=\; e^{N\varsigma} N^{5/2}\, q \left(\int_{\Delta'} \exp\Big(N \cdot V(t,s) - N\varsigma\Big) dt\, ds + O(e^{-N\varepsilon_4}) \right) \tag{46}$$

for some $\varepsilon_4 > 0$, noting that we verify the assumptions of Proposition 4.6 in Lemma 5.8 below.

In order to apply the saddle point method (Proposition 3.5, see also Remark 3.6 and Appendix B) to (46), we consider a critical point of $\hat{V}(t,s)$. From the definition of $\hat{V}(t,s)$, a critical point is a solution of the following equations,

$$\begin{aligned} \frac{\partial}{\partial t}\hat{V}(t,s) &= -\log(1-e^{-2\pi\sqrt{-1}\,t}) - 2\pi\sqrt{-1}\,\big(t+s-\tfrac{1}{2}\big) \;=\; 0, \\ \frac{\partial}{\partial s}\hat{V}(t,s) &= -2\,\log(1-e^{-2\pi\sqrt{-1}\,s}) - 2\pi\sqrt{-1}\,\big(t+2s-1\big) \;=\; 0. \end{aligned} \tag{47}$$

Hence, putting $x = e^{2\pi\sqrt{-1}\,t}$ and $y = e^{2\pi\sqrt{-1}\,s}$,

$$\big(1-\frac{1}{x}\big)xy \;=\; -1, \qquad \big(1-\frac{1}{y}\big)^2 xy^2 \;=\; 1.$$

Therefore,

$$x \;=\; 1-\frac{1}{y}\,, \qquad (y-1)^3 \;=\; y.$$

Let (x_0, y_0) be the solution of these equations mentioned in the introduction. We consider the critical point (t_0, s_0) such that $(\operatorname{Re} t_0, \operatorname{Re} s_0) \in \Delta'$ and $(e^{2\pi\sqrt{-1}\,t_0}, e^{2\pi\sqrt{-1}\,s_0}) = (x_0, y_0)$. It is numerically given by

$$t_0 \;=\; 0.22404487... - \sqrt{-1}\cdot 0.04475430...\,, \qquad s_0 \;=\; 0.16393269... + \sqrt{-1}\cdot 0.06713145...\,.$$

Let (t_c, s_c) be the critical point of $V(t,s)$ which goes to (t_0, s_0) as $N \to \infty$. We consider to apply Proposition 3.5 to $V(t-t_c, s-s_c)$. We consider the expansion

$$V(t,s) \;=\; V(t_c,s_c) + \frac{1}{2} V_{tt}\cdot(t-t_c)^2 + V_{ts}\cdot(t-t_c)(s-s_c) + \frac{1}{2} V_{ss}\cdot(s-s_c)^2 + \cdots \quad (48)$$

at the critical point, where we put

$$V_{tt} \;=\; \frac{\partial^2 V}{\partial t^2}(t_c,s_c), \quad V_{ts} \;=\; \frac{\partial^2 V}{\partial t\, \partial s}(t_c,s_c), \quad V_{ss} \;=\; \frac{\partial^2 V}{\partial s^2}(t_c,s_c).$$

As we show in Section 5.3, we can make a concrete homotopy which moves Δ' to a new domain containing the above critical point in such a way that it satisfies the assumption of the saddle point method.[6] Hence, we obtain the following expansion by applying Proposition 3.5 (see also Remark 3.6 and Appendix B) to (46),

$$\langle\, 5_2\, \rangle_N \;=\; N^{5/2}\, q\, \exp\Big(N\, V(t_c,s_c)\Big)\cdot \frac{2\pi}{N}\big(V_{tt}V_{ss}-V_{ts}^2\big)^{-1/2}\Big(1+\sum_{i=1}^{d}\lambda_i \hbar^i \;+O(\hbar^{d+1})\Big), \quad (49)$$

where λ_i's are given in the proposition, noting that we verify the assumptions of Proposition 3.5 in Lemma 5.7 below. Here, we also note that $e^{N\varsigma}\times O(e^{-N\varepsilon_4})$ in (46) is included in $\exp\big(N\,V(t_c,s_c)\big)\times O(\hbar^{d+1})$ in (49); see Lemma 5.2 below for the behavior of $\exp\big(N\,V(t_c,s_c)\big)$.

We calculate the concrete form of (49), for simplicity, when $d=1$. By Lemmas 5.2, 5.3 and 5.4 below, (49) is rewritten,

$$\begin{aligned}
\langle\, 5_2\, \rangle_N \;&=\; N^{5/2} q \,\times\, e^{N\varsigma}\, e^{-\pi\sqrt{-1}/4}\big(1-y_0\big)^{1/2}\Big(1+(C_1+C_2)\hbar \;+O(\hbar^2)\Big)\times\frac{2\pi}{N} \\
&\qquad \times\frac{1}{2\pi\sqrt{-1}}\big(-2y_0-1\big)^{-1/2}\Big(1+C_3\hbar\;+O(\hbar^2)\Big)\times\Big(1+(C_4+C_5)\hbar\;+O(\hbar^2)\Big) \\
&=\; e^{N\varsigma}\, N^{3/2}\, e^{\pi\sqrt{-1}/4}\Big(\frac{2y_0+1}{y_0-1}\Big)^{-1/2}\Big(1+\hbar\big(1+C_1+C_2+C_3+C_4+C_5\big)\;+O(\hbar^2)\Big),
\end{aligned}$$

where the constants $C_1, \cdots, C_5$ are given in the lemmas. In particular, κ_1 of the theorem is given by

$$\begin{aligned}
\kappa_1 \;&=\; 1+C_1+C_2+C_3+C_4+C_5 \\
&=\; \frac{1}{184}(-16y_0^2+33y_0+9)+\frac{1}{24}(2y_0^2-5y_0)+\frac{1}{46}\big(8y_0^2-17y_0+9\big) \\
&\qquad +\frac{1}{92}(2y_0^2+15y_0)-\frac{1}{3174}(197y_0^2+127y_0+227)+1 \\
&=\; \frac{1}{12696}(1650y_0^2-3498y_0+2197)+1.
\end{aligned}$$

Hence, we obtain the required formula for $d=1$.

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[6]That is, the new domain is in the area $\operatorname{Re}\hat{V}(t,s) \le \varsigma_R - \varepsilon$ except for a neighborhood of the above critical point. In order to obtain this new domain, it is sufficient to push the shaded part in Figure 1 into the imaginary direction by $(-\sqrt{-1}\cdot 0.04475430..., \sqrt{-1}\cdot 0.06713145...)$. For details, see Section 5.3.

For general d, each λ_i of the expansion (49) is given by a linear sum of higher coefficients of the expansion (48) by Proposition 3.5 (see also Remark 3.6 and Appendix B). In the same way as the above case of $d = 1$, such coefficients are given by some polynomials in y_0 with rational coefficients. Hence, each κ_i of the theorem can be given by some polynomial in y_0 with rational coefficients. $\square$

5.2 Calculation of the asymptotic expansion

In this section, we show some lemmas used in the proof of Theorem 1.1.

Let (t_0, s_0) and (t_c, s_c) be the critical points of $\hat{V}(t,s)$ and $V(t,s)$ given in the previous section. We put $x_0 = e^{2\pi\sqrt{-1}\cdot t_0}$, $y_0 = e^{2\pi\sqrt{-1}\cdot s_0}$ and $x_c = e^{2\pi\sqrt{-1}\cdot t_c}$, $y_c = e^{2\pi\sqrt{-1}\cdot s_c}$. In order to show Lemma 5.2 below, we calculate (x_c, y_c) in terms of (x_0, y_0), as follows. We show a proof of the following lemma in Appendix C.

Lemma 5.1. *We can put $x_c = x_0 + x_1\hbar + O(\hbar^2)$ and $y_c = y_0 + y_1\hbar + O(\hbar^2)$ for some $x_1, y_1 \in \mathbb{C}$ in a sufficiently small neighborhood of (x_0, y_0) (i.e., for sufficiently large N). Here, $O(\hbar^2)$ means that the absolute value of the error term is bounded by $C\hbar^2$ with a constant C which is independent of N.*

We calculate x_1 and y_1, as follows. We put $x = e^{2\pi\sqrt{-1}\cdot t}$ and $y = e^{2\pi\sqrt{-1}\cdot s}$. From the definition of $V(t,s)$ and Proposition A.1, we have that

$$\begin{aligned} \frac{\partial}{\partial t}V(t,s) &= -\log\big(1-\frac{1}{x}\big) - 2\pi\sqrt{-1}\,\big(t+s-\frac{1}{2}-\frac{1}{2N}\big) \ +O(\hbar^2), \\ \frac{\partial}{\partial s}V(t,s) &= -2\log\big(1-\frac{1}{y}\big) - 2\pi\sqrt{-1}\,\big(t+2s-1-\frac{1}{2N}\big) \ +O(\hbar^2). \end{aligned} \tag{50}$$

Then, since (t_c, s_c) is a critical point of $V(t,s)$, we have that

$$\begin{cases} \big(1-\dfrac{1}{x_c}\big)(-q^{-1/2}x_c y_c) = 1 \ +O(\hbar^2), \\ \big(1-\dfrac{1}{y_c}\big)^2 q^{-1/2}x_c y_c^2 = 1 \ +O(\hbar^2). \end{cases}$$

These are rewritten,

$$\begin{cases} (1-x_c)y_c = q^{1/2} \ +O(\hbar^2), \\ q^{1/2}y_c = (y_c-1)^2(y_c-q^{1/2}) \ +O(\hbar^2). \end{cases}$$

Hence, putting $x_c = x_0 + x_1\hbar + O(\hbar^2)$ and $y_c = y_0 + y_1\hbar + O(\hbar^2)$ by Lemma 5.1, we have that

$$\begin{cases} x_0 = 1-\dfrac{1}{y_0}, \\ y_0 = (y_0-1)^3, \end{cases} \qquad \begin{cases} -\dfrac{x_1}{1-x_0}+\dfrac{y_1}{y_0} = \dfrac{1}{2}, \\ y_1+\dfrac{1}{2}y_0 = (y_0-1)^2\big(3y_1-\dfrac{1}{2}\big). \end{cases}$$

Therefore, we obtain that

$$y_1 = \frac{y_0^2}{2(2y_0+1)}, \qquad x_1 = -\frac{y_0+1}{2y_0(2y_0+1)}.$$

Further, in order to show Lemmas 5.2, 5.3 and 5.4 below, we calculate the expansion (48) concretely, as follows. Putting $\check{t} = t - t_c$, we expand $\varphi(1-t)/N$ at t_c as follows,

$$\frac{\varphi(1-t)}{N} = \frac{\varphi(1-t_c)}{N} - \frac{\varphi'(1-t_c)}{N}\check{t} + \frac{1}{2}\frac{\varphi''(1-t_c)}{N}\check{t}^2 - \frac{1}{6}\frac{\varphi^{(3)}(1-t_c)}{N}\check{t}^3 + \frac{1}{24}\frac{\varphi^{(4)}(1-t_c)}{N}\check{t}^4 + \cdots,$$

where we can calculate $\varphi^{(k)}(1-t_c)/N$ concretely by Proposition A.1. Hence, putting $\check{t} = t - t_c$ and $\check{s} = s - s_c$, we can show by concrete calculation that $V(t,s)$ is expanded at a critical point (t_c, s_c) in the following form,

$$V(t,s) = V(t_c,s_c) + (\text{degree 2 part}) + (\text{degree 3 part}) + (\text{degree 4 part}) + (\text{degree} \geq 5 \text{ part}), \tag{51}$$

where

$$(\text{degree 2 part}) = 2\pi\sqrt{-1}\Big(\frac{\check{t}^2}{2}\frac{1}{1-x_c} + \frac{\check{s}^2}{2}\frac{1+y_c}{1-y_c} - \frac{(\check{t}+\check{s})^2}{2}\Big) + O(\hbar^2), \tag{52}$$

$$(\text{degree 3 part}) = (2\pi\sqrt{-1})^2\,(y_0-1)\Big(\frac{\check{t}^3}{6}\cdot y_0 + \frac{\check{s}^3}{6}\cdot 2\Big) + O(\hbar), \tag{53}$$

$$(\text{degree 4 part}) = (2\pi\sqrt{-1})^3\Big(\frac{\check{t}^4}{24}(x_0^2+x_0)y_0^3 - \frac{\check{s}^4}{24}2(y_0+1)\Big) + O(\hbar). \tag{54}$$

Now, we show lemmas used in the proof of Theorem 1.1.

Lemma 5.2. *Under the notation in the proof of Theorem 1.1,*

$$\exp\Big(N\,V(t_c,s_c)\Big) = e^{N\varsigma}\,e^{-\pi\sqrt{-1}/4}\big(1-y_0\big)^{1/2}\Big(1 + (C_1+C_2)\hbar + O(\hbar^2)\Big),$$

where we put

$$C_1 = \frac{1}{184}(-16y_0^2 + 33y_0 + 9), \qquad C_2 = \frac{1}{24}(2y_0^2 - 5y_0).$$

Proof. We put $t_c = t_0 + t_1/N + \cdots$ and $s_c = s_0 + s_1/N + \cdots$. Recalling that $x_c = e^{2\pi\sqrt{-1}\cdot t_c}$ and $y_c = e^{2\pi\sqrt{-1}\cdot s_c}$, we have that

$$t_1 = \frac{x_1}{x_0} = -\frac{y_0+1}{2(y_0-1)(2y_0+1)}, \qquad s_1 = \frac{y_1}{y_0} = \frac{y_0}{2(2y_0+1)}.$$

Putting $\check{t} = -t_1/N$ and $\check{s} = -s_1/N$ in the degree 2 part (52) of the expansion (51), we have that

$$V(t_0,s_0) = V(t_c,s_c) + 2\pi\sqrt{-1}\,\Big(\frac{t_1^2}{2N^2}\frac{1}{1-x_0} + \frac{s_1^2}{2N^2}\frac{1+y_0}{1-y_0} - \frac{(t_1+s_1)^2}{2N^2}\Big) + O(\hbar^3).$$

Hence,

$$N\,V(t_c,s_c) = N\,V(t_0,s_0) + C_1\hbar + O(\hbar^2),$$

where we put

$$C_1 \;=\; -\frac{1}{2}\Big(t_1^2\,\frac{1}{1-x_0}+s_1^2\,\frac{1+y_0}{1-y_0}-(t_1+s_1)^2\Big) \;=\; \frac{1}{184}(-16y_0^2+33y_0+9).$$

Hence,

$$\exp\Big(N\,V(t_c,s_c)\Big) \;=\; \exp\Big(N\,V(t_0,s_0)\Big)\cdot\Big(1+C_1\hbar\;+O(\hbar^2)\Big). \tag{55}$$

Further, we calculate $V(t_0,s_0)$, as follows. From the definition of $V(t,s)$, we have that

$$\begin{aligned}V(t_0,s_0) \;=\;& \frac{1}{N}\big(-\varphi(1-t_0)-2\,\varphi(1-s_0)\big)-\frac{1}{2\pi\sqrt{-1}}\frac{\pi^2}{6}\\ &-2\pi\sqrt{-1}\Big(\frac{1}{2}\big(t_0+s_0-\frac{1}{2N}\big)^2+\frac{1}{2}s_0^2-\frac{1}{2}t_0-s_0+\frac{1}{6}\Big)-\frac{3\pi\sqrt{-1}}{4N}+\frac{\pi\sqrt{-1}}{4N^2}.\end{aligned}$$

Since this goes to $\hat V(t_0,s_0)$ as $N\to\infty$, we have that

$$\begin{aligned}\hat V(t_0,s_0) \;=\;& \frac{1}{2\pi\sqrt{-1}}\Big(-\operatorname{Li}_2\big(\frac{1}{x_0}\big)-2\operatorname{Li}_2\big(\frac{1}{y_0}\big)-\frac{\pi^2}{6}\Big)\\ &-2\pi\sqrt{-1}\Big(\frac{1}{2}(t_0+s_0)^2+\frac{1}{2}s_0^2-\frac{1}{2}t_0-s_0+\frac{1}{6}\Big)\\ =\;& \frac{1}{2\pi\sqrt{-1}}\Big(\operatorname{Li}_2(x_0y_0)+\operatorname{Li}_2\big(\frac{1}{x_0y_0}\big)+\operatorname{Li}_2(y_0)-\operatorname{Li}_2\big(\frac{1}{y_0}\big)-\operatorname{Li}_2\big(\frac{1}{x_0}\big)-\frac{\pi^2}{6}\Big) \;=\; \varsigma.\end{aligned}$$

Hence, by Proposition A.1,

$$V(t_0,s_0) \;=\; \varsigma+2\pi\sqrt{-1}\,\frac{t_0+s_0}{2N}-\frac{3\pi\sqrt{-1}}{4N}+\frac{2\pi\sqrt{-1}}{N^2}\Big(\frac{1}{24(x_0-1)}+\frac{1}{12(y_0-1)}\Big)\;+O\big(\frac{1}{N^3}\big).$$

Therefore,

$$\begin{aligned}\exp\Big(N\,V(t_0,s_0)\Big) \;&=\; e^{N\varsigma}e^{-\frac{3}{4}\pi\sqrt{-1}}\,\big(x_0y_0\big)^{1/2}\Big(1+C_2\hbar\;+O(\hbar^2)\Big)\\ &=\; e^{N\varsigma}e^{-\pi\sqrt{-1}/4}\,\big(1-y_0\big)^{1/2}\Big(1+C_2\hbar\;+O(\hbar^2)\Big),\end{aligned}$$

where we put

$$C_2 \;=\; \frac{1}{24(x_0-1)}+\frac{1}{12(y_0-1)} \;=\; \frac{1}{24}(2y_0^2-5y_0).$$

Hence, from the above formula and (55), we obtain the required formula. □

Lemma 5.3. *Under the notation in the proof of Theorem 1.1,*

$$\begin{aligned}V_{tt}V_{ss}-V_{ts}^2 \;&=\; -4\pi^2(-2y_0-1)\Big(1-2C_3\hbar\;+O(\hbar^2)\Big),\\ (V_{tt}V_{ss}-V_{ts}^2)^{-1/2} \;&=\; \frac{1}{2\pi\sqrt{-1}}\big(-2y_0-1\big)^{-1/2}\Big(1+C_3\hbar\;+O(\hbar^2)\Big),\end{aligned}$$

where we put

$$C_3 \;=\; \frac{1}{46}\big(8y_0^2-17y_0+9\big).$$

Proof. From the degree 2 part (52) of the expansion (51), we have that

$$\begin{aligned}
V_{tt} &= \frac{\partial^2 V}{\partial t^2}(t_c, s_c) = 2\pi\sqrt{-1}\Big(\frac{1}{1-x_c} - 1\Big) + O(\hbar^2) = 2\pi\sqrt{-1}\cdot\frac{x_c}{1-x_c} + O(\hbar^2), \\
V_{ts} &= \frac{\partial^2 V}{\partial t\,\partial s}(t_c, s_c) = -2\pi\sqrt{-1} + O(\hbar^2), \\
V_{ss} &= \frac{\partial^2 V}{\partial s^2}(t_c, s_c) = 2\pi\sqrt{-1}\Big(\frac{1+y_c}{1-y_c} - 1\Big) + O(\hbar^2) = 2\pi\sqrt{-1}\cdot\frac{2y_c}{1-y_c} + O(\hbar^2)
\end{aligned}$$

Hence,

$$\begin{aligned}
V_{tt}V_{ss} - V_{ts}^2 &= -4\pi^2\Big(\frac{x_c}{1-x_c}\cdot\frac{2y_c}{1-y_c} - 1\Big) + O(\hbar^2) \\
&= -4\pi^2\Big(\frac{2q^{-1/2}x_c y_c^2}{1-y_c} - 1\Big) + O(\hbar^2) \\
&= -4\pi^2\Big(-2y_c\cdot\frac{1-q^{-1/2}y_c}{1-y_c} - 1\Big) + O(\hbar^2) \\
&= -4\pi^2\Big(-y_c\big(2 + \frac{y_0}{1-y_0}\hbar\big) - 1\Big) + O(\hbar^2) \\
&= -4\pi^2(-2y_0 - 1)\Big(1 - \frac{y_0^2(y_0+2)}{(y_0-1)(2y_0+1)^2}\hbar\Big) + O(\hbar^2) \\
&= -4\pi^2(-2y_0 - 1)\big(1 - 2C_3\hbar\big) + O(\hbar^2),
\end{aligned}$$

where

$$C_3 = \frac{y_0^2(y_0+2)}{2(y_0-1)(2y_0+1)^2} = \frac{1}{46}\big(8y_0^2 - 17y_0 + 9\big).$$

Therefore, we obtain the required formulas. □

Lemma 5.4. *Under the notation in the proof of Theorem 1.1,*

$$\lambda_1 = C_4 + C_5\,,$$

where we put

$$C_4 = \frac{1}{92}(2y_0^2 + 15y_0), \qquad C_5 = -\frac{1}{3174}(197y_0^2 + 127y_0 + 227).$$

Proof. We show the lemma by using Proposition 3.5. As shown in Proposition 3.5, λ_1 consists of two contributions from the degree 3 part and the degree 4 part of the expansion (51).

We calculate the contribution from the degree 4 part of the expansion (51), as follows. As in (54), the degree 4 part of this expansion is given by

$$(2\pi\sqrt{-1})^3\Big(\frac{\check{t}^4}{24}(x_0^2 + x_0)y_0^3 - \frac{\check{s}^4}{24}2(y_0+1)\Big) + O(\hbar). \tag{56}$$

As explained in Proposition 3.5, the contribution from $\check{t}^4$ is calculated as

$$\begin{aligned}
& N\cdot\big(\frac{\partial}{\partial u_1}\big)^4 \exp\left(-\frac{1}{4N}\begin{pmatrix} u_1 \\ u_2\end{pmatrix}^T A^{-1}\begin{pmatrix} u_1 \\ u_2\end{pmatrix}\right)\Bigg|_{u_1=u_2=0} \\
&= \frac{1}{2}N\cdot\big(\frac{\partial}{\partial u_1}\big)^4\left(-\frac{1}{2N}\begin{pmatrix} u_1 \\ u_2\end{pmatrix}^T \begin{pmatrix} V_{tt} & V_{ts} \\ V_{ts} & V_{ss}\end{pmatrix}^{-1}\begin{pmatrix} u_1 \\ u_2\end{pmatrix}\right)^2 \\
&= \frac{1}{8N\,(V_{tt}V_{ss}-V_{ts}^2)^2}\big(\frac{\partial}{\partial u_1}\big)^4\big(V_{ss}u_1^2-2V_{ts}u_1u_2+V_{tt}u_2^2\big)^2 \\
&= \frac{1}{8N\,(V_{tt}V_{ss}-V_{ts}^2)^2}\cdot 4!\cdot V_{ss}^2 \\
&= \frac{4!}{8N\,(2\pi\sqrt{-1})^2\,(2y_0+1)^2}\big(\frac{2y_0}{1-y_0}\big)^2 \;+O(\hbar),
\end{aligned}$$

by using Lemma 5.3 and formulas in the proof of Lemma 5.3. Similarly, the contribution from $\check{s}^4$ is calculated as

$$\begin{aligned}
& \frac{1}{8N\,(V_{tt}V_{ss}-V_{ts}^2)^2}\big(\frac{\partial}{\partial u_2}\big)^4\big(V_{ss}u_1^2-2V_{ts}u_1u_2+V_{tt}u_2^2\big)^2 \\
&= \frac{1}{8N\,(V_{tt}V_{ss}-V_{ts}^2)^2}\cdot 4!\cdot V_{tt}^2 \\
&= \frac{4!}{8N\,(2\pi\sqrt{-1})^2\,(2y_0+1)^2}\big(\frac{x_0}{1-x_0}\big)^2 \;+O(\hbar).
\end{aligned}$$

Hence, the contribution from (56) is equal to

$$\frac{2\pi\sqrt{-1}}{8N\,(2y_0+1)^2}\Big((x_0^2+x_0)y_0^3\big(\frac{2y_0}{1-y_0}\big)^2-2(y_0+1)\big(\frac{x_0}{1-x_0}\big)^2\Big) \;=\; C_4\hbar,$$

where we put

$$C_4 \;=\; \frac{1}{8(2y_0+1)^2}\Big((x_0^2+x_0)y_0^3\big(\frac{2y_0}{1-y_0}\big)^2-2(y_0+1)\big(\frac{x_0}{1-x_0}\big)^2\Big) \;=\; \frac{1}{92}(2y_0^2+15y_0).$$

We calculate the contribution from the degree 3 part of the expansion (51), as follows. As in (53), the degree 3 part of this expansion is given by

$$(2\pi\sqrt{-1})^2\,(y_0-1)\Big(\frac{\check{t}^3}{6}\cdot y_0+\frac{\check{s}^3}{6}\cdot 2\Big) \;+O(\hbar),$$

and, hence, the corresponding degree 6 part is given by

$$\begin{aligned}
& \frac{1}{2}\,(2\pi\sqrt{-1})^4\,(y_0-1)^2\Big(\frac{\check{t}^3}{6}\cdot y_0+\frac{\check{s}^3}{6}\cdot 2\Big)^2 \;+O(\hbar) \\
&= \frac{1}{2}\,(2\pi\sqrt{-1})^4\,(y_0-1)^2\Big(\frac{\check{t}^6}{6^2}\cdot y_0^2+\frac{\check{s}^6}{6^2}\cdot 4+\frac{\check{t}^3\check{s}^3}{6^2}\cdot 4y_0\Big) \;+O(\hbar). \qquad (57)
\end{aligned}$$

The contribution from $\check{t}^6$ is calculated as

$$
\begin{aligned}
& N^2 \cdot \Big(\frac{\partial}{\partial u_1}\Big)^6 \exp\left(-\frac{1}{4N} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^T A^{-1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right)\Bigg|_{u_1=u_2=0} \\
&= \frac{1}{6} N^2 \cdot \Big(\frac{\partial}{\partial u_1}\Big)^6 \left(-\frac{1}{2N} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^T \begin{pmatrix} V_{tt} & V_{ts} \\ V_{ts} & V_{ss} \end{pmatrix}^{-1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right)^3 \\
&= \frac{-1}{48N\,(V_{tt}V_{ss} - V_{ts}^2)^3} \Big(\frac{\partial}{\partial u_1}\Big)^6 \big(V_{ss}u_1^2 - 2V_{ts}u_1u_2 + V_{tt}u_2^2\big)^3 \\
&= \frac{-1}{48N\,(V_{tt}V_{ss} - V_{ts}^2)^3} \cdot 6! \cdot V_{ss}^3 \\
&= \frac{6!}{48N\,(2\pi\sqrt{-1})^3\,(2y_0+1)^3} \Big(\frac{2y_0}{1-y_0}\Big)^3 + O(\hbar),
\end{aligned}
$$

by using Lemma 5.3 and formulas in the proof of Lemma 5.3. Similarly, the contribution from $\check{s}^6$ is calculated as

$$
\begin{aligned}
& \frac{-1}{48N\,(V_{tt}V_{ss} - V_{ts}^2)^3} \Big(\frac{\partial}{\partial u_2}\Big)^6 \big(V_{ss}u_1^2 - 2V_{ts}u_1u_2 + V_{tt}u_2^2\big)^3 \\
&= \frac{-1}{48N\,(V_{tt}V_{ss} - V_{ts}^2)^3} \cdot 6! \cdot V_{tt}^3 \\
&= \frac{6!}{48N\,(2\pi\sqrt{-1})^3\,(2y_0+1)^3} \Big(\frac{x_0}{1-x_0}\Big)^3 + O(\hbar).
\end{aligned}
$$

Similarly, the contribution from $\check{t}^3\check{s}^3$ is calculated as

$$
\begin{aligned}
& \frac{-1}{48N\,(V_{tt}V_{ss} - V_{ts}^2)^3} \Big(\frac{\partial}{\partial u_1}\Big)^3 \Big(\frac{\partial}{\partial u_2}\Big)^3 \big(V_{ss}u_1^2 - 2V_{ts}u_1u_2 + V_{tt}u_2^2\big)^3 \\
&= \frac{-1}{48N\,(V_{tt}V_{ss} - V_{ts}^2)^3} \cdot 6! \cdot \Big(\frac{2}{5}(-V_{ts})^3 + \frac{3}{5}V_{tt}V_{ss}(-V_{ts})\Big) + O(\hbar) \\
&= \frac{6!}{48N\,(2\pi\sqrt{-1})^3\,(2y_0+1)^3} \Big(\frac{2}{5} + \frac{3}{5}\,\frac{x_0}{1-x_0}\,\frac{2y_0}{1-y_0}\Big),
\end{aligned}
$$

by using Lemma 5.3 and formulas in the proof of Lemma 5.3. Hence, the contribution from (57) is equal to

$$
\frac{5\cdot 2\pi\sqrt{-1}\,(y_0-1)^2}{24N\,(2y_0+1)^3}\Big(y_0^2\Big(\frac{2y_0}{1-y_0}\Big)^3 + 4\Big(\frac{x_0}{1-x_0}\Big)^3 + 4y_0\Big(\frac{2}{5} + \frac{3}{5}\,\frac{x_0}{1-x_0}\,\frac{2y_0}{1-y_0}\Big)\Big) = C_5\hbar
$$

where we put

$$
\begin{aligned}
C_5 &= \frac{5(y_0-1)^2}{24(2y_0+1)^3}\Big(y_0^2\Big(\frac{2y_0}{1-y_0}\Big)^3 + 4\Big(\frac{x_0}{1-x_0}\Big)^3 + 4y_0\Big(\frac{2}{5} + \frac{3}{5}\,\frac{x_0}{1-x_0}\,\frac{2y_0}{1-y_0}\Big)\Big) \\
&= -\frac{1}{3174}(197y_0^2 + 127y_0 + 227).
\end{aligned}
$$

Therefore, we obtain the required formula of the lemma. □

5.3 Verifying the assumption of the saddle point method

In this section, in Lemma 5.7, we verify the assumption of the saddle point method (Proposition 3.5, see also Remark 3.6 and Appendix B) when we apply Proposition 3.5 and Remark 3.6 to (46). The arguments of this section are due to Yokota [44].

Let $V(t,s)$ and $\hat{V}(t,s)$ be as in Section 2.2. As shown in Appendix B, $V(t,s)$ uniformly converges to $\hat{V}(t,s)$ on Δ' as $N \to \infty$. Hence, as mentioned in Remarks 3.3 and 3.6 and Appendix B, the saddle point method of the problem can be reduced to the saddle point method of an integral of the form $\int \phi(t,s)\, e^{N\hat{V}(t,s)} dt\, ds$. Therefore, we verify the assumption of the saddle point method for $\hat{V}(t,s)$. We recall that the differentials of $\hat{V}(t,s)$ are given in (47).

In order to show Lemma 5.7 below, we calculate the behavior of the following function,

$$f_{t,s}(\delta_1,\delta_2) \;=\; \operatorname{Re}\hat{V}(t+\delta_1\sqrt{-1},\, s+\delta_2\sqrt{-1}) - \varsigma_R\,.$$

The differentials of this function are given by

$$\begin{aligned}
\frac{\partial}{\partial\delta_1} f_{t,s}(\delta_1,\delta_2) &= \operatorname{Re}\Big(\sqrt{-1}\,\frac{\partial}{\partial t}\hat{V}(t+\delta_1\sqrt{-1},\, s+\delta_2\sqrt{-1})\Big) \\
&= -\operatorname{Im}\Big(-\log\big(1-\frac{1}{x}\big) - 2\pi\sqrt{-1}\,\big(t+s-\frac{1}{2}\big)\Big) \\
&= \operatorname{Arg}\big(1-\frac{1}{x}\big) + 2\pi\big(t+s-\frac{1}{2}\big), \qquad (58) \\
\frac{\partial}{\partial\delta_2} f_{t,s}(\delta_1,\delta_2) &= \operatorname{Re}\Big(\sqrt{-1}\,\frac{\partial}{\partial s}\hat{V}(t+\delta_1\sqrt{-1},\, s+\delta_2\sqrt{-1})\Big) \\
&= -\operatorname{Im}\Big(-2\log\big(1-\frac{1}{y}\big) - 2\pi\sqrt{-1}\,(t+2s-1)\Big) \\
&= 2\Big(\operatorname{Arg}\big(1-\frac{1}{y}\big) + 2\pi\big(\frac{1}{2}t+s-\frac{1}{2}\big)\Big), \qquad (59)
\end{aligned}$$

where $x = e^{2\pi\sqrt{-1}\,(t+\delta_1\sqrt{-1})}$ and $y = e^{2\pi\sqrt{-1}\,(s+\delta_2\sqrt{-1})}$.

Lemma 5.5 (Yokota [44])**.** *Fixing $(t,s)\in\Delta'$ and $\delta_2\in\mathbb{R}$, we regard $f_{t,s}(X,\delta_2)$ as a function of $X\in\mathbb{R}$.*
(1) If $t+s\ge\frac{1}{2}$, then $f_{t,s}(X,\delta_2)$ is monotonically increasing for $X\in\mathbb{R}$.
(2) If $t+s<\frac{1}{2}$, then $f_{t,s}(X,\delta_2)$ has a unique minimal point at $X=g_1(t,s)$, where

$$g_1(t,s) \;=\; \frac{1}{2\pi}\log\frac{\sin 2\pi(t+s)}{\sin 2\pi s}\,,$$

i.e., $f_{t,s}(X,\delta_2)$ is monotonically decreasing for $X<g_1(t,s)$, and is monotonically increasing for $X>g_1(t,s)$.

Proof. We put $x=e^{2\pi\sqrt{-1}\,(t+X\sqrt{-1})}$. Then, $1/x = e^{2\pi X}e^{-2\pi\sqrt{-1}\,t}$. We put $\theta=\operatorname{Arg}(1-\frac{1}{x})$ in this proof. Since $t<\frac{1}{2}$, θ is in the following range,

$$0 \;<\; \theta \;<\; 2\pi\big(\frac{1}{2}-t\big).$$

When $t+s \geq \frac{1}{2}$, we show the lemma, as follows. By (58),

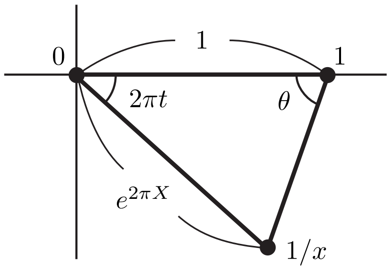
$$\frac{\partial}{\partial X} f_{t,s}(X,\delta_2) \;=\; \theta + 2\pi\big(t+s-\frac{1}{2}\big) \;>\; 0.$$

Therefore, $f_{t,s}(X,\delta_2)$ is monotonically increasing, and (1) holds.

When $t+s < \frac{1}{2}$, we show the lemma, as follows. In this case, by (58),

$$\frac{\partial}{\partial X} f_{t,s}(X,\delta_2) \begin{cases} >0 & \text{if } \;\theta > 2\pi(\frac{1}{2}-t-s), \\ =0 & \text{if } \;\theta = 2\pi(\frac{1}{2}-t-s), \\ <0 & \text{if } \;\theta < 2\pi(\frac{1}{2}-t-s). \end{cases}$$

Further, θ and X are related as shown in the following picture.



Hence, X is monotonically increasing as a function of θ, and they satisfy that

$$\frac{e^{2\pi X}}{\sin\theta} \;=\; \frac{1}{\sin(\pi - 2\pi t - \theta)}.$$

This is rewritten,

$$X \;=\; \frac{1}{2\pi}\log\frac{\sin\theta}{\sin(\pi-2\pi t-\theta)}.$$

Therefore,

$$\frac{\partial}{\partial X} f_{t,s}(X,\delta_2) \begin{cases} >0 & \text{if } \;X > g_1(t,s), \\ =0 & \text{if } \;X = g_1(t,s), \\ <0 & \text{if } \;X < g_1(t,s), \end{cases}$$

where we put

$$g_1(t,s) \;=\; \frac{1}{2\pi}\log\frac{\sin 2\pi(\frac{1}{2}-t-s)}{\sin 2\pi s} \;=\; \frac{1}{2\pi}\log\frac{\sin 2\pi(t+s)}{\sin 2\pi s}.$$

Hence, (2) holds. □

Lemma 5.6 (Yokota [44])**.** *Fixing $(t,s)\in\Delta'$ and $\delta_1\in\mathbb{R}$, we regard $f_{t,s}(\delta_1,Y)$ as a function of $Y\in\mathbb{R}$.*
(1) If $t+2s\geq 1$, then $f_{t,s}(\delta_1,Y)$ is monotonically increasing for $Y\in\mathbb{R}$.
(2) If $t+2s<1$, then $f_{t,s}(\delta_1,Y)$ has a unique minimal point at $Y=g_2(t,s)$, where

$$g_2(t,s) \;=\; \frac{1}{2\pi}\log\frac{\sin\pi(t+2s)}{\sin\pi t},$$

i.e., $f_{t,s}(\delta_1,Y)$ is monotonically decreasing for $Y<g_2(t,s)$, and is monotonically increasing for $Y>g_2(t,s)$.

Proof. We put $y = e^{2\pi\sqrt{-1}\,(s+Y\sqrt{-1})}$. Then, $1/y = e^{2\pi Y} e^{-2\pi\sqrt{-1}\,s}$. We put $\theta = \operatorname{Arg}(1-\frac{1}{y})$ in this proof. Since $s<\frac{1}{2}$, θ is in the following range,

$$0 \;<\; \theta \;<\; 2\pi\,\big(\frac{1}{2}-s\big).$$

When $t+2s\geq 1$, we show the lemma, as follows. By (59),

$$\frac{\partial}{\partial Y} f_{t,s}(\delta_1, Y) \;=\; 2\Big(\theta + \pi\,\big(t+2s-1\big)\Big) \;>\; 0.$$

Hence, $f_{t,s}(\delta_1, Y)$ is monotonically increasing, and (1) holds.

When $t+2s<1$, we show the lemma, as follows. In this case, by (59),

$$\frac{\partial}{\partial Y} f_{t,s}(\delta_1, Y) \begin{cases} >0 & \text{if } \ \theta > 2\pi(\frac{1}{2}-\frac{1}{2}t-s), \\ =0 & \text{if } \ \theta = 2\pi(\frac{1}{2}-\frac{1}{2}t-s), \\ <0 & \text{if } \ \theta < 2\pi(\frac{1}{2}-\frac{1}{2}t-s). \end{cases}$$

Further, similarly as in the proof of Lemma 5.5, θ and Y are related by

$$Y \;=\; \frac{1}{2\pi}\log\frac{\sin\theta}{\sin(\pi-2\pi s-\theta)}\,.$$

Since Y is monotonically increasing as a function of θ,

$$\frac{\partial}{\partial Y} f_{t,s}(\delta_1, Y) \begin{cases} >0 & \text{if } \ Y > g_2(t,s), \\ =0 & \text{if } \ Y = g_2(t,s), \\ <0 & \text{if } \ Y < g_2(t,s), \end{cases}$$

where we put

$$g_2(t,s) \;=\; \frac{1}{2\pi}\log\frac{\sin 2\pi(\frac{1}{2}-\frac{1}{2}t-s)}{\sin\big(\pi-2\pi s-2\pi(\frac{1}{2}-\frac{1}{2}t-s)\big)} \;=\; \frac{1}{2\pi}\log\frac{\sin\pi(t+2s)}{\sin\pi t}\,.$$

Hence, we obtain the lemma. □

Lemma 5.7 (Yokota [44])**.** *When we apply Proposition 3.5 (saddle point method) to (46), the assumption of Proposition 3.5 holds.*

Proof. We show that there exists a homotopy Δ'_δ $(0\leq\delta\leq 1)$ between $\Delta'_0=\Delta'$ and Δ'_1 such that

$$(t_c, s_c)\in\Delta'_1, \tag{60}$$

$$\Delta'_1 - \{(t_c,s_c)\} \;\subset\; \big\{(t,s)\in\mathbb{C}^2 \;\big|\; \operatorname{Re}\hat{V}(t,s) < \varsigma_R\big\}, \tag{61}$$

$$\partial\Delta'_\delta \;\subset\; \big\{(t,s)\in\mathbb{C}^2 \;\big|\; \operatorname{Re}\hat{V}(t,s) < \varsigma_R\big\}. \tag{62}$$

For a sufficiently large $R>0$, we put

$$\hat{g}_1(t,s) \;=\; \begin{cases} \max\{-R,\, g_1(t,s)\} & \text{if } \ t+s<\frac{1}{2}, \\ -R & \text{if } \ t+s\geq\frac{1}{2}, \end{cases}$$

$$\hat{g}_2(t,s) \;=\; \begin{cases} \max\{-R,\, g_2(t,s)\} & \text{if } \; t+2s<1, \\ -R & \text{if } \; t+2s \ge 1. \end{cases}$$

We note that, since $g_1(t,s) \to -\infty$ as $t+s \to \frac{1}{2}$, $\hat{g}_1(t,s)$ is continuous, and similarly, since $g_2(t,s) \to -\infty$ as $t+2s \to 1$, $\hat{g}_2(t,s)$ is continuous. We put

$$\Delta'_\delta \;=\; \big\{ \big(t+\delta\cdot\hat{g}_1(t,s)\sqrt{-1},\; s+\delta\cdot\hat{g}_2(t,s)\sqrt{-1}\big) \in \mathbb{C}^2 \;\big|\; (t,s)\in\Delta' \big\}.$$

We show (62), as follows. From the definition of Δ',

$$\partial\Delta' \;\subset\; \big\{ (t,s)\in\mathbb{C}^2 \;\big|\; \operatorname{Re}\hat{V}(t,s) < \varsigma_{R} \big\}.$$

Further, by Lemmas 5.5 and 5.6,

$$\operatorname{Re}\hat{V}\big(t+\delta\cdot\hat{g}_1(t,s)\sqrt{-1},\; s+\delta\cdot\hat{g}_2(t,s)\sqrt{-1}\big) \;\le\; \hat{V}(t,s)$$

for any $\delta\in[0,1]$ and any $(t,s)\in\Delta'$. Hence, (62) holds.

We show (60) and (61), as follows. Consider the following functions

$$\begin{aligned} F(t,s,X,Y) \;&=\; \operatorname{Re}\hat{V}\big(t+X\sqrt{-1},\; s+Y\sqrt{-1}\big), \\ h(t,s) \;&=\; F\big(t,\; s,\; \hat{g}_1(t,s),\; \hat{g}_2(t,s)\big). \end{aligned}$$

When $t+2s\ge 1$, $-h(t,s)$ is sufficiently large (because we let R be sufficiently large), and (61) holds in this case. When $t+2s<1$, it is shown from the definitions of $g_1(t,s)$ and $g_2(t,s)$ that $\frac{\partial F}{\partial X}=0$ at $X=g_1(t,s)$ and $\frac{\partial F}{\partial Y}=0$ at $Y=g_2(t,s)$. Hence, $\operatorname{Im}\frac{\partial\hat{V}}{\partial t} = \operatorname{Im}\frac{\partial\hat{V}}{\partial s}=0$ at $\big(t+g_1(t,s)\sqrt{-1},\; s+g_2(t,s)\sqrt{-1}\big)$. Further, $\frac{\partial h}{\partial t}=\operatorname{Re}\frac{\partial\hat{V}}{\partial t}$ and $\frac{\partial h}{\partial s}=\operatorname{Re}\frac{\partial\hat{V}}{\partial s}$ at $\big(t+g_1(t,s)\sqrt{-1},\; s+g_2(t,s)\sqrt{-1}\big)$. Therefore, when (t,s) is a critial point of $h(t,s)$, $\big(t+g_1(t,s)\sqrt{-1},\; s+g_2(t,s)\sqrt{-1}\big)$ is a critical point of $\hat{V}$. It follows that $h(t,s)$ has a unique maximal point at $(t,s)=(\operatorname{Re}t_c,\operatorname{Re}s_c)$. Therefore, (60) and (61) hold. □

5.4 Verifying the assumption of the Poisson summation formula

In this section, in Lemma 5.8, we verify the assumption of the Poisson summation formula (Proposition 4.6, see also Remark 4.8 and Appendix B) when we apply Proposition 4.6 and Remark 4.8 to (14). As in the previous section, we consider $\hat{V}(t,s)$ instead of $V(t,s)$ in this section.

We calculate a critical point $(t_r,s_r)\in\Delta'$ of $\operatorname{Re}\hat{V}(t,s)$, as follows. Putting $x=e^{2\pi\sqrt{-1}\cdot t}$ and $y=e^{2\pi\sqrt{-1}\cdot s}$, we have that

$$\begin{aligned} \frac{\partial}{\partial t}\operatorname{Re}\hat{V}(t,s) \;&=\; \operatorname{Re}\Big(-\log\big(1-\frac{1}{x}\big)\Big), \\ \frac{\partial}{\partial s}\operatorname{Re}\hat{V}(t,s) \;&=\; \operatorname{Re}\Big(-2\,\log\big(1-\frac{1}{y}\big)\Big), \end{aligned}$$

since $t,s\in\mathbb{R}$. Hence, putting $x_r=e^{2\pi\sqrt{-1}\cdot t_r}$ and $y_r=e^{2\pi\sqrt{-1}\cdot s_r}$,

$$\big|\,x_r-1\,\big| \;=\; \big|\,y_r-1\,\big| \;=\; 1,$$

noting that $|x_r| = |y_r| = 1$. Therefore, $x_r = y_r = e^{2\pi\sqrt{-1}/3}$, and $\operatorname{Re} V(t,s)$ has a unique maximal point $(\frac{1}{6}, \frac{1}{6})$ on Δ'. Its maximal value is given by

$$\begin{aligned}\operatorname{Re} \hat{V}\big(\frac{1}{6}, \frac{1}{6}\big) - \varsigma_R &= \operatorname{Re} \frac{1}{2\pi\sqrt{-1}}\big(\mathrm{Li}_2(e^{\pi\sqrt{-1}/3}) - 2\,\mathrm{Li}_2(e^{-\pi\sqrt{-1}/3})\big) - \varsigma_R \\ &= 0.03448931080... .\end{aligned}$$

Hence,

$$\operatorname{Re} \hat{V}(t,s) - \varsigma_R \;\le\; 0.03448931080... , \tag{63}$$

for any $(t,s) \in \Delta'$.

Lemma 5.8. *When we apply Proposition 4.6 to (14), the assumptions of Proposition 4.6 hold.*

Proof. We verify the assumptions (41) and (42) for $i = 1$ in Lemmas 5.9 and 5.10 below, and verify the assumptions for $i = 2$ in Lemmas 5.11 and 5.12 below. The other assumptions of Proposition 4.6 can be verified easily. □

Lemma 5.9. *The assumption (41) holds for $i = 1$.*

Proof. As for the assumption (41) for $i = 1$, we show that $\partial\Delta'$ is null-homotopic in

$$\big\{(t + \delta\sqrt{-1},\, s) \in \mathbb{C}^2 \;\big|\; (t,s) \in \Delta',\ \delta \ge 0,\ \operatorname{Re} V(t + \delta\sqrt{-1}, s) < \varsigma_R + 2\pi\delta\big\}.$$

To show it, we show that the following disk bounds $\partial\Delta'$ in the above domain,

$$\big\{(t + \delta_0\sqrt{-1},\, s) \in \mathbb{C}^2 \;\big|\; (t,s) \in \Delta'\big\} \;\cup\; \big\{(t + \delta\sqrt{-1},\, s) \in \mathbb{C}^2 \;\big|\; (t,s) \in \partial\Delta',\ \delta \in [0, \delta_0]\big\}.$$

We put

$$F_{t,s}(\delta) \;=\; \operatorname{Re} V(t + \delta\sqrt{-1}, s) - \varsigma_R - 2\pi\delta$$

in this proof. Then, it is sufficient to show that

$$F_{t,s}(\delta_0) \;<\; 0 \quad \text{for any } (t,s) \in \Delta', \text{ and} \tag{64}$$

$$F_{t,s}(\delta) \;<\; 0 \quad \text{for any } (t,s) \in \partial\Delta' \text{ and } \delta \in [0, \delta_0], \tag{65}$$

for some $\delta_0 > 0$.

To show these, we estimate the differential of $F_{t,s}(\delta)$, as follows. The differential of $F_{t,s}(\delta)$ is given by

$$\begin{aligned}\frac{d}{d\delta} F_{t,s}(\delta) &= \operatorname{Re}\Big(\sqrt{-1}\,\frac{\partial}{\partial t}\hat{V}(t + \delta\sqrt{-1}, s)\Big) - 2\pi \\ &= -\operatorname{Im}\Big(-\log\big(1 - \frac{1}{x}\big) - 2\pi\sqrt{-1}\,\big(t + s - \frac{1}{2}\big)\Big) - 2\pi \\ &= \operatorname{Arg}\big(1 - \frac{1}{x}\big) + 2\pi\,\big(t + s - \frac{3}{2}\big),\end{aligned}$$

where we put $x = e^{2\pi\sqrt{-1}(t+\delta\sqrt{-1})}$. Since $0.04 \le t \le 0.4$, it is shown that $\operatorname{Arg}(1 - \frac{1}{x})$ is in the following range,

$$0 \;<\; \operatorname{Arg}\big(1 - \frac{1}{x}\big) \;<\; \pi.$$

Therefore, since $t+s \leq 0.6$,

$$\frac{d}{d\delta} F_{t,s}(\delta) \;<\; 2\pi\,\big(\frac{1}{2}+0.6-\frac{3}{2}\big) \;=\; -2\pi\cdot 0.4.$$

We show (64), as follows. We have that

$$F_{t,s}(\delta_0) \;=\; F_{t,s}(0) + \int_0^{\delta_0} \frac{d}{d\delta} F_{t,s}(\delta)\, d\delta \;<\; F_{t,s}(0) - 2\pi\cdot 0.4\cdot \delta_0\,.$$

Further, by (63),

$$F_{t,s}(0) \;=\; \operatorname{Re}\hat{V}(t,s) - \varsigma_R \;\leq\; 0.03448931080...\;.$$

Hence, (64) is satisfied for a sufficiently large δ_0.

We show (65), as follows. From the definition of Δ', we have that $F_{t,s}(0)<0$ for any $(t,s)\in\partial\Delta'$. Since $\frac{d}{d\delta}F_{t,s}(\delta)<0$ as shown above, it is shown similarly as above that $F_{t,s}(\delta)<0$ for any $\delta\geq 0$. Hence, (65) is satisfied. □

Lemma 5.10. *The assumption (42) holds for $i=1$.*

Proof. We put

$$F_{t,s}(\delta) \;=\; \operatorname{Re} V(t-\delta\sqrt{-1}, s) - \varsigma_R - 2\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.9, it is sufficient to show that there exists $\varepsilon>0$ such that

$$\frac{d}{d\delta} F_{t,s}(\delta) \;<\; -\varepsilon, \tag{66}$$

for any $(t,s)\in\Delta'$.

We calculate this as

$$\begin{aligned}\frac{d}{d\delta} F_{t,s}(\delta) \;&=\; \operatorname{Re}\Big(-\sqrt{-1}\,\frac{\partial}{\partial t}\hat{V}(t-\delta\sqrt{-1},s)\Big) - 2\pi \\ &=\; \operatorname{Im}\Big(-\log\big(1-\frac{1}{x}\big) - 2\pi\sqrt{-1}\,\big(t+s-\frac{1}{2}\big)\Big) - 2\pi \\ &=\; -\operatorname{Arg}\big(1-\frac{1}{x}\big) - 2\pi\,\big(t+s+\frac{1}{2}\big),\end{aligned}$$

where we put $x=e^{2\pi\sqrt{-1}(t-\delta\sqrt{-1})}$. Since $0.04\leq t\leq 0.4$, it is shown that $\operatorname{Arg}(1-\frac{1}{x})$ is in the following range,

$$0 \;<\; \operatorname{Arg}\big(1-\frac{1}{x}\big) \;<\; \pi\,.$$

Therefore, since $t+s\geq 0.09$,

$$\frac{d}{d\delta} F_{t,s}(\delta) \;<\; -2\pi\,\big(0.09+\frac{1}{2}\big) \;=\; -2\pi\cdot 0.59,$$

and (66) is satisfied. □

Lemma 5.11. *The assumption (41) holds for $i=2$.*

Proof. We put

$$F_{t,s}(\delta) \;=\; \operatorname{Re} V(t, s+\delta\sqrt{-1}) - \varsigma_{_R} - 2\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.9, it is sufficient to show that there exists $\varepsilon > 0$ such that

$$\frac{d}{d\delta} F_{t,s}(\delta) \;<\; -\varepsilon, \tag{67}$$

for any $(t,s) \in \Delta'$.

To show these, we calculate the differential of $F_{t,s}(\delta)$ as

$$\begin{aligned}\frac{d}{d\delta} F_{t,s}(\delta) \;&=\; \operatorname{Re}\Big(\sqrt{-1}\,\frac{\partial}{\partial s} V(t, s+\delta\sqrt{-1})\Big) - 2\pi \\ &=\; -\operatorname{Im}\Big(-2\,\log\big(1-\frac{1}{y}\big) - 2\pi\sqrt{-1}\,\big(t+2s-1\big)\Big) - 2\pi \\ &=\; 2\operatorname{Arg}\big(1-\frac{1}{y}\big) + 2\pi\,\big(t+2s-2\big),\end{aligned}$$

where we put $y = e^{2\pi\sqrt{-1}(s+\delta\sqrt{-1})}$. Since $0.05 \le s \le 0.4$, it is shown that $\operatorname{Arg}(1-\frac{1}{y})$ is in the following range,

$$0 \;<\; \operatorname{Arg}\big(1-\frac{1}{y}\big) \;<\; 2\pi\,\big(\frac{1}{2}-s\big).$$

Therefore, since $t \le 0.4$,

$$\frac{d}{d\delta} F_{t,s}(\delta) \;<\; 2\cdot 2\pi\,\big(\frac{1}{2}-s\big) + 2\pi\,\big(t+2s-2\big) \;=\; 2\pi(t-1) \;\le\; -2\pi\cdot 0.6,$$

and (67) is satisfied. □

Lemma 5.12. *The assumption (42) holds for $i=2$.*

Proof. We put

$$F_{t,s}(\delta) \;=\; \operatorname{Re} V(t, s-\delta\sqrt{-1}) - \varsigma_{_R} - 2\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.9, it is sufficient to show that there exists $\varepsilon > 0$ such that

$$\frac{d}{d\delta} F_{t,s}(\delta) \;<\; -\varepsilon, \tag{68}$$

for any $(t,s) \in \Delta'$.

We calculate this as

$$\begin{aligned}\frac{d}{d\delta} F_{t,s}(\delta) \;&=\; \operatorname{Re}\Big(-\sqrt{-1}\,\frac{\partial}{\partial s} V(t, s-\delta\sqrt{-1})\Big) - 2\pi \\ &=\; \operatorname{Im}\Big(-2\,\log\big(1-\frac{1}{y}\big) - 2\pi\sqrt{-1}\,\big(t+2s-1\big)\Big) - 2\pi \\ &=\; -2\operatorname{Arg}\big(1-\frac{1}{y}\big) - 2\pi\,\big(t+2s\big),\end{aligned}$$

where we put $y = e^{2\pi\sqrt{-1}(s-\delta\sqrt{-1})}$. Since $0.05 \le s \le 0.4$, it is shown that $\operatorname{Arg}(1-\frac{1}{y})$ is in the following range,

$$0 \;<\; \operatorname{Arg}\big(1-\frac{1}{y}\big) \;<\; \pi\,.$$

Therefore, since $t+s \ge 0.09$ and $s \ge 0.05$,

$$\frac{d}{d\delta}F_{t,s}(\delta) \;<\; -2\pi\,\big(0.09+0.05\big) \;=\; -2\pi\cdot 0.14,$$

and (68) is satisfied. □

A Properties of $\varphi(t)$

In this appendix, we review some basic properties of $\varphi(t)$.

We put $\hbar = 2\pi\sqrt{-1}/N$, and put

$$\Phi_d(z) \;=\; \operatorname{Li}_2\big(z\big) + \sum_{1\le k\le d} \hbar^{2k} c_{2k}\cdot\Big(z\frac{d}{dz}\Big)^{2k-2}\frac{z}{1-z}\,,$$

where we define c_{2k} by

$$\frac{y/2}{\sinh(y/2)} \;=\; \sum_{k\ge 0} c_{2k}\, y^{2k}.$$

Proposition A.1. *We fix any sufficiently small $\delta>0$ and any $M>0$. Let d be any non-negative integer. Then, in the domain*

$$\big\{t\in\mathbb{C} \;\big|\; \delta \le \operatorname{Re}t \le 1-\delta,\quad |\operatorname{Im}t|\le M\big\}, \tag{69}$$

$\varphi(t)$ and $\varphi^{(k)}(t)$ are presented by

$$\varphi(t) \;=\; \frac{N}{2\pi\sqrt{-1}}\,\Phi_d(e^{2\pi\sqrt{-1}\,t}) \;+O\big(\frac{1}{N^{2d+1}}\big), \tag{70}$$

$$\varphi^{(k)}(t) \;=\; \frac{N}{2\pi\sqrt{-1}}\big(\frac{d}{dt}\big)^k\Phi_d(e^{2\pi\sqrt{-1}\,t}) \;+O\big(\frac{1}{N^{2d+1}}\big), \tag{71}$$

for each $k>0$, where $O(1/N^{2d+1})$ means the error term whose absolute value is bounded by C/N^{2d+1} for some $C>0$, which is independent of t (but possibly dependent on δ).

In particular, $\frac{1}{N}\varphi(t)$ uniformly converges to $\frac{1}{2\pi\sqrt{-1}}\operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t})$ in the domain (69), and $\frac{1}{N}\varphi'(t)$ uniformly converges to $-\log(1-e^{2\pi\sqrt{-1}\,t})$ in the domain (69).

Proof. We show (70). We have that

$$\varphi(t) \;=\; \int_{-\infty}^{\infty}\frac{Ne^{(2t-1)x}}{4x^2\sinh x}\cdot\frac{x/N}{\sinh(x/N)}\,dx \;=\; \int_{-\infty}^{\infty}\frac{Ne^{(2t-1)x}}{4x^2\sinh x}\sum_{k\ge 0}c_{2k}\big(\frac{2x}{N}\big)^{2k}dx. \tag{72}$$

We put the "$k\!=\!0$" part of (72) to be

$$f(t) \;=\; \int_{-\infty}^{\infty} \frac{N e^{(2t-1)x}}{4x^2 \sinh x}\, dx.$$

To calculate $f(t)$, we consider the following contour,

$$\big[-(m+\tfrac{1}{2})\pi, -1\big] \;\cup\; \big\{ z \in \mathbb{C} \;\big|\; |z| = 1,\ \operatorname{Im} z \geq 0 \big\} \\ \cup\; \big[1, (m+\tfrac{1}{2})\pi\big] \;\cup\; \big\{ z \in \mathbb{C} \;\big|\; |z| = m + \tfrac{1}{2},\ \operatorname{Im} z \geq 0 \big\}.$$

Then, the integrand has poles at $n\pi\sqrt{-1}$ $(n = 1, 2, \cdots, m)$ in the region bounded by the contour. Hence,

$$\begin{aligned} f(t) \;&=\; \lim_{m\to\infty} \sum_{n=1}^{m} 2\pi\sqrt{-1} \operatorname*{Res}_{x=n\pi\sqrt{-1}} \frac{N e^{(2t-1)x}}{4x^2 \sinh x} \\ &=\; 2\pi\sqrt{-1} \sum_{n=1}^{\infty} \frac{-N}{4\pi^2} \frac{(e^{2\pi\sqrt{-1}\,t})^n}{n^2} \;=\; \frac{N}{2\pi\sqrt{-1}} \operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t}). \end{aligned}$$

Further, for general $k > 0$, the "k" part of (72) is calculated as

$$\frac{c_{2k}}{N^{2k}}\, f^{(2k)}(t) \;=\; \frac{N}{2\pi\sqrt{-1}} \cdot \frac{c_{2k}}{N^{2k}} \Big(\frac{d}{dt}\Big)^{2k} \operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t}) \;=\; \frac{N}{2\pi\sqrt{-1}} \cdot \hbar^{2k} c_{2k} \Big(z\frac{d}{dz}\Big)^{2k-2} \frac{z}{1-z}.$$

Therefore, we can put

$$\varphi(t) \;=\; \frac{N}{2\pi\sqrt{-1}}\, \Phi_d(e^{2\pi\sqrt{-1}\,t}) \;+ R_d,$$

where R_d is the error term. We estimate it, as follows. Since $|(y/2)/\sinh(y/2)|$ is bounded for $y \in \mathbb{R}$, there exists $M_0 > 0$ such that

$$\Big| \frac{1}{y^{2d+2}} \Big(\frac{y/2}{\sinh y/2} \;-\; \sum_{k=0}^{d} c_{2k} y^{2k} \Big) \Big| \;\leq\; M_0$$

for any $y \in \mathbb{R}$. Hence, putting $y = 2x/N$,

$$\begin{aligned} |R_d| \;&=\; \Big| \varphi(t) \;-\; \frac{N}{2\pi\sqrt{-1}}\, \Phi_d(e^{2\pi\sqrt{-1}\,t}) \Big| \\ &=\; \Big| \int_{-\infty}^{\infty} \frac{N e^{(2t-1)x}}{4x^2 \sinh x} \Big(\frac{x/N}{\sinh(x/N)} \;-\; \sum_{k=0}^{d} c_{2k} \Big(\frac{2x}{N}\Big)^{2k} \Big)\, dx \Big| \\ &\leq\; \int_{-\infty}^{\infty} \Big| \frac{N e^{(2t-1)x}}{4x^2 \sinh x} \Big| \cdot M_0 \Big|\frac{2x}{N}\Big|^{2d+2} dx \\ &=\; \frac{2^{2d} M_0}{N^{2d+1}} \Bigg(\int_1^{\infty} \frac{e^{(2\operatorname{Re} t-1)x}}{|\sinh x|}\, x^{2d}\, dx + \int_{-\infty}^{-1} \frac{e^{(2\operatorname{Re} t-1)x}}{|\sinh x|}\, x^{2d}\, dx + \int_{\substack{|z|=1 \\ \operatorname{Im} z \geq 0}} \frac{e^{\operatorname{Re}((2t-1)z)}}{|\sinh z|} |dz| \Bigg). \end{aligned} \tag{73}$$

For $x \geq 1$, the integrand of (73) is bounded by $M_1 e^{-2\delta x} x^{2d}$ for some $M_1 > 0$. For $x \leq -1$, the integrand of (73) is bounded by $M_2 e^{2\delta x} x^{2d}$ for some $M_2 > 0$. For $|z| = 1$, the integrand of (73) is bounded by a constant. Hence, the values of the integrals of (73) is bounded by a constant. Therefore, $|R_d|$ is bounded by C/N^{2d+1} for some $C > 0$, which is independent of t. Hence, we obtain (70).

We obtain (71) by similar arguments for $\varphi^{(k)}(t)$. □

Lemma A.2. *For any* $t \in \mathbb{C}$ *with* $0 < \mathrm{Re}\, t < 1$,

$$\varphi(t) + \varphi(1-t) \;=\; 2\pi\sqrt{-1}\Big(-\frac{N}{2}\big(t^2 - t + \frac{1}{6}\big) + \frac{1}{24N}\Big).$$

Proof. By definition,

$$\varphi(t) + \varphi(1-t) \;=\; \int_\gamma \frac{e^{(2t-1)x} + e^{(1-2t)x}}{4x\, \sinh x\, \sinh(x/N)}\, dx.$$

Since this integrand is an odd function, if the contour was $\mathbb{R}$, the integral would vanish. In this case, since the contour γ avoids the pole at $x = 0$, the contribution from the residue at this pole survives. By expanding the numerator and the denominator of the integrand as power series of x concretely, we can calculate this residue, and obtain the required formula. □

Lemma A.3 (Kashaev)**.**

$$\varphi\big(\frac{1}{2N}\big) \;=\; \frac{N}{2\pi\sqrt{-1}}\,\frac{\pi^2}{6} + \frac{1}{2}\log N + \frac{\pi\sqrt{-1}}{4} - \frac{\pi\sqrt{-1}}{12N}\,,$$

$$\varphi\big(1-\frac{1}{2N}\big) \;=\; \frac{N}{2\pi\sqrt{-1}}\,\frac{\pi^2}{6} - \frac{1}{2}\log N + \frac{\pi\sqrt{-1}}{4} - \frac{\pi\sqrt{-1}}{12N}\,.$$

Proof. It is sufficient to show that

$$\varphi\big(\frac{1}{2N}\big) + \varphi\big(1-\frac{1}{2N}\big) \;=\; 2\pi\sqrt{-1}\Big(-\frac{N}{12} + \frac{1}{4} - \frac{1}{12N}\Big), \tag{74}$$

$$\varphi\big(\frac{1}{2N}\big) - \varphi\big(1-\frac{1}{2N}\big) \;=\; \log N. \tag{75}$$

We obtain (74) from Lemma A.2 by putting $t = 1/2N$.

We show (75), as follows. By (3) and (8),

$$\exp\Big(\varphi\big(\frac{1}{2N}\big) - \varphi\big(1-\frac{1}{2N}\big)\Big) \;=\; N. \tag{76}$$

Further, from the definition of $\varphi(t)$,

$$\varphi\big(\frac{1}{2N}\big) - \varphi\big(1-\frac{1}{2N}\big) \;=\; \int_\gamma \frac{e^{(1-1/N)x} - e^{(1/N-1)x}}{4x\, \sinh x\, \sinh(x/N)}\, dx.$$

By a similar argument as in the proof of Lemma A.2, we can see that the residue of this integrand at 0 vanishes. Hence, the value of this integral is real, and we obtain (75) from (76). □

B Convergence of $V(t,s)$ to $\hat{V}(t,s)$

When we apply the saddle point method and the Poisson summation formula to $V(t,s)$, it is necessary to note how $V(t,s)$ converges to $\hat{V}(t,s)$ as $N \to \infty$. In this appendix, we verify that this convergence is suitable in those applications.

We recall that in Section 2.2 we put $V(t,s)$ and $\hat{V}(t,s)$ by

$$\begin{aligned} V(t,s) \;=\; & \frac{1}{N}\big(- \varphi(1-t) - 2\,\varphi(1-s)\big) - \frac{1}{2\pi\sqrt{-1}}\frac{\pi^2}{6} \\ & - 2\pi\sqrt{-1}\Big(\frac{1}{2}\big(t+s-\frac{1}{2N}\big)^2 + \frac{1}{2}s^2 - \frac{1}{2}t - s + \frac{1}{6}\Big) - \frac{3\pi\sqrt{-1}}{4N} + \frac{\pi\sqrt{-1}}{4N^2} \end{aligned}$$

and

$$\begin{aligned} \hat{V}(t,s) \;=\; & \frac{1}{2\pi\sqrt{-1}}\Big(- \mathrm{Li}_2(e^{-2\pi\sqrt{-1}\,t}) - 2\,\mathrm{Li}_2(e^{-2\pi\sqrt{-1}\,s}) - \frac{\pi^2}{6}\Big) \\ & - 2\pi\sqrt{-1}\,\Big(\frac{1}{2}t^2 + s^2 + ts - \frac{1}{2}t - s + \frac{1}{6}\Big). \end{aligned}$$

Lemma B.1. *Let m be any non-negative integer. Then, in the domain Δ'' of* (12)*, $V(t,s)$ is presented by the following form*

$$V(t,s) \;=\; \hat{V}(t,s) + V_1(t,s)\frac{1}{N} + V_2(t,s)\frac{1}{N^2} + \cdots + V_m(t,s)\frac{1}{N^m} + + R_m(t,s)\frac{1}{N^{m+1}}$$

where $V_i(t,s)$'s are holomorphic functions independent of N, and $|R_m(t,s)|$ is bounded by a constant which is independent of N.

In particular, $V(t,s)$ uniformly converges to $\hat{V}(t,s)$ in the domain Δ''.

Proof. We obtain the required presentation by applying Proposition A.1 to $V(t,s)$. In particular, by Proposition A.1, $\frac{1}{N}\varphi(t)$ uniformly converges to $\frac{1}{2\pi\sqrt{-1}}\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t})$ as $N \to \infty$ in the domain Δ''. Hence, we obtain the lemma. □

Lemma B.2. *Let i and j be any non-negative integers. Then, in the domain Δ'', $\big|\frac{\partial^{i+j}}{\partial^i t \partial^j s}V(t,s)\big|$ is bounded by a constant which is independent of N.*

Proof. By Proposition A.1, $\big|\frac{\partial^{i+j}}{\partial^i t\partial^j s}V(t,s)\big|$ uniformly converges to $\big|\frac{\partial^{i+j}}{\partial^i t\partial^j s}\hat{V}(t,s)\big|$ as $N \to \infty$ in the domain Δ''. Since Δ'' is compact, $\big|\frac{\partial^{i+j}}{\partial^i t\partial^j s}\hat{V}(t,s)\big|$ is bounded in Δ''. Hence, we obtain the lemma. □

When we apply the saddle point method, we need to show that $V(t,s)$ satisfies the condition of Remark 3.6, noting that $\hat{V}(t,s)$ satisfies the assumption of the saddle point method as shown in Section 5.3. In fact, we can show the condition of Remark 3.6 for $V(t,s)$ by Lemma B.1.

When we apply the Poisson summation formula, we need to show that $V(t,s)$ satisfies the condition of Remark 4.8, noting that $\hat{V}(t,s)$ satisfies the assumption of the Poisson summation formula as shown in Section 5.4. In fact, we can show the condition of Remark 4.8 for $V(t,s)$ by Lemmas B.1 and B.2.

C Critical points of $V(t,s)$ and $\hat{V}(t,s)$

In this appendix, we show a proof of Lemma 5.1.

We put $\hbar = 2\pi\sqrt{-1}/N$, and use the notation in Section 5.2. By Proposition A.1, we can put

$$\frac{1}{N}\,\varphi'(1-t) \;=\; -\log(1-e^{-2\pi\sqrt{-1}\,t}) + r(t,\hbar)\,\hbar^2$$

for some smooth function $r(t,\hbar)$ of t and $\hbar$. Then, from the definition of $V(t,s)$, we have that

$$\begin{aligned}
\frac{\partial}{\partial t}V(t,s) &= -\log\big(1-\frac{1}{x}\big) - 2\pi\sqrt{-1}\,\big(t+s-\frac{1}{2}-\frac{1}{2N}\big) \;+ r(t,\hbar)\,\hbar^2, \\
\frac{\partial}{\partial s}V(t,s) &= -2\,\log\big(1-\frac{1}{y}\big) - 2\pi\sqrt{-1}\,\big(t+2s-1-\frac{1}{2N}\big) \;+ 2\,r(s,\hbar)\,\hbar^2.
\end{aligned}$$

Hence, a critical point of $V(t,s)$ is a solution of

$$\begin{cases} \big(1-\dfrac{1}{x}\big)(-q^{-1/2}xy) \;=\; e^{r(t,\hbar)\,\hbar^2}, \\ \big(1-\dfrac{1}{y}\big)^2 q^{-1/2}xy^2 \;=\; e^{2\,r(s,\hbar)\,\hbar^2}. \end{cases}$$

This is rewritten as

$$\begin{cases} (1-x)y \;=\; q^{1/2}e^{r(t,\hbar)\,\hbar^2}, \\ x(y-1)^2 \;=\; q^{1/2}e^{2\,r(s,\hbar)\,\hbar^2}. \end{cases}$$

Further, by putting

$$\begin{aligned}
F(t,s,\hbar) &= (1-x)y \;-\; q^{1/2}e^{r(t,\hbar)\,\hbar^2}, \\
G(t,s,\hbar) &= x(y-1)^2 \;-\; q^{1/2}e^{2\,r(s,\hbar)\,\hbar^2},
\end{aligned}$$

the above system of equations is rewritten as

$$\begin{cases} F(t,s,\hbar) \;=\; 0, \\ G(t,s,\hbar) \;=\; 0. \end{cases} \tag{77}$$

We note that $(t_0, s_0, 0)$ is a solution of (77).

Proof of Lemma 5.1. It is sufficient to show that there exists a smooth solution $\big(f(\hbar), g(\hbar), \hbar\big)$ of (77) in a sufficiently small neighborhood of $(t_0, s_0, 0)$. Hence, by the implicit function theorem, it is sufficient to show that

$$\det\begin{pmatrix} \frac{\partial}{\partial t}F(t_0,s_0,0) & \frac{\partial}{\partial s}F(t_0,s_0,0) \\ \frac{\partial}{\partial t}G(t_0,s_0,0) & \frac{\partial}{\partial s}G(t_0,s_0,0) \end{pmatrix} \;\neq\; 0. \tag{78}$$

Since $x = e^{2\pi\sqrt{-1}\,t}$, $\frac{d}{dt} = 2\pi\sqrt{-1}\,x\,\frac{d}{dx}$. Similarly, $\frac{d}{ds} = 2\pi\sqrt{-1}\,y\,\frac{d}{dy}$. Therefore, from the definitions of $F(t,s,\hbar)$ and $G(t,s,\hbar)$, we have that

$$\frac{1}{2\pi\sqrt{-1}}\,\frac{\partial}{\partial t}F(t_0,s_0,0) \;=\; x\,\frac{\partial}{\partial x}\big((1-x)y\big)\big|_{\hbar=0} \;=\; -x_0y_0\,,$$

$$\frac{1}{2\pi\sqrt{-1}}\frac{\partial}{\partial s}F(t_0,s_0,0) \;=\; y\frac{\partial}{\partial y}\big((1-x)y\big)\big|_{\hbar=0} \;=\; (1-x_0)y_0\,,$$
$$\frac{1}{2\pi\sqrt{-1}}\frac{\partial}{\partial t}G(t_0,s_0,0) \;=\; x\frac{\partial}{\partial x}\big(x(y-1)^2\big)\big|_{\hbar=0} \;=\; x_0(y_0-1)^2,$$
$$\frac{1}{2\pi\sqrt{-1}}\frac{\partial}{\partial s}G(t_0,s_0,0) \;=\; y\frac{\partial}{\partial y}\big(x(y-1)^2\big)\big|_{\hbar=0} \;=\; 2x_0y_0(y_0-1).$$

Hence, the determinant of (78) is calculated as

$$-4\pi^2\cdot\det\begin{pmatrix}-x_0y_0 & (1-x_0)y_0\\ x_0(y_0-1)^2 & 2x_0y_0(y_0-1)\end{pmatrix} \;=\; 4\pi^2\cdot x_0y_0(y_0-1)(x_0y_0+x_0+y_0-1)$$
$$=\; 4\pi^2\cdot(2y_0+1)(y_0-1)^3/y_0 \;=\; 4\pi^2\cdot(2y_0+1)\;\neq\;0,$$

since $x_0=1-1/y_0$ and $(y_0-1)^3=y_0$. Therefore, we obtain (78), completing the proof of the lemma. □

D Proof of Lemma 2.1

In this appendix, we give a proof of Lemma 2.1.

Proof of Lemma 2.1. We put

$$\Lambda(t) \;=\; \mathrm{Re}\Big(\frac{1}{2\pi\sqrt{-1}}\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t})\Big)$$

for $t\in\mathbb{R}$. We note that this function has period 1, and $\Lambda(-t)=-\Lambda(t)$. Since

$$\Lambda'(t) \;=\; -\log\big(2\sin\pi t\big)$$

for $0<t<1$, the behavior of $\Lambda(t)$ is as follows.

t	0	$\cdots$	$\frac{1}{6}$	$\cdots$	$\frac{1}{2}$	$\cdots$	$\frac{5}{6}$	$\cdots$	1
$\Lambda(t)$	0	$\nearrow$	$\Lambda(\frac{1}{6})$	$\searrow$	0	$\searrow$	$-\Lambda(\frac{1}{6})$	$\nearrow$	0

Here, $\Lambda(\frac{1}{6})=0.161533...$. See Figure 6 for the graph of $\Lambda(t)$.

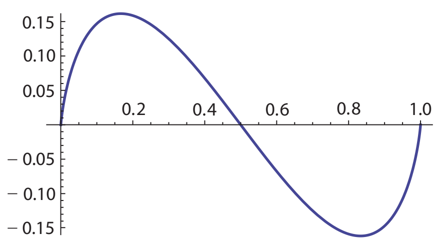


Figure 6: The graph of $\Lambda(t)$ for $0\le t\le 1$

We consider the following domain,

$$\big\{(t,s)\in\mathbb{R}^2 \;\big|\; \mathrm{Re}\,\hat{V}(t,s)\;\ge\;\varsigma_R\big\}. \tag{79}$$

For $t, s \in \mathbb{R}$,

$$\operatorname{Re} \hat{V}(t,s) \;=\; -\Lambda(-t) - 2\,\Lambda(-s) \;=\; \Lambda(t) + 2\,\Lambda(s).$$

Since $2\Lambda(\frac{1}{6}) - \varsigma_R = -0.127043... < 0$, the values of $\Lambda(t)$ and $\Lambda(s)$ must be positive when (t,s) is in the domain (79). Hence, the domain (79) is included in the following area,

$$\big\{ (t,s) \in \mathbb{R}^2 \;\big|\; 0 \le t \le 0.5, \quad 0 \le s \le 0.5 \big\}.$$

We assume that $0 \le t \le 0.5$ and $0 \le s \le 0.5$ in the following of this proof.

We consider the minimal value $t_{\min}$ of t such that (t,s) is in the domain (79). It satisfies that $\Lambda(t_{\min}) + 2\Lambda(\frac{1}{6}) = \varsigma_R$. Since $\Lambda(0.04) + 2\Lambda(\frac{1}{6}) - \varsigma_R = -0.031768... < 0$, we have that $0.04 < t_{\min}$. Hence, the domain (11) is included in the area $0.04 \le t$ for some sufficiently small $\varepsilon > 0$. Further, in similar ways, we can show that the domain (11) is included in the area

$$\big\{ (t,s) \in \mathbb{R}^2 \;\big|\; 0.04 \le t \le 0.4, \quad 0.05 \le s \le 0.4 \big\}$$

for some sufficiently small $\varepsilon > 0$. We assume that (t,s) is in the above area in the following of this proof.

By expanding $\Lambda(t)$ at $t = 0.3$, we can show that

$$\Lambda(t) \;\le\; \Lambda(0.3) + \Lambda'(0.3)(t - 0.3).$$

If (t,s) is in the domain (79),

$$\varsigma_R \;\le\; \Lambda(t) + 2\Lambda(s) \;\le\; \Lambda(0.3) + \Lambda'(0.3)(t-0.3) + 2\big(\Lambda(0.3) + \Lambda'(0.3)(s-0.3)\big).$$

Hence,

$$t + 2s \;\le\; \frac{\varsigma_R - 3\Lambda(0.3)}{\Lambda'(0.3)} + 0.9.$$

Further, since $t \le 0.4$,

$$t + s \;\le\; \frac{1}{2}\Big(\frac{\varsigma_R - 3\Lambda(0.3)}{\Lambda'(0.3)} + 1.3 \Big) \;=\; 0.571668... \;<\; 0.6.$$

Therefore, the domain (11) is included in Δ' for some sufficiently small $\varepsilon > 0$, as required. □

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