Perturbative invariants of
3-manifolds with the first Betti number 1

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Abstract

It is known that perturbative invariants of rational homology 3-spheres can be formulated by using arithmetic perturbative expansion of quantum invariants of them. However, we could not make arithmetic perturbative expansion of quantum invariants for 3-manifolds with positive Betti numbers by the same method.

In this paper, we explain how to make arithmetic perturbative expansion of quantum $SO(3)$ invariants of 3-manifolds with the first Betti number 1. Further, motivated by this expansion, we formulate perturbative invariants of such 3-manifolds. We show some properties of the perturbative invariants, which imply that their coefficients are independent invariants.

In the late 1980s, Witten [Wi] proposed topological invariants of a closed 3-manifold M for a simple compact Lie group G, what we call quantum G invariant, which is formally presented by a path integral whose Lagrangian is the Chern-Simons functional of G connections on M. There are two approaches to obtain mathematically rigorous information from a path integral: the operator formalism and the perturbative expansion. Motivated by the operator formalism of the Chern-Simons path integral, Reshetikhin and Turaev [ReT] gave the first rigorous mathematical construction of quantum invariants, as linear sums of quantum invariants of framed links. After that, rigorous constructions of quantum invariants were obtained by various approaches; in particular, Kirby and Melvin [KiM] formulated the quantum $SO(3)$ invariant, which we denote by $\tau_r^{SO(3)}(M)$; it is defined to be a linear sum of the quantum $\mathfrak{sl}_2$ invariant (the colored Jones polynomial) of framed links at an rth roots of unity. On the other hand, the perturbative expansion of the Chern-Simons path integral suggests that we can formulate perturbative invariants which describe asymptotic behavior of quantum invariants at $r \to \infty$; in fact, it is known (see, e.g., [O3]) that we can formulate perturbative invariants of rational homology 3-spheres based on arithmetic perturbative expansion of quantum invariants of them.

We review the construction of the perturbative $SO(3)$ invariant of a rational homology 3-sphere M, as follows. Let p be an odd prime, and put $\zeta = \exp(2\pi\sqrt{-1}/p)$. Since it is known (by Murakami [M]) that $\tau_p^{SO(3)}(M) \in \mathbb{Z}[\zeta]$, we can make an expansion,

$$\tau_p^{SO(3)}(M) = a_{p,0} + a_{p,1}(\zeta - 1) + a_{p,2}(\zeta - 1)^2 + \cdots + a_{p,N}(\zeta - 1)^N,$$

with some integers $a_{p,n}$'s. Though this expansion is not unique, $(a_{p,n} \bmod p) \in \mathbb{Z}/p\mathbb{Z}$ is uniquely determined by the value of $\tau_p^{SO(3)}(M)$, since $\mathbb{Z}[\zeta]$ is isomorphic to $\mathbb{Z}[q]/T(q)$ where

$$T(q) = \frac{q^p - 1}{q - 1} = \binom{p}{1} + \binom{p}{2}(q - 1) + \binom{p}{3}(q - 1)^2 + \cdots + \binom{p}{p}(q - 1)^{p-1}.$$

The author showed in [O1] that there exist $\lambda_n(M) \in \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{m}]$, where $m = \max\{2n+1, |H_1(M; \mathbb{Z})|\}$ and $|H_1(M; \mathbb{Z})|$ denotes the order of $H_1(M; \mathbb{Z})$, such that

$$a_{p,n} \underset{(p)}{\equiv} \left(\frac{|H_1(M;\mathbb{Z})|}{p}\right) \lambda_n(M)$$

for any odd prime $p \geq \max\{2n+3, |H_1(M; \mathbb{Z})|\}$, and defined the perturbative $SO(3)$ invariant of M by

$$\tau(M) = \lambda_0(M) + \lambda_1(M)(q-1) + \lambda_2(M)(q-1)^2 + \cdots \in \mathbb{Q}[[q-1]].$$

In particular, by results of Murakami [M], we can see that the first two coefficients are presented by "(semi-)classical" invariants,

$$\lambda_0(M) = \frac{1}{|H_1(M;\mathbb{Z})|}, \qquad \lambda_1(M) = \frac{6\,\lambda(M)}{|H_1(M;\mathbb{Z})|},$$

where $\lambda(M)$ denotes the Casson-Walker invariant ([Wa]) of M. Further, when M is obtained from S^3 by f surgery along a knot K, $\tau(M)$ is presented by Gaussian integral of the following form ([Ro1, Ro3], see also [BGRT, O3]),

$$\tau(M) = (\text{constant}) \cdot \int_{n \in \mathbb{R}} q^{f(n^2-1)/4} [n]^2 \left(\text{perturbative expansion of } J_n(K; q)\right) dn,$$

where $[n] = (q^{n/2} - q^{-n/2})/(q^{1/2} - q^{-1/2})$, and $J_n(K; q)$ denotes the colored Jones polynomial of K, whose perturbative expansion is a power series in $q - 1$ with coefficients of polynomials in q and q^n. Note that, when $f = 0$, this integral diverges.

In this paper, motivated by the arithmetic expansion of $\tau_p^{SO(3)}(M)$, we formulate a perturbative invariant of a 3-manifold M with the first Betti number 1,

$$\tau(M; c) = \lambda_0(M; c) + \lambda_1(M; c)(q-1) + \lambda_2(M; c)(q-1)^2 + \cdots \in \mathbb{C}[[q-1]],$$

where c is 0 or a zero of the Alexander polynomial $\Delta(t)$ of M, and prove that it is a topological invariant of M (Theorem 1.1). When M is obtained from S^3 by 0 surgery along a knot K, instead of the $f = 0$ case of the above Gaussian integral, we consider the following integral,

$$(\text{constant}) \cdot \int_{t \in \{|z|=1\} \subset \mathbb{C}} \left(\text{loop expansion of } [n]^2 J_n(K; q)\,|_{q^n \to t}\right) \frac{dt}{t},$$

where the loop expansion of $J_n(K; q)$ is a power series in $q - 1$ with coefficients of rational functions of q^n whose denominators are powers of the Alexander polynomial of K, and our formulation of the perturbative invariant is obtained as the residue of this integral,

$$\tau(M; c) = (\text{constant}) \cdot \operatorname*{Res}_{t=c} \frac{1}{t} \left(\text{loop expansion of } [n]^2 J_n(K; q)\,|_{q^n \to t}\right).$$

The 0th coefficient $\lambda_0(M; c)$ is presented by the Alexander polynomial of M (Proposition 3.3), noting that the Alexander polynomial can be regarded as the $\mathbb{Z}$ equivariant version of the order of the first homology group. Further, when M is obtained from S^3 by 0

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surgery along a knot K, $\lambda_1(M; c)$ is presented by the 2-loop polynomial of K, which can be regarded as "$\mathbb{Z}$ equivariant Casson invariant" of the infinite cyclic cover of the complement of K (see [O5]). $\lambda_1(M; c)$ can be regarded as another extension of the Casson-Walker invariant than the Casson-Walker-Lescop invariant ([Les1, Les2]), for 3-manifolds with the first Betti number 1. The coefficients of the arithmetic expansion of $\tau_p^{SO(3)}(M)$ are presented by linear sums of 1 and $\big((1+c_i)/(1-c_i)\big)^p$, where $c_1^{\pm 1}, \cdots, c_n^{\pm 1}$ are the zeros of the Alexander polynomial. As we explain in Section 2, the space of such series can be described by the "Tamagawa map",

$$\Big(\sum_{i=1}^{n} \big(\frac{1+c_i}{1-c_i}\big)^{ap+b} \Big)_{p:\,\mathrm{primes}} \quad \longmapsto \quad \sum_{i=1}^{n} \big(\frac{1+c_i}{1-c_i}\big)^a \otimes \big(\frac{1+c_i}{1-c_i}\big)^b \quad \in \mathbb{C} \underset{\mathbb{Q}}{\otimes} \mathbb{C}.$$

Through this map, we can regard

$$\sum_{c=0,c_1,\cdots,c_n} \frac{1+c}{1-c} \otimes \frac{1-c}{1+c} \tau(M; c) \ \in (\mathbb{C} \underset{\mathbb{Q}}{\otimes} \mathbb{C})[[q-1]]$$

as an arithmetic perturbative expansion of $\tau_p^{SO(3)}(M)$.

An idea of constructing our perturbative invariants is to consider equivariant invariants ([O5, O6, O7]). As mentioned above, the Alexander polynomial and the 2-loop polynomial can be regarded as equivariant invariants. More generally, the loop expansion of the Kontsevich invariant of a knot can be regarded as the "$\mathbb{Z}$ equivariant LMO invariant" of the infinite cyclic cover of the knot complement. As its $\mathfrak{sl}_2$ reduction, the loop expansion of the colored Jones polynomial of a knot can be regarded as the "$\mathbb{Z}$ equivariant perturbative $SO(3)$ invariant" of the infinite cyclic cover of the knot complement. In this sense, our perturbative invariants of 3-manifolds with the first Betti number 1 can be regarded as "$\mathbb{Z}$ equivariant perturbative invariants" of the infinite cyclic cover of such 3-manifolds. By considering equivariant invariants of covering spaces, we can formulate perturbative invariants of such 3-manifolds, though we could not define perturbative invariants of such 3-manifolds by the method for homology 3-spheres.

It is known that, though the LMO invariant ([LMO]) is enough powerful to be expected to classify integral homology 3-spheres, it is weak for 3-manifolds with positive first Betti numbers; in fact, when $b_1(M) > 0$, the value of the LMO invariant of M can be determined from "classical" invariants such as the cohomology ring, the Alexander polynomial and the Casson-Walker-Lescop invariant of M (see [O3]). On the other hand, Proposition 3.2 suggests that our perturbative invariant is as fine as the perturbative $SO(3)$ invariant for homology 3-spheres in the sense that the perturbative $SO(3)$ invariant of a homology 3-sphere N can be determined from the values of our perturbative invariants of $M \# N$ and M for a 3-manifold M with the first Betti number 1. Moreover, Proposition 3.4 implies that the coefficients of our perturbative invariants are independent invariants. Since the LMO invariant dominates all perturbative invariants for homology 3-spheres (see [O3]), it is expected that there exists a refinement of the LMO invariant; this will be discussed in [O9]. Out perturbative invariants would be the $\mathfrak{sl}_2$ reduction of such a refinement of the LMO invariant.

It is still an open problem to interpret our perturbative invariants from the Chern-Simons path integral. It might be necessary to consider a $\mathbb{Z}$ equivariant version of the

Chern-Simons theory. It might be related to the fact that the space of irreducible $SU(2)$ representations of the fundamental group of a knot complement branches from the space of reducible representations at zeros of the Alexander polynomial of the knot.

The paper is organized as follows. In Section 1, we define perturbative invariants for 3-manifolds with the first Betti number 1, and show their concrete values for some examples. In Section 2, we explain how we can describe the arithmetic perturbative expansion of $\tau_p^{SO(3)}(M)$ in terms of the perturbative invariants. In Section 3, we show some properties of the perturbative invariants. In Section 4, we review the loop expansions of the Kontsevich invariant and the colored Jones polynomials, which are used in the proof of the main theorem. In Section 5, we prove the main theorem, which states the topological invariance of the perturbative invariants. In Section 6, we calculate arithmetic limits of some rational functions of roots of unity, which we use when we calculate arithmetic expansion of quantum invariants.

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Notation

We denote by $\Delta(t)$ the Alexander polynomial of a knot K or a 3-manifold M with $b_1(M) = 1$; they are equal when M is obtained from S^3 by 0 surgery along K. We normalize $\Delta(t)$ in the way how $\Delta(t) = \Delta(t^{-1})$ and $\Delta(1)$ is equal to the order of the torsion subgroup of $H_1(M; \mathbb{Z})$. Further, we normalize the colored Jones polynomial $J_n(K; q)$ in the way how $[n] J_n(K; q)$ is equal to the quantum $(\mathfrak{sl}_2, V_n)$ invariant for any knot K with 0 framing, which is the invariant defined from the irreducible n-dimensional representation V_n of the quantum group of $\mathfrak{sl}_2$ (see, *e.g.*, [O3]).

1   Definition of the perturbative invariants

In this section, we define perturbative invariants for 3-manifolds with the first Betti number 1, motivated by the arithmetic perturbative expansion of the quantum $SO(3)$ invariant. We show concrete values of the perturbative invariants for some examples and for the case where the Alexander polynomial has small degree.

Before explaining the general case, for simplicity, we explain the definition when M is obtained from S^3 by surgery along a knot K with 0 framing. Let p be an odd prime, and put $\zeta = \exp(2\pi\sqrt{-1}/p)$. Then, the quantum $SO(3)$ invariant $\tau_p^{SO(3)}(M)$ of M is presented by

$$\tau_p^{SO(3)}(M) = \frac{1}{c_0} \sum_{\substack{1 \le n < p \\ n \, \text{is odd}}} [n]_\zeta^2 \, J_n(K; \zeta),$$

where $J_n(K; q)$ denotes the colored Jones polynomial of K, and $[n]_\zeta = (\zeta^{n/2} - \zeta^{-n/2})/(\zeta^{1/2} -$

$\zeta^{-1/2}$), and

$$c_0 = \sum_{\substack{1 \le n < p \\ n \text{ is odd}}} [n]_\zeta^2 = \frac{1}{2} \sum_{n \in \mathbb{Z}/p\mathbb{Z}} \frac{\zeta^n + \zeta^{-n} - 2}{\zeta + \zeta^{-1} - 2} = \frac{-p}{\zeta + \zeta^{-1} - 2}.$$

Further, as shown in [Ro4] (see also Remark 4.2), the loop expansion of the colored Jones polynomial is presented by

$$J_n(K; q) = \sum_{\ell=0}^{\infty} \frac{P_\ell(q^n)}{\Delta(q^n)^{2\ell+1}} (q-1)^\ell, \tag{1}$$

where $P_\ell(t) \in \mathbb{Z}[t^{\pm 1}]$ and, in particular, $P_0(t) = 1$, and $\Delta(t)$ is the Alexander polynomial of K, which is equal to the Alexander polynomial of M. Hence, for each n,

$$J_n(K; \zeta) = \sum_{\ell=0}^{N} \frac{P_\ell(\zeta^n)}{\Delta(\zeta^n)^{2\ell+1}} (\zeta-1)^\ell + O\big((\zeta-1)^{N+1}; \mathbb{Z}_{(p)}[\zeta]\big).$$

By substituting this to the formula of $\tau_p^{SO(3)}(M)$,

$$\tau_p^{SO(3)}(M) = \frac{1}{c_0} \sum_{\substack{1 \le n < p \\ n \text{ is odd}}} \frac{\zeta^n + \zeta^{-n} - 2}{\zeta + \zeta^{-1} - 2} \sum_{\ell=0}^{N} \frac{P_\ell(\zeta^n)}{\Delta(\zeta^n)^{2\ell+1}} (\zeta-1)^\ell + O\big((\zeta-1)^{N+1}\big)$$

$$= -\frac{1}{2} \sum_{\ell=0}^{N} \Big(\frac{1}{p} \sum_{n \in \mathbb{Z}/p\mathbb{Z}} \frac{(\zeta^n + \zeta^{-n} - 2)\, P_\ell(\zeta^n)}{\Delta(\zeta^n)^{2\ell+1}} \Big) (\zeta-1)^\ell + O\big((\zeta-1)^{N+1}\big).$$

Further, by Proposition 6.1, $\tau_p^{SO(3)}(M) \in \mathbb{Z}_{(p)}[\zeta]$ and

$$\tau_p^{SO(3)}(M) = -\frac{1}{2} \sum_{\ell=0}^{N} \Big(\sum_{c=0, c_1, \cdots, c_n} \big(\frac{1+c}{1-c}\big)^p \operatorname*{Res}_{t=c} \frac{(1-t^{-1})^2 P_\ell(t)}{\Delta(t)^{2\ell+1}} \Big) (\zeta-1)^\ell + O\big((\zeta-1)^{\min\{N+1, p-1\}}\big), \tag{2}$$

for all but finitely many primes p, where $c_1, c_1^{-1}, \cdots, c_n, c_n^{-1}$ are the zeros of $\Delta(t)$. Motivated by this formula, we define the *perturbative invariant* of M at c by

$$\tau(M; c) = -\frac{1}{2} \cdot \frac{1+c}{1-c} \sum_{\ell=0}^{\infty} \Big(\operatorname*{Res}_{t=c} \frac{(1-t^{-1})^2 P_\ell(t)}{\Delta(t)^{2\ell+1}} \Big) (q-1)^\ell \in \mathbb{C}[[q-1]], \tag{3}$$

where c is 0 or a zero of the Alexander polynomial $\Delta(t)$ of M.

In general, we define the perturbative invariant, as follows. Let M be a 3-manifold with the first Betti number 1. Then, M can be obtained from a rational homology 3-sphere N by surgery along a null-homologous knot K with 0 framing in N. By Proposition 4.1,

$$W_{\mathfrak{sl}_2, V_n}\big(Z^{\mathrm{LMO}}(N, K)\big) = [n] \sum_{\ell=0}^{\infty} \frac{P_\ell(q^n)}{\Delta(q^n)^{2\ell+1}} (q-1)^\ell$$

for some $P_\ell(t) \in \mathbb{Q}[t^{\pm 1}]$, where $\Delta(t)$ is the Alexander polynomial of K, which is equal to the Alexander polynomial of M; in particular, $P_0(t)$ $\big(= \Delta(1)\big)$ is equal to the order of the torsion part of $H_1(M; \mathbb{Z})$. By using these $P_\ell(t)$, we define the *perturbative invariant* $\tau(M; c)$ at c by (3), where c is 0 or a zero of the Alexander polynomial $\Delta(t)$ of M. By Remark 4.2, this definition fits the previous definition when $N = S^3$.

Theorem 1.1. $\tau(M; c)$ *does not depend on the choice of N and K, i.e., $\tau(M; c)$ is a topological invariant of a 3-manifold M with $b_1(M) = 1$.*

We show the proof of the theorem in Section 5.

By definition, $\tau(M; 0) \in \mathbb{Q}[[q-1]]$. By Proposition 1.6, if the irreducible factors of $\Delta(t)$ are of degree ≤ 1, then $\tau(M; c) \in \mathbb{Q}[[q-1]]$. By Proposition 3.1, $\tau(M; c) = \tau(M; c^{-1})$ for a zero c of $\Delta(t)$. We denote by $\lambda_\ell(M; c)$ the ℓth coefficient of $\tau(M; c)$, that is, $\tau(M; c) = \sum_{\ell=0}^{\infty} \lambda_\ell(M; c)(q-1)^\ell$ and

$$\lambda_\ell(M; c) = -\frac{1}{2} \cdot \frac{1+c}{1-c} \operatorname*{Res}_{t=c} \frac{(1-t^{-1})^2 \, P_\ell(t)}{\Delta(t)^{2\ell+1}} \quad \in \mathbb{C}.$$

Note that $\lambda_\ell(M; 0) \in \mathbb{Q}$, and, if the irreducible factors of $\Delta(t)$ are of degree ≤ 1, then $\lambda_\ell(M; c) \in \mathbb{Q}$.

Example 1.2. By definition, $\tau(S^1 \times S^2; 0) = 1$, since $S^1 \times S^2$ is obtained from S^3 by 0 surgery along the trivial knot. Hence, by Proposition 3.2, for a rational homology 3-sphere N, $\tau\big((S^1 \times S^2) \# N; 0\big)$ is equal to the product of the perturbative $SO(3)$ invariant ([O1]) of N and the order of $H_1(N; \mathbb{Z})$.

Example 1.3. For the 3-manifold obtained from S^3 by 0 surgery along the (a, b) torus knot,

$$\tau(M; 0) = \begin{cases} -\dfrac{1}{2}\, q & \text{if } (a, b) = (2, 3), \\ 0 & \text{otherwise,} \end{cases}$$

$$\tau(M; c) = -\frac{(1-c^{-1})^2}{2\,\Delta'(c)} q^{(a-a^{-1})(b-b^{-1})/4},$$

where c is a zero of $\Delta(t) = (t^{ab/2} - t^{-ab/2})(t^{1/2} - t^{-1/2})/(t^{a/2} - t^{-a/2})(t^{b/2} - t^{-b/2})$ and we regard the right-hand sides of the formulas as in $\mathbb{Q}[[q-1]]$ by expanding them into power series in $q - 1$.

Proof. Rozansky [Ro2] showed that the loop expansion of the colored Jones polynomial is presented by

$$J_n(K; q^n) = \frac{q^{(ab-a/b-b/a)/4}}{t^{1/2} - t^{-1/2}} \sum_{m=0}^{\infty} \frac{1}{m!} \Big(\frac{\log q}{ab}\Big)^m \Big(t\frac{d}{dt}\Big)^{2m} \frac{t^{1/2} - t^{-1/2}}{\Delta(t)},$$

where the equality holds by putting $t = q^n$ after we calculate the right-hand side. Hence, by definition,

$$\tau(M; c) = -\frac{q^{(ab-a/b-b/a)/4}}{2} \cdot \frac{1+c}{1-c} \sum_{m=0}^{\infty} \frac{1}{m!} \Big(\frac{\log q}{ab}\Big)^m \operatorname*{Res}_{t=c} \frac{t^{1/2} - t^{-1/2}}{t} \Big(t\frac{d}{dt}\Big)^{2m} \frac{t^{1/2} - t^{-1/2}}{\Delta(t)}.$$

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The residue of this formula is calculated recursively by using

$$\operatorname*{Res}_{t=c} (t^{1/2} \mp t^{-1/2}) \frac{d}{dt}\big(t\,\varphi(t)\big) = -\frac{1}{2}\operatorname*{Res}_{t=c}(t^{1/2}\pm t^{-1/2})\varphi(t), \tag{4}$$

where we obtain this formula from

$$\frac{d}{dt}\Big((t^{1/2}\mp t^{-1/2})\big(t\,\varphi(t)\big)\Big) = \frac{1}{2}(t^{1/2}\pm t^{-1/2})\varphi(t) + (t^{1/2}\mp t^{-1/2})\frac{d}{dt}\big(t\,\varphi(t)\big),$$

since the residue of the differential of some function equals 0. It follows that

$$\tau(M;c) = -\frac{q^{(ab-a/b-b/a)/4}}{2}\cdot\frac{1+c}{1-c}\sum_{m=0}^{\infty}\frac{1}{m!}\Big(\frac{\log q}{ab}\Big)^m \times \frac{1}{4^m}\operatorname*{Res}_{t=c}\frac{t+t^{-1}-2}{t\,\Delta(t)}$$

$$= -\frac{q^{(a-a^{-1})(b-b^{-1})/4}}{2}\cdot\frac{1+c}{1-c}\operatorname*{Res}_{t=c}\frac{t+t^{-1}-2}{t\,\Delta(t)}\,.$$

Therefore, by calculating this residue in the same way as the proof of Proposition 3.3, we obtain the required formulas. □

Example 1.4. Rozansky [Ro2] calculated "approximation" of loop polynomials of some knots. If his "approximate" formulas would be the exact ones, we can calculate our invariants by using them for 3-manifolds obtained from S^3 by 0 surgery along those knots. For example, for the 3-manifold M obtained from S^3 by 0 surgery along the figure-eight knot,

$$\tau(M;0) \underset{?}{=} \frac{1}{2} + 0(q-1) + 0(q-1)^2 + \cdots,$$

$$\tau(M;\tfrac{3+\sqrt{5}}{2}) \underset{?}{=} -\frac{1}{2} + 0(q-1) - \frac{1}{125}(q-1)^2 + \cdots.$$

Further, for the 3-manifold M obtained from S^3 by 0 surgery along the 5_2 knot,

$$\tau(M;0) \underset{?}{=} -\frac{1}{4} + \frac{5}{16}(q-1) - \frac{23}{64}(q-1)^2 + \cdots,$$

$$\tau(M;\tfrac{3+\sqrt{-7}}{4}) \underset{?}{=} \frac{1}{4} - \frac{31}{112}(q-1) + \frac{915}{3126}(q-1)^2 + \cdots.$$

The 0th and 1st coefficients of the right-hand sides are correct, since we can calculate the 2-loop polynomial exactly for arbitrarily given knot; see [Kr2, O4, O8].

1.1 The case where $\Delta(t)$ is of degree 0

In this case, $\Delta(t) = b$, where b is the order of the torsion subgroup of $H_1(M;\mathbb{Z})$. We can calculate the perturbative invariant from the loop polynomial by

$$\lambda_\ell(M;0) = -\frac{1}{2}\operatorname*{Res}_{t=0}\frac{(t+t^{-1}-2)\,P_\ell(t)}{b^{2\ell+1}\,t}$$

$$= -\frac{1}{2\,b^{2\ell+1}}\big(\text{constant term of }(t+t^{-1}-2)P_\ell(t)\big).$$

Remark 1.5. When $H_1(M;\mathbb{Z}) \cong \mathbb{Z}$ and the Alexander polynomial $\Delta(t)$ of M equals 1, independently of the author, Kazuo Habiro and Thang Le constructed an invariant, presented by "Habiro expansion", which is an expansion of the form

$$\sum_{n=0}^{\infty} a_n (q-1)(q^2-1)\cdots(q^n-1).$$

It is an equivalent invariant to $\tau(M;0)$ for such 3-manifolds in the sense that their invariant is uniquely determined by $\tau(M;0)$. Further, they showed that the value of each $\tau_p^{SO(3)}(M)$ is obtained from their invariant by substituting a root of unity to q. It follows that $\tau(M;0)$ is universal among quantum invariants $\tau_p^{SO(3)}(M)$ for such 3-manifolds.

1.2 The case where the irreducible factors of $\Delta(t)$ are of degree 1

In this case, we show Propositions 1.6 and 1.7, in this subsection.

Proposition 1.6. *If the irreducible factors of $\Delta(t)$ are of degree 1, $\tau(M;c) \in \mathbb{Q}[[q-1]]$.*

Proof. It is sufficient to show that $\lambda_\ell(M;c) \in \mathbb{Q}$ for a zero c of $\Delta(t)$, where c is a zero of a factor $1 - a(t + t^{-1} - 2)$ of $\Delta(t)$. By definition,

$$\lambda_\ell(M;c) = -\frac{1}{2\,b_0^{2\ell+1}} \cdot \frac{1+c}{1-c} \operatorname*{Res}_{t=c} \frac{(t + t^{-1} - 2)\,P_\ell(t)}{t\,\Delta(t)^{2\ell+1}}.$$

It can be presented by a linear sum of

$$\frac{1+c}{1-c} \operatorname*{Res}_{t=c} \frac{(t+t^{-1}-2)^m}{t\left(1 - a(t+t^{-1}-2)\right)^{m+1}}.$$

Further, by Lemma 6.5, this is equal to

$$(-1)^m 2 \binom{2m-1}{m}\left(\frac{1-c}{1+c}\right)^{2m} = \frac{(-1)^m 2}{(4a+1)^m}\binom{2m-1}{m} \in \mathbb{Q},$$

since $4a + 1 = (1+c)^2/(1-c)^2$. Hence, $\lambda_\ell(M;c) \in \mathbb{Q}$. $\square$

Proposition 1.7. *If $\Delta(t)$ is of degree 1, the following (1) and (2) holds, where we put $\Delta(t) = b_0 - b_1(t + t^{-1} - 2)$ with non-zero integers b_0, b_1, and c is a zero of $\Delta(t)$.*
(1) The 0th coefficients of the perturbative invariants are presented by

$$\lambda_0(M;0) = \frac{b_0}{2b_1}, \qquad \lambda_0(M;c) = -\frac{b_0}{2b_1}.$$

(2) Putting the 2-loop polynomial by $P_1(t) = f(t)\Delta(t)^3 + a_2\Delta(t)^2 + a_1\Delta(t) + a_0$, the 1st coefficients of the perturbative invariants are presented by

$$\lambda_1(M;0) = -\frac{1}{2}\Big(\text{constant term of } (t + t^{-1} - 2)f(t)\Big) + \frac{a_2}{2b_1},$$

$$\lambda_1(M;c) = -\frac{a_2}{2b_1} + \frac{a_1}{b_0(4b_1 + b_0)} + \frac{a_0(b_1 + b_0)}{b_0^2(4b_1 + b_0)^2}.$$

Proof. We obtain (1) by Proposition 3.3.

We show (2), as follows. By definition, putting $z = t + t^{-1} - 2$,

$$\lambda_1(M;0) = -\frac{1}{2} \operatorname*{Res}_{t=0} \frac{z \, P_1(t)}{t \, \Delta(t)^3} = -\frac{1}{2} \operatorname*{Res}_{t=0} \frac{z}{t} \Big(f(t) + \frac{a_2}{b_0 - b_1 z} + \frac{a_1}{(b_0 - b_1 z)^2} + \frac{a_0}{(b_0 - b_1 z)^3} \Big).$$

Since

$$\operatorname*{Res}_{t=0} \frac{z}{t \, (b_0 - b_1 z)^\ell} = \begin{cases} -\dfrac{1}{b_1} & \text{if } \ell = 1, \\ 0 & \text{if } \ell > 1, \end{cases}$$

we obtain the required formula of $\lambda_1(M;0)$. Similarly, we have that

$$\lambda_1(M;c) = -\frac{1}{2} \cdot \frac{1+c}{1-c} \operatorname*{Res}_{t=c} \frac{z}{t} \Big(f(t) + \frac{a_2}{b_0 - b_1 z} + \frac{a_1}{(b_0 - b_1 z)^2} + \frac{a_0}{(b_0 - b_1 z)^3} \Big).$$

Since

$$\frac{1+c}{1-c} \operatorname*{Res}_{t=c} \frac{z}{t \, (b_0 - b_1 z)^\ell} = \begin{cases} \dfrac{1}{b_1} & \text{if } \ell = 1, \\ -\dfrac{2}{b_0(4b_1 + b_0)} & \text{if } \ell = 2, \\ -\dfrac{2(b_1 + b_0)}{b_0^2(4b_1 + b_0)^2} & \text{if } \ell = 3, \end{cases}$$

we obtain the required formula of $\lambda_1(M;c)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2 Arithmetic perturbative expansion of $\tau_p^{SO(3)}(M)$

In this section, we explain how we can describe the arithmetic perturbative expansion of $\tau_p^{SO(3)}(M)$. We have two descriptions; one is $\tau(M; \sigma)$ which describes the behavior of the series $\big(\hat{\lambda}_{p,\ell}(M)\big)$ for certain subsequences of p, and the other is $\tau(M)$ which is formulated in the space to identify the set of the series $\big(\hat{\lambda}_{p,\ell}(M)\big)$.

We calculate the arithmetic perturbative expansion of $\tau_p^{SO(3)}(M)$ under the following assumption; this is a technical assumption (see Remark 2.3).

Assumption 2.1. M is a 3-manifold obtained from S^3 by 0 surgery along a knot K.

In this case, as mentioned before, $\tau_p^{SO(3)}(M) \in \mathbb{Z}_{(p)}[\zeta]$. We expand this as a polynomial in $(\zeta - 1)$,

$$\tau_p^{SO(3)}(M) = \sum_{\ell=0}^{p-2} \hat{\lambda}_{p,\ell}(M)(\zeta - 1)^\ell + O\big((\zeta - 1)^{p-1}; \mathbb{Z}_{(p)}[\zeta]\big) \tag{5}$$

for some $\hat{\lambda}_{p,\ell}(M)$. This expansion is not unique, but $\big(\hat{\lambda}_{p,\ell}(M)$ modulo $p\big)$ is uniquely determined from $\tau_p^{SO(3)}(M)$, so we regard $\hat{\lambda}_{p,\ell}(M)$ as in the prime field $\mathbb{F}_p$. Further, by (2),

$$\hat{\lambda}_{p,\ell}(M) \underset{(p)}{\equiv} \sum_c \Big(\frac{1+c}{1-c}\Big)^{p-1} \lambda_\ell(M;c) \in \mathbb{Z}_{(p)} \tag{6}$$

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for all but finitely many primes p (to be precise, for an odd prime p such that $p > \ell + 1$ and $p > 1 + \deg P_k(t) - (2k+1)\deg \Delta(t)$ for any $k \leq \ell$ and $\Delta(1)$ is not divisible by p). Let $\mathcal{F}$ be the minimal splitting field of $\Delta(t)$, which is a Galois extension of $\mathbb{Q}$. For an element σ of the Galois group of $\mathcal{F}/\mathbb{Q}$, we put

$$\tau(M;\sigma) = \sum_{c=0,c_1,\cdots,c_n} \frac{1+\sigma(c)}{1-\sigma(c)} \cdot \frac{1-c}{1+c} \tau(M;c) \quad \in \mathcal{F}[[q-1]] \ \subset \ \mathbb{C}[[q-1]],$$

where $c_1, c_1^{-1}, \cdots, c_n, c_n^{-1}$ are the zeros of $\Delta(t)$. In particular, its ℓth coefficient is given by

$$\lambda_\ell(M;\sigma) = \sum_{c=0,c_1,\cdots,c_n} \frac{1+\sigma(c)}{1-\sigma(c)} \cdot \frac{1-c}{1+c} \lambda_\ell(M;c) \quad \in \mathcal{F} \ \subset \ \mathbb{C}.$$

Let $\mathcal{O}$ be the integer ring of $\mathcal{F}$, and let $\mathfrak{p}$ be a non-zero prime ideal of $\mathcal{O}$. Let $\mathcal{O}_{(\mathfrak{p})}$ denote the subring of $\mathcal{F}$ consisting of elements of the form a/b for $a \in \mathcal{O}$ and $b \in \mathcal{O} - \mathfrak{p}$. Then, $\lambda_\ell(M;\sigma) \in \mathcal{O}_{(\mathfrak{p})}$ for $\mathfrak{p}$ with $\Delta(1) \notin \mathfrak{p}$.

Proposition 2.2. *Under Assumption 2.1, let $\mathcal{F}$ and $\mathcal{O}$ be as above. Then, for all but finitely many prime ideals $\mathfrak{p}$ of $\mathcal{O}$,*

$$\hat{\lambda}_{p,\ell}(M) \ \underset{(\mathfrak{p})}{\equiv} \ \lambda_\ell(M;\sigma) \ \in \mathcal{O}_{(\mathfrak{p})}$$

where p is the prime given by $\mathfrak{p} \cap \mathbb{Z} = (p)$, and we determine $\sigma \in \mathrm{Gal}(\mathcal{F}/\mathbb{Q})$ from $\mathfrak{p}$ in the way mentioned in Section 2.1. Hence, the value of $\lambda_\ell(M;\sigma) \in \mathcal{F}$ is uniquely determined by the series $\big(\hat{\lambda}_{p,\ell}(M)\big)_{p:\,\mathrm{primes}}$, since there are infinitely many $\mathfrak{p}$ for each σ.

A more concrete assumption of $\mathfrak{p}$ for the proposition is that $\Delta(1) \notin \mathfrak{p}$, and $p > \ell + 1$, and $p > 1 + \deg P_k(t) - (2k+1)\deg \Delta(t)$ for any $k \leq \ell$.

Remark 2.3. Rozansky informed the author that we can show the loop expansion (1) of the colored Jones polynomial with integer coefficients for knots in integral homology 3-spheres; see [Ro5]. By using this, Assumption 2.1 can be replaced with the assumption that $H_1(M;\mathbb{Z}) \cong \mathbb{Z}$.

In the following of this section, we explain how to describe the arithmetic perturbative expansion of $\tau_p^{SO(3)}(M)$, by identifying the series $\big(\hat{\lambda}_{p,\ell}(M)\big)_{p:\,\mathrm{primes}}$, under Assumption 2.1 and, for simplicity, the following assumption.

Assumption 2.4. The Alexander polynomial $\Delta(t)$ of M does not have a multiple zero.

From (6) and the definition of $\lambda_\ell(M;c)$, we have that

$$\hat{\lambda}_{p,\ell}(M) \ \underset{(p)}{\equiv} \ \lambda_\ell(M;0) - \frac{1}{2} \sum_{i=1}^{n} \Big(\frac{1+c_i}{1-c_i}\Big)^p \mathop{\mathrm{Res}}_{t=c_i} \frac{(1-t^{-1})^2 P_\ell(t)}{\Delta(t)^{2\ell+1}}$$

for all but finitely many primes p, where we put $\Delta(t) = \prod_{i=1}^{n}\big(1 - \alpha_i(t + t^{-1} - 2)\big)$ with distinct $\alpha_i \in \mathbb{C}$, and c_i is a zero of $1 - \alpha_i(t + t^{-1} - 2)$. Hence, $\hat{\lambda}_{p,\ell}(M)$ can be presented

by a $\mathbb{Q}$-linear sum of 1 and

$$\sum_{i=1}^{n} \Big(\frac{1+c_i}{1-c_i}\Big)^p \operatorname*{Res}_{t=c_i} \frac{f_m(\alpha_i)}{t\big(1-\alpha_i(t+t^{-1}-2)\big)^m}$$

for some polynomials $f_m(\alpha) \in \mathbb{Q}[\alpha]$, because, for example, putting $z = t + t^{-1} - 2$,

$$\frac{1}{\Delta(t)} = \sum_{i=1}^{n} \frac{\alpha_i^{n-1}}{\prod_{j\neq i}(\alpha_i - \alpha_j)} \frac{1}{1-\alpha_i z}$$

$$= \frac{1}{d}\sum_{i=1}^{n} \alpha_i^{n-1} \prod_{j\neq i}(\alpha_i - \alpha_j) \prod_{\substack{j,j'\neq i \\ j\neq j'}}(\alpha_j - \alpha_{j'})^2 \frac{1}{1-\alpha_i z} = \sum_{i=1}^{n} \frac{f(\alpha_i)}{1-\alpha_i z}$$

for some polynomial $f(\alpha) \in \mathbb{Q}[\alpha]$, where $d = \prod_{i\neq j}(\alpha_i - \alpha_j)^2 \in \mathbb{Q}$, since a symmetric polynomial in α_j's ($j \neq i$) can be presented by a polynomial in α_i with rational coefficients. Therefore, in the same way as in the proof of Proposition 1.6, $\hat{\lambda}_{p,\ell}(M)$ can be presented by a $\mathbb{Q}$-linear sum of 1 and

$$\sum_{i=1}^{n} \Big(\frac{1+c_i}{1-c_i}\Big)^{p-1} g(\alpha_i)$$

for some polynomial $g(\alpha) \in \mathbb{Q}[\alpha]$. Putting $\epsilon_i = (1+c_i)/(1-c_i)$ (and, hence, $4\alpha_i + 1 = \epsilon_i^2$), $\hat{\lambda}_{p,\ell}(M)$ can be presented by a $\mathbb{Q}$-linear sum of 1 and $\sum_{i=1}^{n} \epsilon_i^{p+b}$ for $b = 1, 3, \cdots, 2n-1$.

Let us regard the series $\big(\hat{\lambda}_{p.\ell}(M)\big)_{p:\,\mathrm{primes}}$ as in

$$\mathbb{F} = \prod_{\mathrm{primes}\,p} \mathbb{F}_p \Big/ \bigoplus_{\mathrm{primes}\,p} \mathbb{F}_p,$$

i.e., we consider the series $\big(\hat{\lambda}_{p.\ell}(M)\big)_{p:\,\mathrm{primes}} \in \prod_{\mathrm{primes}\,p} \mathbb{F}_p$ modulo the equivalence that $(a_p) \sim (a_p')$ when $a_p = a_p'$ for all but finitely many primes p. Note that $\mathbb{F}$ forms a vector space over $\mathbb{Q}$. In $\mathbb{F}$, the series $\big(\hat{\lambda}_{p.\ell}(M)\big)_{p:\,\mathrm{primes}}$ can be presented by a linear sum of 1 and $\big(\sum_i \epsilon_i^{p+b}\big)_{p:\,\mathrm{primes}}$ for $b = 1, 3, \cdots, 2n-1$. Here, $\pm\epsilon_1, \cdots, \pm\epsilon_n$ equals the zeros of $F(x^2)$, where we set $\nabla(z) \in \mathbb{Q}[z]$ by $\Delta(t) = \nabla(z) = \prod_i(1-\alpha_i z)$ putting $z = t + t^{-1} - 2$, and set $F(x^2) \in \mathbb{Q}[x^2]$ by $F(x^2) = z^n \nabla(1/z) = \prod_i(z - \alpha_i)$ putting $x^2 = 4z + 1$.

Further, by the Tamagawa map given in Section 2.1, such a linear sum in $\mathbb{F}$ is identified in $\mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C}$, and, in particular, the Tamagawa map takes $\big(\hat{\lambda}_{p.\ell}(M)\big)_{p:\,\mathrm{primes}}$ to

$$-\frac{1}{2}\sum_{c=0,c_1,\cdots,c_n} \frac{1+c}{1-c} \otimes \operatorname*{Res}_{t=c} \frac{(1-t^{-1})^2 P_\ell(t)}{\Delta(t)^{2\ell+1}} = \sum_{c=0,c_1,\cdots,c_n} \frac{1+c}{1-c} \otimes \frac{1-c}{1+c}\lambda(M;c) \ \in \mathbb{C}\underset{\mathbb{Q}}{\otimes}\mathbb{C}. \quad (7)$$

So, we put

$$\tau(M) = \sum_{c=0,c_1,\cdots,c_n} \frac{1+c}{1-c} \otimes \frac{1-c}{1+c}\tau(M;c) \ \in (\mathbb{C}\underset{\mathbb{Q}}{\otimes}\mathbb{C})[[q-1]], \quad (8)$$

and regard it as the arithmetic perturbative expansion of $\tau_p^{SO(3)}(M)$. It can be shown from the topological invariance of $\tau_p^{SO(3)}(M)$ (without using Theorem 1.1) that this is a topological invariant of M, when M satisfies Assumptions 2.1 and 2.4.

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2.1 The Tamagawa map

The arguments of this subsection are due to Akio Tamagawa.

As before, let $\mathcal{F}$ be the minimal splitting field of $\Delta(t)$, let $\mathcal{O}$ be the integer ring of $\mathcal{F}$, and let $\mathfrak{p}$ be a non-zero prime ideal of $\mathcal{O}$. Further, let $k_{\mathfrak{p}}$ denote the residue field $\mathcal{O}/\mathfrak{p}$, which is a Galois extension of the prime field $\mathbb{F}_p$. Let $\mathrm{Fr}_{\mathfrak{p}}$ denote the Frobenius map $k_{\mathfrak{p}} \to k_{\mathfrak{p}}$ taking x to x^p. For all but finitely many $\mathfrak{p}$, the middle map of the following line is an isomorphism,

$$\mathrm{Fr}_{\mathfrak{p}} \in \mathrm{Gal}(k_{\mathfrak{p}}/\mathbb{F}_p) \xleftarrow{\cong} \big\{ \sigma \in \mathrm{Gal}(\mathcal{F}/\mathbb{Q}) \mid \sigma(\mathfrak{p}) = \mathfrak{p} \big\} \subset \mathrm{Gal}(\mathcal{F}/\mathbb{Q}).$$

By this map, we associate $\mathrm{Fr}_{\mathfrak{p}} = \sigma \in \mathrm{Gal}(\mathcal{F}/\mathbb{Q})$ with $\mathfrak{p}$. It is known as Chebotarev density theorem that, for each $\sigma \in \mathrm{Gal}(\mathcal{F}/\mathbb{Q})$, there are infinitely many $\mathfrak{p}$ such that $\mathrm{Fr}_{\mathfrak{p}} = \sigma$.

As before, let $\pm\epsilon_1, \cdots, \pm\epsilon_n$ be the zeros of $F(x^2)$. Tamagawa constructed a map

$$\Big(\sum_{i=1}^{n} \epsilon_i^{ap+b} \Big)_{p:\,\mathrm{primes}} \longmapsto \sum_{i=1}^{n} \epsilon_i^a \otimes \epsilon_i^b \in \mathcal{F} \underset{\mathbb{Q}}{\otimes} \mathcal{F} \subset \mathbb{C} \underset{\mathbb{Q}}{\otimes} \mathbb{C},$$

and showed that it induces an isomorphism between vector spaces,

$$\mathrm{span}_{\mathbb{Q}}\Big\{ \Big(\sum_{i=1}^{n} \epsilon_i^{ap+b} \Big)_p \ \Big| \ a+b \text{ is even} \Big\} \xrightarrow{\cong} \mathrm{span}_{\mathbb{Q}}\Big\{ \sum_{i=1}^{n} \epsilon_i^a \otimes \epsilon_i^b \ \Big| \ a+b \text{ is even} \Big\}. \tag{9}$$

$$\cap \qquad\qquad\qquad\qquad\qquad\qquad \cap$$

$$\mathbb{F} \qquad\qquad\qquad\qquad\qquad\qquad \mathcal{F} \underset{\mathbb{Q}}{\otimes} \mathcal{F}$$

A key of his construction of the isomorphism is to check the correspondence of the isomorphism at the middle part of the following maps using Chebotarev density theorem.

$$\mathcal{F} \underset{\mathbb{Q}}{\otimes} \mathbb{F} \xrightarrow{\cong} \prod_{\mathfrak{p}} k_{\mathfrak{p}} \Big/ \bigoplus_{\mathfrak{p}} k_{\mathfrak{p}} \longleftarrow \mathcal{F} \underset{\mathbb{Q}}{\otimes} \mathcal{F}$$

$$\cup \qquad\qquad\qquad\qquad \cup$$

$$\big(\epsilon^p \delta \big)_{\mathfrak{p}} \longleftarrow\!\shortmid \epsilon \otimes \delta$$

Remark 2.5. This remark is on a personal history of the research of this paper. In an early stage of the research, the author observed that the arithmetic perturbative expansion of $\tau_p^{SO(3)}(M)$ can be presented by a linear sum of $1, \gamma_0, \cdots, \gamma_{n-1}$ where $\gamma_j = \sum_{i=1}^{n} \epsilon_i^{p+2j-1}$ and $\epsilon_i = (1+c_i)/(1-c_i)$ for the zeros $c_1^{\pm 1}, \cdots, c_n^{\pm 1}$ of $\Delta(t)$, and formulated results such as results in Sections 2.2 and 2.3, when the irreducible factors of the Alexander polynomial $\Delta(t)$ is of degree ≤ 2. In order to formulate results when irreducible factors of $\Delta(t)$ is of general degree, it was a problem to identify the space $W = \mathrm{span}_{\mathbb{Q}}\{1, \gamma_0, \cdots, \gamma_{n-1}\}$. The author asked Akio Tamagawa how to identify W, and Tamagawa soon constructed the map (9) for $a = 1$. Motivated by this map, the author presented the coefficients of the linear sum in terms of integral, showing Proposition 6.1, and observed that the image of the linear sum by the Tamagawa map (9) is presented by the tensor product (7) for the ℓth coefficient of the arithmetic perturbative expansion of $\tau_p^{SO(3)}(M)$. After that, the author directly proved Theorem 1.1, which implies that the second factor of this tensor product is also a topological invariant, by introducing the t-through relation.

2.2 The case where $\Delta(t)$ is of degree 0

In this case, the arithmetic perturbative expansion of $\tau_p^{SO(3)}(M)$ is described (without using the Tamagawa map), as follows.

As a particular case of (6), we have

Proposition 2.6. *Under Assumption 2.1, for all but finitely many primes p,*

$$\hat{\lambda}_{p,\ell}(M) \underset{(p)}{\equiv} \lambda_\ell(M; 0).$$

That is, for such p,

$$\tau_p^{SO(3)}(M) \ = \ \tau(M; 0)^{(\leq \ell)} \ + O\big((\zeta - 1)^{\ell+1}; \mathbb{Z}_{(p)}[\zeta]\big),$$

where we put $\tau(M; \cdot)^{(\leq \ell)} = \sum_{k=0}^\ell \lambda_k(M; \cdot)(q - 1)^k$.

A more concrete assumption of p for the proposition is that b_0 is not divisible by p and $p > \ell + 1$ and $p > 1 + \deg P_k(t)$ for any $k \leq \ell$.

Therefore, as an alternative form of (8), $\tau(M) = 1 \otimes \tau(M; 0)$ is regarded as the arithmetic perturbative expansion of $\tau_p^{SO(3)}(M)$.

2.3 The case where the irreducible factors of $\Delta(t)$ are of degree 1

In this case, the arithmetic perturbative expansion of $\tau_p^{SO(3)}(M)$ is described, as follows.

As a particular case of (6), we have

Proposition 2.7. *Let M be a 3-manifold whose Alexander polynomial is presented by $\Delta(t) = \prod_{i=n}^n \big(1 - a_i(t + t^{-1} - 2)\big)$ with $a_i \in \mathbb{Q}$, satisfying Assumptions 2.1 and 2.4. Then, for all but finitely many primes p,*

$$\hat{\lambda}_{p,\ell}(M) \underset{(p)}{\equiv} \lambda(M; 0) + \sum_{i=1}^n \Big(\tfrac{4a_i+1}{p}\Big) \lambda(M; c_i),$$

where c_i is a zero of $1 - a_i(t + t^{-1} - 2)$. That is, for such p,

$$\tau_p^{SO(3)}(M) = \tau(M; 0)^{(\leq \ell)} + \sum_{i=1}^n \Big(\tfrac{4a_i+1}{p}\Big) \tau(M; c_i)^{(\leq \ell)} \ + O\big((\zeta - 1)^{\ell+1}; \mathbb{Z}_{(p)}[\zeta]\big).$$

A more concrete assumption of p for the proposition is that $\Delta(\pm 1)$ are not divisible by p and $p > \ell + 1$ and $p > \deg P_k(t) - 2\ell$ for any $k \leq \ell$.

We put

$$\varepsilon_k = \Big(\big(\tfrac{k}{2}\big), \big(\tfrac{k}{3}\big), \big(\tfrac{k}{5}\big), \cdots\Big) \in \mathbb{F},$$

and denote by $\mathcal{R}_1$ the subring of $\mathbb{F}$ generated by ε_k for non-zero integers k, ignoring primes p for which the Legendre symbol $\big(\tfrac{k}{p}\big)$ is not defined. Then,

$$\mathcal{R}_1 = \mathbb{Q}[\varepsilon_{-1}, \varepsilon_2, \varepsilon_3, \varepsilon_5, \cdots] / (\varepsilon_{-1}^2 = \varepsilon_2^2 = \varepsilon_3^2 = \varepsilon_5^2 = \cdots = 1) \tag{10}$$

by the Dirichlet prime number theorem which implies that there are infinitely many primes with same values for Legendre symbols. We can verify that the relations among ε_k in (10) are sufficient, since $\mathcal{R}_1$ is embedded into $\mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C}$ by the Tamagawa map (9),

$$\mathcal{R}_1 \longrightarrow \mathbb{C} \underset{\mathbb{Q}}{\otimes} \mathbb{C}, \qquad \varepsilon_k \longmapsto \sqrt{k} \otimes \frac{1}{\sqrt{k}},$$

which is an algebra homomorphism in this domain. Therefore, as an alternative form of (8),

$$\tau(M; 0) + \sum_{i=1}^{n} \varepsilon_{4a_i+1} \tau(M; c_i) \ \in \ \mathcal{R}_1[[q-1]]$$

is regarded as the arithmetic perturbative expansion of $\tau_p^{SO(3)}(M)$.

3 Properties of the perturbative invariants

In this section, we show some properties of the perturbative invariants. As before, we let M be a 3-manifold with the first Betti number 1.

Proposition 3.1. *For a zero c of $\Delta(t)$, $\tau(M; c) = \tau(M; c^{-1})$.*

Proof. It is sufficient to show that $\lambda_\ell(M; c) = \lambda_\ell(M; c^{-1})$. By definition, $\lambda_\ell(M; c)$ is presented by the form,

$$\frac{1+c}{1-c} \operatorname*{Res}_{t=c} \frac{\varphi(t)}{t} \ = \ \frac{1+c}{1-c} \cdot \frac{1}{2\pi\sqrt{-1}} \int_{\gamma(c)} \varphi(t) \frac{dt}{t}$$

for some rational function $\varphi(t)$ with $\varphi(t) = \varphi(t^{-1})$, where $\gamma(c)$ is a small loop winding around c. Further, $\lambda_\ell(M; c^{-1})$ is calculated by the same formula replacing the variable by $t = 1/u$. Since $dt/t = -du/u$ and $(1+c)/(1-c) = -(1+c^{-1})/(1-c^{-1})$, we obtain that $\lambda_\ell(M; c) = \lambda_\ell(M; c^{-1})$. $\qquad\square$

Proposition 3.2. *For a rational homology 3-sphere N,*

$$\tau(M \# N; c) = \tau(M; c)\,\tau(N)\,|H_1(N; \mathbb{Z})|,$$

where $M \# N$ denotes the connected sum of M and N, c is 0 or a zero of the Alexander polynomial of M, and $\tau(N) \in \mathbb{Q}[[q-1]]$ denotes the perturbative $SO(3)$ invariant ([O1]) of N.

Proof. By the connected sum of N, the perturbative invariant $\tau(M; c)$ changes by the multiple of

$$W_{\mathfrak{sl}_2}\big(Z^{\mathrm{LMO}}(N)\big) = |H_1(N; \mathbb{Z})|\,\tau(N) \ \in \mathbb{Q}[[q-1]],$$

where the equality is derived from the universality of the LMO invariant among perturbative invariants of rational homology 3-spheres [O2]. Hence, we obtain the proposition. $\qquad\square$

Proposition 3.3.

$$\lambda_0(M; 0) = \begin{cases} 1 & \text{if } \deg \Delta(t) = 0, \\ -\dfrac{\Delta(1)}{\Delta''(1)} & \text{if } \deg \Delta(t) = 1, \\ 0 & \text{if } \deg \Delta(t) > 1. \end{cases}$$

For a zero c of $\Delta(t)$ of multiplicity $m \leq 2$,

$$\lambda_0(M; c) = \begin{cases} \dfrac{(1 - c^{-2})\Delta(1)}{2\,\Delta'(c)} & \text{if } m = 1, \\ \left(\dfrac{2(c+1)}{c^3} + \dfrac{(c^{-2} - 1)\Delta'''(c)}{3\,\Delta''(c)}\right)\dfrac{\Delta(1)}{\Delta''(c)} & \text{if } m = 2. \end{cases}$$

Proof. We calculate $\lambda_0(M; 0)$, as follows. By definition,

$$\lambda_0(M; 0) = -\frac{\Delta(1)}{2} \operatorname*{Res}_{t=0} \frac{t + t^{-1} - 2}{t\,\Delta(t)}.$$

If $\deg \Delta(t) \leq 1$, putting $\Delta(t) = b_0$ or $\Delta(t) = b_1(t + t^{-1}) + b_0$, the required formula can be shown concretely. If $\deg \Delta(t) > 1$, putting $\Delta(t) = b_n(t^n + t^{-n}) + \cdots$, the function of the residue has the form $(t^n - 2t^{n-1} + t^{n-2})/(b_n + \cdots + b_n t^{2n})$, and it has no pole at $t = 0$. Hence, its residue equals 0.

We calculate $\lambda_0(M; c)$, as follows. By definition,

$$\lambda_0(M; c) = -\frac{\Delta(1)}{2} \cdot \frac{1 + c}{1 - c} \operatorname*{Res}_{t=c} \frac{t + t^{-1} - 2}{t\,\Delta(t)}.$$

If c is a zero of $\Delta(t)$ of multiplicity 1, the residue is calculated as

$$\lim_{t \to c} \frac{(t - c)(t + t^{-1} - 2)}{t\,\Delta(t)} = \lim_{t \to c} \frac{t + t^{-1} - 2}{t\,\Delta'(t)} = \frac{(c - 1)^2}{c^2\,\Delta'(c)},$$

and we obtain the required formula. If c is a zero of $\Delta(t)$ of multiplicity 2, putting $\Delta(t) = (t - c)^2 f(t)$, the residue is calculated as

$$\operatorname*{Res}_{t=c} \left(\frac{t + t^{-1} - 2}{t(t - c)^2 f(t)} - \frac{c + c^{-1} - 2}{c(t - c)^2 f(c)}\right) = \lim_{t \to c} \frac{1}{t - c}\left(\frac{t + t^{-1} - 2}{t\,f(t)} - \frac{c + c^{-1} - 2}{c\,f(c)}\right)$$

$$= \frac{d}{dt}\left(\frac{t + t^{-1} - 2}{t\,f(t)}\right)\Big|_{t=c}.$$

Hence, noting that $\Delta''(c) = 2f(c)$ and $\Delta'''(c) = 6f'(c)$, we obtain the required formula. $\square$

3.1 Clasper surgery formula

In this section, we show a surgery formula of the perturbative invariants under clasper surgery; for claspers, see [H1]. It follows from this surgery formula that the coefficients of the perturbative invariants are independent invariants.

Let F be a Seifert surface of a knot K. The Seifert form $H_1(F) \otimes H_1(F) \to \mathbb{R}$ is defined by taking $a \otimes b$ to the linking number of a and b^+, where b^+ denotes the puss-off of b in the normal direction of F. It is presented by a Seifert matrix, fixing a basis of $H_1(F)$. We denote by e_x, e_y the vectors presenting cohomology classes $x, y \in H^1(F)$ for the basis. The scaler $e_x^T \left(t^{1/2}V - t^{-1/2}V^T\right)^{-1} e_y$ depends only on the Seifert form and $x, y \in H^1(F)$, independently of the choice of a basis of $H_1(F)$. The Alexander polynomial of the knot is given by $\Delta(t) = \det\left(t^{1/2}V - t^{-1/2}V^T\right)$. A leaf of a clasper in the complement of a Seifert surface F of a knot is associated with a cohomology class in $H^1(F)$ counting cycles as
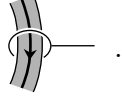
Let K be a knot with 0 framing in an integral homology 3-sphere, and let M be the 3-manifold obtained from the integral homology 3-sphere by surgery along K. Consider a graph clasper C of the following form, embedded in the complement of a Seifert surface F of a knot K. Let x, y be cohomology classes in $H^1(F)$ associated with the leaves of the graph clasper.
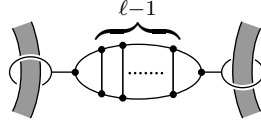
Proposition 3.4. *Let M and C be as above, and let M_C denote the 3-manifold obtained from M by surgery along C. Then, the change of the perturbative invariants by the clasper surgery is presented by*

$$\lambda_\ell(M_C; c) - \lambda_\ell(M; c) = -\frac{1}{2} \cdot \frac{1+c}{1-c} \operatorname*{Res}_{t=c} \frac{2^{\ell+2}(t^{1/2} - t^{-1/2})^3}{t\,\Delta(t)} e_x^T \left(t^{1/2}V - t^{-1/2}V^T\right)^{-1} e_y,$$

$$\lambda_k(M_C; c) = \lambda_k(M; c) \quad \text{for any } k < \ell.$$

Proof. In the same way as [O8, Proposition 4.17], we have that

$$Z^{(\ell\text{-loop})}\Big(\ \Big) - Z^{(\ell\text{-loop})}\Big(\ \Big) \qquad \Big(\ \Big) = \overbrace{}^{F(t)/\Delta(t)} ,$$

where $Z^{(\ell\text{-loop})}(K)$ denotes the ℓ-loop part of $\log \chi^{-1} Z(K)$, and

$$\frac{F(t)}{\Delta(t)} = -(t^{1/2} - t^{-1/2})\, e_x^T \left(t^{1/2}V - t^{-1/2}V^T\right)^{-1} e_y\,.$$

Hence, by this clasper surgery, $P_\ell(t)/\Delta(t)^{2\ell+1}$ changes by

$$\frac{2^{\ell+2}(t^{1/2} - t^{-1/2})}{\Delta(t)} e_x^T \left(t^{1/2}V - t^{-1/2}V^T\right)^{-1} e_y\,,$$

keeping $P_k(t)$ unchanged for any $k < \ell$. Therefore, from the definition of the perturbative invariants, we obtain the proposition. $\square$

We can obtain similar surgery formulas for other forms of the graph clasper, in the same way as in [O8].

3.2 Stability of the perturbative invariants for finite cyclic covers

There is a natural d-fold cyclic cover M_d of a 3-manifold M with $b_1(M) = 1$ (defined below). By regarding the perturbative invariant of M_d as an invariant of M (assuming that $b_1(M_d) = 1$), we have an infinite series of invariants of M. In this section, we show that this series is stable for sufficiently large d in the cases of Propositions 3.5 and 3.6.

For a positive integer d, let M_d be the d-fold cyclic cover of M induced by the homomorphism

$$\pi_1(M) \longrightarrow H_1(M; \mathbb{Z}) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/d\mathbb{Z},$$

where the first map is the abelianization, the second map is the map ingoring the torsion part, and the third map is the projection. We assume that $b_1(M_d) = 1$, i.e., the Alexander polynomial $\Delta(t)$ of M does not have a zero of a dth root of unity. We choose a rational homology 3-sphere N and a null-homologous knot K in N such that M is obtained from N by 0 surgery along K. Let N_d be the d-fold cyclic cover of N branched along N induced by the homomorphism $\pi_1(N - K) \to H_1(N; \mathbb{Z}) \oplus \mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$, where the second map is the projection from $\mathbb{Z}$. Let K_d be the knot of the branch set in N_d. Then, M_d is obtained from N_d by 0 surgery along K_d. Further, the Alexander polynomial of K_d is presented by

$$\Delta_{K_d}(t) = \pm \prod_{\zeta^d = 1} \Delta(\zeta \, t^{1/d}),$$

where we choose the sign so that $\Delta_{K_d}(1) > 0$. Note that, for a zero c of $\Delta(t)$, c^d is a zero of $\Delta_{K_d}(t)$.

We define the total signature of a 3-manifold M with $b_1(M) = 1$, as follows. As in Section 5, M is obtained from S^3 by surgery along some framed link $K_0 \cup L$ such that K_0 is the trivial knot and each component of L is null-homologous in $S^3 - K_0$. Let $\tilde{L}$ be the preimage of L in the d-fold cyclic cover of S^3 branched along K_0, and let A and A_d be the linking matrices of L and $\tilde{L}$ respectively. We define the *total d-signature* σ_d of M by

$$\sigma_d(M) = \sigma(A_d) - d \, \sigma(A),$$

where $\sigma(\cdot)$ denotes the signature of a symmetric matrix. We can verify, by Proposition 5.1, that this is an invariant of M.

Proposition 3.5. *When $\Delta(t)$ is of degree 0, for any positive integer ℓ, the degree $\leq \ell$ part of $\left(q^{-(3/4)\sigma_d(M)} \tau(M_d; 0)\right)^{1/d}$ is stable (i.e., constant) for sufficiently large d.*

Hence, the stable part is presented by

$$\tilde{\tau}(M; 0) = \lim_{d \to \infty} \left(q^{-(3/4)\sigma_d(M)} \tau(M_d; 0)\right)^{1/d} \in \mathbb{Q}[[q - 1]].$$

Proof of Proposition 3.5. When $\Delta(t)$ is of degree 0, the loop expansion (see Section 4.1) of $Z^{\mathrm{LMO}}(N, K)$ is presented by

$$\chi^{-1} Z^{\mathrm{LMO}}(N, K) = Z^{\mathrm{LMO}}(N) \sqcup \Omega \sqcup \exp_{\sqcup} \left(\begin{array}{c} \text{linear sum of connected Jacobi diagrams} \\ \text{with labels of polynomials in } t^{\pm 1} \end{array} \right).$$

Its d-fold cover is calculated in the way shown in [GK3]; in particular, in this case,

$$\chi^{-1} Z^{\mathrm{LMO}}(N_d, K_d) \ \sim \ e^{\sigma_d(M)\theta/16} \sqcup Z^{\mathrm{LMO}}(N)^d \sqcup \Omega \sqcup \exp_{\sqcup}\Big(d \cdot \big(\ {}^{\text{constant terms of}}_{\text{the above linear sum}}\ \big)\Big),$$

where θ is the Jacobi diagram of the θ graph, and this formula means that, for any positive integer ℓ, the $(\le \ell)$-loop parts of both sides are equal for sufficiently large d. Hence, the $\mathfrak{sl}_2$ reduction of $\chi^{-1} Z^{\mathrm{LMO}}(N_d, K_d) \sqcup \Omega^{-1}$ is stably presented by

$$q^{(3/4)\sigma_d(M)} \Big(W_{\mathfrak{sl}_2}\big(Z^{\mathrm{LMO}}(N)\big) \cdot \exp\big(W_{\mathfrak{sl}_2}(\text{the constant terms})\big)\Big)^d, \tag{11}$$

and

$$\tau(M_d; 0) \ \sim \ \text{``(11)''} \cdot \Big(-\frac{1}{2}\Big) \operatorname*{Res}_{t=0} (1 - t^{-1})^2 \ = \ \text{``(11)''}.$$

Therefore, the degree $\le \ell$ part of $\big(q^{-(3/4)\sigma_d(M)} \tau(M_d; 0)\big)^{1/d}$ is stable for sufficiently large d. $\qquad\square$

Proposition 3.6. *Suppose that $\Delta(t)$ does not have a multiple zero, and let $c_1^{\pm 1}, \cdots, c_n^{\pm 1}$ be the zeros of $\Delta(t)$. Let c be 0 or a zero of $\Delta(t)$. Let d be a positive integer satisfying that $\Delta(t)$ does not have a zero of a dth root of unity (i.e., $b_1(M_d) = 1$). Then,*
(1) $\lambda_0(M_d; c^d)$ is presented by a rational function of $c_1^d, \cdots, c_n^d$.
(2) $\big(\lambda_1(M_d; c^d) - \frac{3}{4}\sigma_d(M)\lambda_0(M_d; c^d)\big)/d$ is stable, in the sense that it can be presented by a rational function of $c_1^d, \cdots, c_n^d$ for sufficiently large d satisfying the above condition.

Hence, the stable part can be presented by

$$\tilde{\lambda}_1(M; c) = \Big(\big(\lambda_1(M_d; c^d) - \frac{3}{4}\sigma_d(M)\lambda_0(M_d; c^d)\big)/d \quad \text{for such } d \gg 0 \Big),$$

where this is a rational function of $c_1^d, \cdots, c_n^d$.

Proof of Proposition 3.6. We show (1), as follows. We put $\Delta(t) = b \prod_i \big(1 - a_i(t + t^{-1} - 2)\big)$, where $a_i^{-1} = c_i + c_i^{-1} - 2$. Then, $\Delta_{K_d}(t)/\Delta_{K_d}(1) = \prod_i \big(1 - \tilde{a}_i(t + t^{-1} - 2)\big)$, where $\tilde{a}_i^{-1} = c_i^d + c_i^{-d} - 2$. Hence, by Proposition 3.3, we obtain (1).

We show (2), in the following of this proof. Similarly as the proof of Proposition 3.5,

$$\chi^{-1} Z^{\mathrm{LMO}}(N_d, K_d) \ \sim \ e^{\sigma_d(M)\theta/16} \sqcup Z^{\mathrm{LMO}}(N)^d \sqcup \Omega \sqcup \exp_{\sqcup}\Big(\overbrace{}^{-\frac{1}{2}\log\big(\Delta_{K_d}(t)/\Delta_{K_d}(1)\big)} \Big)$$
$$\sqcup \big(1 + (\text{2-loop terms}) + \cdots\big)$$
$$= \ e^{\sigma_d(M)\theta/16} \sqcup \Omega \sqcup \exp_{\sqcup}\Big(\overbrace{}^{-\frac{1}{2}\log\big(\Delta_{K_d}(t)/\Delta_{K_d}(1)\big)} \Big) \sqcup \big(1 + (\text{2-loop part}) + \cdots\big),$$

where the 2-loop part is determined by the 2-loop polynomial $\Theta_{K_d}(t_1, t_2)$ (see [GK3, O8]),

$$\frac{\Theta_{K_d}(t_1, t_2)}{\Delta_{K_d}(t_1)\Delta_{K_d}(t_2)\Delta_{K_d}(t_1 t_2)} = \frac{1}{d} \sum_{\zeta_1^d = \zeta_2^d = 1} \frac{\Theta(\zeta_1 t_1^{1/d}, \zeta_2 t_2^{1/d})}{\Delta(\zeta_1 t_1^{1/d})\Delta(\zeta_2 t_2^{1/d})\Delta(\zeta_1\zeta_2 t_1^{1/d} t_2^{1/d})}, \tag{12}$$

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where $\Theta(t_1, t_2)$ denotes the 2-loop polynomial of K. Hence, the $\mathfrak{sl}_2$ reduction of $\chi^{-1} Z^{\mathrm{LMO}}(N_d, K_d) \sqcup \Omega^{-1}$ is stably presented by

$$q^{(3/4)\sigma_d(M)} \Big(\frac{\Delta_{K_d}(1)}{\Delta_{K_d}(t)} + \frac{\Theta_{K_d}(t, 1)}{\Delta_{K_d}(t)^3}(q - 1) + \cdots \Big).$$

Further,

$$\lambda_0(M_d; c^d) = -\frac{1}{2} \cdot \frac{1 + c^d}{1 - c^d} \operatorname*{Res}_{t=c} \frac{(1 - t^{-1})^2 \Delta_{K_d}(1)}{\Delta_{K_d}(t)},$$

$$\lambda_1(M_d; c^d) = \frac{3}{4} \sigma_d(M) \lambda_0(M_d; c^d) - \frac{1}{2} \cdot \frac{1 + c^d}{1 - c^d} \operatorname*{Res}_{t=c} \frac{(1 - t^{-1})^2 \Theta_{K_d}(t, 1)}{\Delta_{K_d}(t)^3}.$$

Therefore, it is sufficient to show that $\Theta_{K_d}(t, 1)/d$ is presented by a rational function of $c_1^d, \cdots, c_n^d$ for $d \gg 0$.

We show it, in the following of this proof. For example, when $\Delta(t) = 1 - a(t + t^{-1} - 2)$,

$$\Delta_{K_d}(t) = a^d(c^d + c^{-d} - t - t^{-1})$$

for a zero c of $\Delta(t)$. Further, since

$$\frac{\Delta_{K_d}(s^d)}{\Delta(s)} = a^{d-1} s^{-d+1} \frac{(s^d - c^d)(s^d - c^{-d})}{(s - c)(s - c^{-1})} = \frac{a^{d-1} s^{-d+1}}{c - c^{-1}} \Big(\frac{s^d - c^d}{s - c}(s^d - c^{-d}) - \frac{s^d - c^{-d}}{s - c^{-1}}(s^d - c^d) \Big),$$

we can calculate $\Theta_{K_d}(t_1, t_2)$ by (12) putting $s = \zeta_i t_i^{1/d}$. For example, when $\Theta(t_1, t_2) = 1$, the summand of (12) is presented by

$$\frac{d \, a^{3d-3}}{(c - c^{-1})^3} \prod_{i=1}^{3} \Big(\frac{s_i^d - c^d}{s_i - c}(s_i^d - c^{-d}) - \frac{s_i^d - c^{-d}}{s_i - c^{-1}}(s_i^d - c^d) \Big),$$

where we put $s_i = \zeta_i t_1^{1/d}$, and $s_1 s_2 s_3 = 1$. Since the sum of (12) picks up the terms of the form $s_1^{n_1 d} s_2^{n_2 d}$ for $n_1, n_2 \in \mathbb{Z}$, we pick up the following term, for example, from a part of the above product,

$$\frac{d \, a^{3d-3}}{(c - c^{-1})^3} \cdot \frac{s_1^d - c^d}{s_1 - c} \frac{s_2^d - c^d}{s_2 - c} \frac{s_3^d - c^d}{s_3 - c} \cdot (s_1^d - c^{-d})(s_2^d - c^{-d})(s_3^d - c^{-d})$$

$$\rightsquigarrow \quad \frac{d \, a^{3d-3}}{(c - c^{-1})^3} \cdot \frac{c^{3d} - 1}{c^3 - 1} \cdot (t_1 - c^{-d})(t_2 - c^{-d})(t_3 - c^{-d}).$$

Hence, in this way, we can show that

$$\frac{\Theta_{K_d}(t_1, t_2)}{d} = \frac{a^{3d-3}(c+1)(c^d-1)^2}{(c^2-1)^3(c^3-1)c^{d-4}} \Big(\frac{(c^{d-1}-1)(c^{d+1}-1)}{c^d}(t_1 t_2 + t_1 + t_2 + t_1^{-1} + t_2^{-1} + t_1^{-1} t_2^{-1})$$

$$+ \frac{(c^d - 1)^2}{c^d} \big((c^d + c^{-d})(c + c^{-1} - 1) + 4c + 4c^{-1} + 2 \big) \Big).$$

For general $\Theta(t_1, t_2)$ and $\Delta(t)$, we can similarly show that $\Theta_{K_d}(t, 1)/d$ is presented by a rational function of $c_1^d, \cdots, c_n^d$ for $d \gg 0$. $\square$

4 The loop expansion

In this section, we review the loop expansions of the Kontsevich invariant and the colored Jones polynomials in Section 4.1 and 4.2 respectively. The latter can be obtained from the former by $\mathfrak{sl}_2$ reduction; we explain its concrete procedure in Section 4.2.

4.1 The loop expansion of the Kontsevich invariant

In this subsection, we review the loop expansion of the Kontsevich invariant and how we obtain it by the rational version of the Aarhus integral.

The Kontsevich invariant is defined in the space of Jacobi diagrams on S^1, which we define as follows. For a 1-manifold X, a *Jacobi diagram* on X is the manifold X together with a uni-trivalent graph such that univalent vertices of the graph are distinct points on X and each trivalent vertex is vertex-oriented, where a *vertex-oriented* trivalent vertex is a trivalent vertex such that a cyclic order of the three edges around the trivalent vertex is fixed. In figures we draw X by thick lines and the uni-trivalent graphs by thin lines, in such a way that each trivalent vertex is vertex-oriented in the counterclockwise order. We define the *degree* of a Jacobi diagram to be half the number of univalent and trivalent vertices of the uni-trivalent graph of the Jacobi diagram. We denote by $\mathcal{A}(X)$ the quotient vector space spanned by Jacobi diagrams on X subject to the following relations, called *the AS, IHX, and STU relations* respectively,

The *Kontsevich invariant $Z(K)$* [Ko] of a knot K is defined to be in $\mathcal{A}(S^1)$; for details of its constructions, see, *e.g.*, [O3].

The loop expansion of the Kontsevich invariant is defined in the space of open Jacobi diagrams. An *open Jacobi diagram* is a vertex-oriented uni-trivalent graph. We denote by $\mathcal{A}(*)$ the quotient vector space spanned by open Jacobi diagrams subject to the AS and IHX relations. The *Poincare-Birkhoff-Witt isomorphism $\chi : \mathcal{A}(*) \to \mathcal{A}(\downarrow)$* is defined by

$$\raisebox{-1em}{\includegraphics{eq13}} \quad (13)$$

for any diagram D, where the box denotes the symmetrizer,

$$\raisebox{-1em}{\includegraphics{eq14}} = \frac{1}{n!}\Big(\quad + \quad + \quad + \cdots \Big). \quad (14)$$

A label of a power series $f(\hbar) = c_0 + c_1\hbar + c_2\hbar^2 + c_3\hbar^3 + \cdots$ implies that

$$\left.\Big)\right|^{f(\hbar)} = c_0 \;\Big) \;+\; c_1 \;\Big|\!\!-\!\!\hbar \;+\; c_2 \;\Big|\!\!\genfrac{}{}{0pt}{}{\hbar}{\hbar} \;+\; c_3 \;\Big|\!\!\genfrac{}{}{0pt}{}{\hbar}{\genfrac{}{}{0pt}{}{\hbar}{\hbar}} \;+\; \cdots . \qquad (15)$$

Note that $\left.\right|^{f(\hbar)} = \left.\right|^{f(-\hbar)}$ by the AS relation, in the notation of this paper. Any open Jacobi diagram can be presented by a trivalent graph with labels on its edges. It is known [Ro6, Kr1, GK2] that the Kontsevich invariant of a knot K has a presentation, called the *loop expansion*,

$$\chi^{-1}Z(K) = \Omega \;\sqcup\; \exp_\sqcup\Big(\underset{-\frac{1}{2}\log\Delta(t)}{\bigodot} + \overset{\text{finite}}{\underset{i}{\sum}} \underset{\substack{p_{i,1}(t)/\Delta(t)\\ p_{i,2}(t)/\Delta(t)\\ p_{i,3}(t)/\Delta(t)}}{\theta}$$
$$+ \big(\text{terms of } (\geq 3)\text{-loop presented in the same way}\big)\Big),$$

where we put $t = e^\hbar$, and $\exp_\sqcup$ denotes the exponential with respect to the disjoint-union product of open Jacobi diagrams, $\Delta(t)$ denotes the Alexander polynomial of K, and $p_{i,j}(t)$ is a polynomial in $t^{\pm 1}$, and we put

$$\Omega = \chi^{-1}Z\big(\text{the trivial knot}\big) = \underset{\frac{1}{2}\log\frac{\sinh(\hbar/2)}{\hbar/2}}{\bigodot} .$$

Similarly, it can be shown that the LMO invariant of a pair of a rational homology 3-sphere N and a null-homologous knot K in N is also presented by the loop expansion,

$$\chi^{-1}Z^{\mathrm{LMO}}(N,K) \;=\; Z^{\mathrm{LMO}}(N) \;\sqcup\; \Omega \;\sqcup\; \exp_\sqcup\Big(\underset{-\frac{1}{2}\log\big(\Delta(t)/\Delta(1)\big)}{\bigodot} + \overset{\text{finite}}{\underset{i}{\sum}} \underset{\substack{p_{i,1}(t)/\Delta(t)\\ p_{i,2}(t)/\Delta(t)\\ p_{i,3}(t)/\Delta(t)}}{\theta}$$
$$+ \big(\text{terms of } (\geq 3)\text{-loop presented in the same way}\big)\Big).$$

We briefly review how we obtain the loop expansion; for detailed and precise arguments, see [Kr1, GK2]. The Kontsevich invariant of the long Hopf link is presented by

$$\chi^{-1}Z\Big(\text{⌇} \Big) = \Big\downarrow t \Big\downarrow t \;\cdots\; \Big\uparrow t^{-1} \Big\uparrow t^{-1} \;\sqcup\; \Omega, \qquad (16)$$

where we put $t = e^\hbar$. A pair of a rational homology 3-sphere N and a null-homologous knot K in N can be obtained from (S^3, K_0) by surgery along K for some framed link $K_0 \cup L$ in S^3 such that K_0 is the trivial knot and each component of L is null-homologous in the complement of K_0. Let A be the equivariant linking matrix [KY, GK1] of a lift of

L in the infinite cyclic cover of $S^3 - K_0$; its entries are in $\mathbb{Z}[t^{\pm 1}]$. Then, the Kontsevich invariant of $K_0 \cup L$ can be presented by the form,

$$\chi^{-1} Z(K_0 \cup L) \;=\; \Omega \sqcup \exp\Big(\frac{1}{2} \sum_{i,j} \overset{A_{ij}}{\underset{y_i \quad y_j}{\bigwedge}} \Big) \sqcup R,$$

where R is a linear sum of open Jacobi diagrams with labels of polynomials in $t^{\pm 1}$ and with at least 3 univalent vertices; this presentation follows from (16). The rational version of the Aarhus integral takes it to

$$\chi^{-1} Z^{\mathrm{LMO}}(N, K) \;=\; c_+^{-\sigma_+} c_-^{-\sigma_-} \int^{\mathrm{rat}} \chi^{-1} Z(K_0 \cup L) dY$$

$$= \; c_+^{-\sigma_+} c_-^{-\sigma_-} \sqcup \Omega \sqcup \exp_{\sqcup}\Big(\overset{-\frac{1}{2}\log\left(\Delta(t)/\Delta(1)\right)}{\boxed{}} \Big) \sqcup \Big\langle \exp\Big(-\frac{1}{2} \sum_{i,j} \overset{\partial y_j \quad \partial y_i}{\underset{(A^{-1})_{ij}}{\bigcup}} \Big), R \Big\rangle,$$

where $c_\pm \in \mathcal{A}(\emptyset)$ are some normalization constants, $\sigma_\pm$ are the number of positive and negative eigenvalues of the linking matrix of L, and the bracket is defined by

$$\langle D_1, D_2 \rangle = \begin{cases} \raisebox{-1.2em}{\includegraphics{box}} & \text{if the number of } \partial y_i\text{-legs of } D_1 \text{ equals} \\ & \quad \text{the number of } y_i\text{-legs of } D_2 \text{ for each } i, \\ 0 & \text{otherwise,} \end{cases}$$

for open Jacobi diagrams D_1 with ∂y_i-legs and D_2 with y_i-legs. Here, a shaded box means the sum of all ways of connecting the left lines to the right lines, which is equal to $n!$ times (14).

4.2 The loop expansion of the colored Jones polynomial

In this subsection, we explain the loop expansion of the colored Jones polynomial (Proposition 4.1 and Remark 4.2) is obtained from the loop expansion of the Kontsevich invariant by $\mathfrak{sl}_2$ reduction.

The $\mathfrak{sl}_2$ reduction is a procedure reducing an open Jacobi diagram with labels of rational functions to a rational function, in the following way. We resolve trivalent vertices by

$$\raisebox{-1em}{\includegraphics{Y}} \underset{(\mathfrak{sl}_2)}{\equiv} 2h\Big(\;\big)\big(\; \Big)\Big(- \raisebox{-0.5em}{\includegraphics{X}} \Big), \tag{17}$$

where h is a variable satisfying that $q = e^h$, noting that the $\mathfrak{sl}_2$ weight system of both sides are equal; see [CV, O3]. By using this formula recursively, we have that

$$\overset{\hbar^k}{\boxed{}} = \overset{k \text{ legs}}{\boxed{}} \underset{(\mathfrak{sl}_2)}{\equiv} \begin{cases} \big(\sqrt{2C} \cdot h \big)^k + \big(-\sqrt{2C} \cdot h \big)^k & \text{for } k > 0, \\ 3 & \text{for } k = 0, \end{cases}$$

where C denotes the diagram consisting of a single arc. Hence, for a function $\varphi(t)$ which has an expansion $\varphi(e^\hbar) \in \mathbb{Q}[[\hbar]]$,
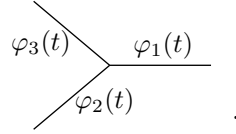
$$\overset{\varphi(t)}{\boxed{}} \underset{(\mathfrak{sl}_2)}{\equiv} \varphi(\hat{t}) + \varphi(\hat{t}^{-1}) + \varphi(1),$$

where we put $t = e^{\hbar}$ and $\hat{t} = e^{\sqrt{2C} \cdot h}$. In a similar way, we can also show (see [O4]) that

$$
\underset{(\mathfrak{sl}_2)}{\equiv} 2h \sum_{\{i,j,k\}=\{1,2,3\}} \varphi_i(\hat{t})\, \varphi_j(\hat{t}^{-1})\, \varphi_k(1), \tag{18}
$$

$$
\underset{(\mathfrak{sl}_2)}{\equiv} 2h\big(\varphi(\hat{t}) - \varphi(\hat{t}^{-1})\big)\big(\psi(\hat{t}) - \psi(\hat{t}^{-1})\big), \tag{19}
$$

for rational functions $\varphi_i(t)$, $\varphi(t)$ and $\psi(t)$. In general, we can decrease the number of trivalent vertices, as follows. Consider a trivalent vertex,

By putting $\hat{\varphi}_i(t) = \varphi_i(t) - \varphi_i(1)$, the above diagram is equal to the sum of diagrams obtained from it by replacing each $\varphi_i(t)$ with $\hat{\varphi}_i(t)$ or $\varphi_i(1)$. If we choose $\hat{\varphi}_i(t)$ for all i, the diagram vanishes, since

$$
\underset{(\mathfrak{sl}_2)}{\equiv} 0
$$

by (17). Hence, at least one of $\varphi_i(t)$'s is replaced with $\varphi_i(1)$. By applying (17) to the edge corresponding to $\varphi_i(1)$, we can decrease the number of trivalent vertices. In this way, we can reduce a Jacobi diagram with labels of rational functions of t to a rational function of $\hat{t}$ by $\mathfrak{sl}_2$ reduction. Further,

$$
\exp_{\sqcup}\left(\overset{-\frac{1}{2}\log\left(\Delta(t)/\Delta(1)\right)}{\bigcirc} \right) \underset{(\mathfrak{sl}_2)}{\equiv} \exp\left(-\log\left(\Delta(\hat{t})/\Delta(1)\right)\right) = \frac{\Delta(1)}{\Delta(\hat{t})}.
$$

Therefore, since the $\mathfrak{sl}_2$ reduction of $Z^{\mathrm{LMO}}(N) \in \mathcal{A}(\emptyset)$ is given by a power series in $\mathbb{Q}[[h]] \cong \mathbb{Q}[[q-1]]$, the $\mathfrak{sl}_2$ reduction of the loop expansion of $\chi^{-1} Z^{\mathrm{LMO}}(N, K)$ is presented by

$$
\chi^{-1} Z^{\mathrm{LMO}}(N, K) \sqcup \Omega^{-1} \underset{(\mathfrak{sl}_2)}{\equiv} \sum_{\ell=0}^{\infty} \frac{\hat{P}_\ell(\hat{t})}{\Delta(\hat{t})^{2\ell+1}} (q-1)^{\ell}
$$

for some polynomials $\hat{P}_\ell(\hat{t}) \in \mathbb{Q}[\hat{t}^{\pm 1}]$.

Proposition 4.1. *Let K be a null-homologous knot in a rational homology 3-sphere N. Then, $W_{\mathfrak{sl}_2, V_n}\big(Z^{\mathrm{LMO}}(N, K)\big)$ is presented by the form,*

$$
W_{\mathfrak{sl}_2, V_n}\big(Z^{\mathrm{LMO}}(N, K)\big) = [n] \sum_{\ell=0}^{\infty} \frac{P_\ell(q^n)}{\Delta(q^n)^{2\ell+1}} (q-1)^{\ell}, \tag{20}
$$

for some polynomials $P_\ell(t) \in \mathbb{Q}[t^{\pm 1}]$, where $\Delta(t)$ is the Alexander polynomial of K; in particular, $P_0(t) \big(= \Delta(1)\big)$ is equal to the order of $H_1(N; \mathbb{Z})$.

Proof. Since $\chi(\Omega)$ is the Kontsevich invariant of the trivial knot, $W_{\mathfrak{sl}_2, V_n}\big(\chi(\Omega)\big) = [n]$. Hence,

$$\Omega \underset{(\mathfrak{sl}_2)}{\equiv} \frac{e^{\sqrt{2C}\cdot h/2} - e^{-\sqrt{2C}\cdot h/2}}{\sqrt{2C}\cdot h} \overset{W_{\mathfrak{sl}_2, V_n} \circ \chi}{\longmapsto} [n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}.$$

By replacing h with kh,

$$\frac{1}{k} \cdot \frac{\hat{t}^{k/2} - \hat{t}^{-k/2}}{\hat{t}^{1/2} - \hat{t}^{-1/2}} \cdot \Omega \longmapsto [n] \cdot \frac{t^{k/2} - t^{-k/2}}{t^{1/2} - t^{-1/2}} \cdot \frac{1}{[k]},$$

where we put $\hat{t} = e^{\sqrt{2C}\cdot h}$ and $t = q^n$. Therefore,

$$\varphi(\hat{t})\,\Omega \longmapsto [n] \cdot \frac{1}{t^{1/2} - t^{-1/2}} \cdot \frac{D_\alpha}{[D_\alpha]}\Big((t^{1/2} - t^{-1/2})\varphi(t)\big|_{t^{1/2} \to e^\alpha}\Big)$$

$$= [n] \cdot \frac{1}{t^{1/2} - t^{-1/2}} \cdot \frac{D}{[D]}\big((t^{1/2} - t^{-1/2})\varphi(t)\big),$$

where we put $D_\alpha = \dfrac{d}{d\alpha}$ and $D = 2t\dfrac{d}{dt}$, regarding $\dfrac{D}{[D]} = \dfrac{e^{h/2} - e^{-h/2}}{e^{hD/2} - e^{-hD/2}}D$ as a power series in $(hD)^2$ and h. Hence, we can determine $P_\ell(t)$ by

$$\sum_\ell \frac{P_\ell(t)}{\Delta(t)^{2\ell+1}}(q-1)^\ell = \frac{1}{t^{1/2} - t^{-1/2}} \cdot \frac{D}{[D]}\Big((t^{1/2} - t^{-1/2})\sum_\ell \frac{\hat{P}_\ell(t)}{\Delta(t)^{2\ell+1}}(q-1)^\ell\Big), \quad (21)$$

where we can show that $P_\ell(t) \in \mathbb{Q}[t^{\pm 1}]$, since

$$\frac{1}{t^{1/2} - t^{-1/2}} \cdot (hD)^{2m}\Big((t^{1/2} - t^{-1/2})\frac{\hat{P}_\ell(t)}{\Delta(t)^{2\ell+1}}(q-1)^\ell\Big)$$

is presented by a linear sum of the form $\frac{f_k(t)}{\Delta(t)^{2k+1}}(q-1)^k$ for some polynomials $f_k(t) \in \mathbb{Q}[t^{\pm 1}]$. $\square$

Remark 4.2. In particular, when $N = S^3$,

$$W_{\mathfrak{sl}_2, V_n}\big(Z^{\mathrm{LMO}}(S^3, K)\big) = W_{\mathfrak{sl}_2, V_n}\big(Z(K)\big) = [n]\, J_n(K; q).$$

Hence, in this case, Proposition 4.1 implies the loop expansion of the colored Jones polynomial,

$$J_n(K; q) = \sum_{\ell=0}^\infty \frac{P_\ell(q^n)}{\Delta(q^n)^{2\ell+1}}(q-1)^\ell.$$

This formula was shown by Rozansky; he further shows that $P_\ell(t) \in \mathbb{Z}[t^{\pm 1}]$ in [Ro4].

Remark 4.3. In the defining relation (2) of the perturbative invariant, we can replace $P_\ell(t)$ with $\hat{P}_\ell(t)$, since

$$\operatorname*{Res}_{t=c} \frac{(1 - t^{-1})^2\, P_\ell(t)}{\Delta(t)^{2\ell+1}} = \operatorname*{Res}_{t=c} \frac{(1 - t^{-1})^2\, \hat{P}_\ell(t)}{\Delta(t)^{2\ell+1}}, \quad (22)$$

where this equality is obtained as follows. By (4),

$$\operatorname*{Res}_{t=c} \frac{t^{1/2} \mp t^{-1/2}}{t} D \, \varphi(t) = - \operatorname*{Res}_{t=c} \frac{t^{1/2} \pm t^{-1/2}}{t} \varphi(t)$$

for $D = 2 \, t \, \dfrac{d}{dt}$. Hence,

$$\operatorname*{Res}_{t=c} \frac{t^{1/2} - t^{-1/2}}{t} D^2 \, \varphi(t) = \operatorname*{Res}_{t=c} \frac{t^{1/2} - t^{-1/2}}{t} \varphi(t).$$

Therefore, when we consider the residue of (21) of the form (22), we can replace D^2 with 1, and, hence, we obtain (22).

5   Proof of Theorem 1.1

In this section, we prove Theorem 1.1, which states the topological invariance of the perturbative invariants of 3-manifolds with the first Betti number 1. We show "Kirby theorem" for surgery presentations of such 3-manifolds in Proposition 5.1, and prove Theorem 1.1 by introducing the t-through relation. To prove the theorem, we also show some properties of the t-through relation.

Let M be a closed 3-manifold with the first Betti number 1. We choose a framed link L' in S^3 such that M is obtained from S^3 by surgery along L'. Then, $H^1(M;\mathbb{Z})$ is presented by the kernel of the homomorphism $\mathbb{Z}^n \to \mathbb{Z}^n$ given by the multiplication of the linking matrix A of L', and, hence, a cohomology class in $H^1(M;\mathbb{Z})$ is presented by $\mathbf{a} \in \mathbb{Z}^n$ such that $A\mathbf{a} = 0$. We can assume, without loss of generality, that all entries in the first row and the first column of A are equal to 0 and $\mathbf{a} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^T$, by changing L' by handle slide moves if necessary. This implies that the first component of L' has 0 framing and the other components are null-homologous in the complement of the first component. Further, we can assume that the first component of L' is the trivial knot K_0, by replacing the first component with its surgery presentation. It follows, by putting $L' = K_0 \cup L$, that, for a 3-manifold M with the first Betti number 1, we can choose a framed link $K_0 \cup L$ in S^3 as a surgery presentation of M such that K_0 is the trivial knot with 0 framing and each component of L is null-homologous in the complement of K_0.
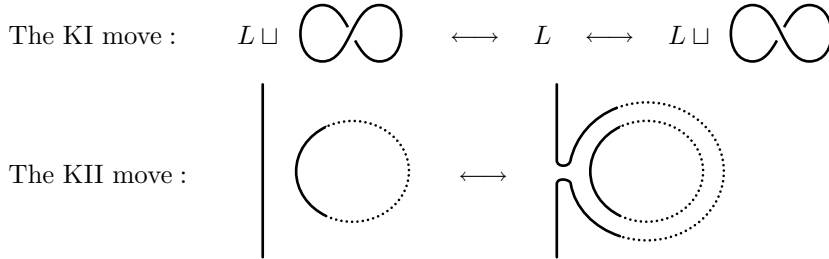

Figure 1: The KI and KII moves

Proposition 5.1. *Let K_0 be the trivial knot with 0 framing in S^3, and let L and L' be framed links in $S^3 - K_0$ such that each component of them is null-homologous in $S^3 - K_0$, and let $S^3_{K_0 \cup L}$ and $S^3_{K_0 \cup L'}$ denote the 3-manifolds obtained from S^3 by surgery along $K_0 \cup L$ and $K_0 \cup L'$ respectively. Then, $S^3_{K_0 \cup L}$ and $S^3_{K_0 \cup L'}$ are homeomorphic if and only if L and L' are related by a sequence of the KI and KII moves on the link and the KII moves over K_0, where the KI and KII moves are as shown in Figure 1.*

Proof. We denote by L_{K_0} and L'_{K_0} the framed links in $S^2 \times S^1$ obtained from L and L' by surgery along K_0; note that they are null-homotopic in $S^2 \times S^1$. Let $W_{L_{K_0}}$ and $W_{L'_{K_0}}$ be the 4-manifolds obtained from $S^2 \times S^1 \times I$ by attaching 2-handles along L_{K_0} and L'_{K_0} respectively. Then, $\partial W_{L_{K_0}} \cong \partial W_{L'_{K_0}} \cong (S^2 \times S^1) \sqcup \overline{M}$. Further, the diagram

$$
\begin{array}{ccc}
\pi_1(M) & \xrightarrow{\;\;=\;\;} & \pi_1(M) \\
\chi \downarrow & & \downarrow \chi \\
\pi_1(W_{L_{K_0}}) \;\cong\; \mathbb{Z} \;\cong\; & & \pi_1(W_{L'_{K_0}})
\end{array}
$$

commutes, and $H_4(\mathbb{Z}) \cong H_4(S^1) = 0$. Hence, by [FR, Theorem 6], L_{K_0} and L'_{K_0} are related by the KI and KII moves. Therefore, L and L' are related by the KI and KII moves on the link and the KII moves over K_0. $\square$

To show the invariance under the KII move over K_0, we introduce the *t-through relation* among Jacobi diagrams with labels of polynomials in t. We define the *t-through relation* to be the relation generated by

$$
\begin{array}{c}
\overset{*}{\raisebox{0pt}{$\bigcirc$}} \\
D
\end{array}
\;\sim\; 0,
$$

where D is a Jacobi diagram with labels of polynomials in t, and we define a diagram with the marking to be the sum of diagrams obtained from the original diagram by connecting the marking at the right side of each t; for example,

This operation is well-defined, since, for the equality,

we have that

Lemma 5.2. *Let K_0 be the trivial knot with 0 framing in S^3, and let L and L' be framed links in $S^3 - K_0$ such that each component of them is null-homologous in $S^3 - K_0$. If L and L' are related by the KII move over K_0, then $\chi^{-1} Z(K_0 \cup L) \sqcup \Omega^{-1}$ and $\chi^{-1} Z(K_0 \cup L') \sqcup \Omega^{-1}$ are related by the t-through relation.*

Proof. From the definition of the Kontsevich invariant (see, *e.g.*, [O3]), we have that

$$Z\Big(\begin{array}{c}\includegraphics\end{array} \Big) - Z\Big(\begin{array}{c}\includegraphics\end{array} \Big) = - \begin{array}{c}\includegraphics\end{array} + \frac{1}{2} \begin{array}{c}\includegraphics\end{array} - \cdots ,$$

where

$$\begin{array}{c}\includegraphics\end{array} = \begin{array}{c}\includegraphics\end{array} + \begin{array}{c}\includegraphics\end{array} + \cdots + \begin{array}{c}\includegraphics\end{array} .$$

Further, from (16) and the definition of the t-through relation, we have that

$$\chi^{-1} Z\Big(\begin{array}{c}\includegraphics\end{array} \Big) \sqcup \Omega^{-1} \;\sim\; \chi^{-1} Z\Big(\begin{array}{c}\includegraphics\end{array} \Big) \sqcup \Omega^{-1}.$$

This implies the lemma. $\qquad\square$

Proof of Theorem 1.1. We recall the construction of the perturbative invariant $\tau(M; c)$, where M is a 3-manifold with the first Betti number 1 and c is a zero of the Alexander polynomial $\Delta(t)$ of M. As mentioned before, we choose a framed link $K_0 \cup L$ in S^3 as a surgery presentation of M such that K_0 is the trivial knot with 0 framing and each component of L is null-homologous in the complement of K_0. Let N be the 3-manifold obtained from S^3 by surgery along L. By Proposition 4.1, the loop expansion of $W_{\mathfrak{sl}_2, V_n}\big(Z^{\mathrm{LMO}}(N, K)\big)$ is presented by (20) for some polynomials $P_\ell(t) \in \mathbb{Q}[t^{\pm 1}]$. By using these $P_\ell(t)$, we defined the perturbative invariant by (3). Further, by Remark 4.3, we can replace $P_\ell(t)$ with $\hat{P}_\ell(t)$, *i.e.*,

$$\tau(M; c) = -\frac{1}{2} \cdot \frac{1+c}{1-c} \sum_{\ell=0}^{\infty} \Big(\operatorname*{Res}_{t=c} \frac{(1 - t^{-1})^2 \, \hat{P}_\ell(t)}{\Delta(t)^{2\ell+1}} \Big) (q - 1)^\ell,$$

where $\hat{P}_\ell(t) \in \mathbb{Q}[t^{\pm 1}]$ are determined by

$$\sum_{\ell=0}^{\infty} \frac{\hat{P}_\ell(\hat{t})}{\Delta(\hat{t})^{2\ell+1}} (q-1)^\ell \underset{(\mathfrak{sl}_2)}{\equiv} \chi^{-1} Z^{\mathrm{LMO}}(N, K) \sqcup \Omega^{-1}$$

$$= c_+^{-\sigma_+} c_-^{-\sigma_-} \int^{\mathrm{rat}} \chi^{-1} Z(K_0 \cup L) dY \;\sqcup\; \Omega^{-1}. \tag{23}$$

By Proposition 5.1, it is sufficient to show the invariance of $\tau(M; c)$ under the orientation change of K_0 and under the KI and KII moves on L and under the KII moves over K_0. By the orientation change of K_0, labels of $\chi^{-1} Z(K_0 \cup L)$ changes by replacing t with t^{-1}. Since the $\mathfrak{sl}_2$ reduction is invariant under this change, we obtain the invariance of $\tau(M; c)$ under the orientation change of K_0. The invariance of (23) under the KI and KII moves on L is obtained from basic properties of the Aarhus integral; the invariance

under the KI move is obtained since the change of the Aarhus integral under the KI move cancels with the change of the normalization factor $c_+^{-\sigma_+} c_-^{-\sigma_-}$, and the invariance under the KII move is obtained since the KII move can be presented by certain variable change of the Aarhus integral and the Aarhus integral is unchanged under such variable change; for details, see [BGRT, GK2]. Hence, it is sufficient to show the invariance of $\tau(M; c)$ under the KII move over K_0.

When we change $K_0 \cup L$ by the KII move over K_0, $\chi^{-1} Z(K_0 \cup L) \sqcup \Omega^{-1}$ changes by the t-through relation by Lemma 5.2. Further, by Lemma 5.7, $\chi^{-1} Z^{\mathrm{LMO}}(N, K) \sqcup \Omega^{-1}$ changes by the t-through relation on Jacobi diagrams with labels of rational functions. Hence, the change of $\chi^{-1} Z^{\mathrm{LMO}}(N, K) \sqcup \Omega^{-1}$ is presented by

$$
\exp_{\sqcup}\Big(\overset{-\frac{1}{2}\log\,(\Delta/\Delta_1)}{\fbox{ }}\Big)\; \underset{D}{\bigcirc}^{*}\;=\;\exp_{\sqcup}\Big(\overset{-\frac{1}{2}\log\,(\Delta/\Delta_1)}{\fbox{ }}\Big)\Big(-\frac{1}{2}\overset{t\,\Delta'/\Delta}{\bigcirc}\!\!-\!\!\underset{D}{\bigcirc}\;+\;\overset{*}{\underset{D}{\bigcirc}}\Big)
$$

for some diagram D with labels of rational functions, where we put $\Delta = \Delta(t)$, $\Delta_1 = \Delta(1)$, and $\Delta' = \frac{d}{dt}\Delta(t)$. By $\mathfrak{sl}_2$ reduction, it is sufficient to consider the case where

$$
\underset{D}{\bigcirc}^{*}\;\underset{(\mathfrak{sl}_2)}{\equiv}\;\overset{\varphi_1(t)}{\bigcirc}\,\overset{\varphi_2(t)}{\bigcirc}\,\cdots\,\overset{\varphi_n(t)}{\bigcirc}\,\overset{\psi(t)}{\bigcirc}^{*}
$$

for some rational functions $\varphi_i(t)$ and $\psi(t)$ satisfying that $\varphi_i(t) = \varphi_i(t^{-1})$, $\varphi_i(1) = 0$ and $\psi(t) = -\psi(t^{-1})$. In this case,

$$
\overset{t\,\Delta'/\Delta}{\bigcirc}\!\!-\!\!\underset{D}{\bigcirc}\;\underset{(\mathfrak{sl}_2)}{\equiv}\;8\,\frac{\hat{t}\,\Delta'(\hat{t})\,\psi(\hat{t})}{\Delta(\hat{t})}\prod_i 2\,\varphi_i(\hat{t}),
$$

$$
\overset{*}{\underset{D}{\bigcirc}}\;\underset{(\mathfrak{sl}_2)}{\equiv}\;8\,\hat{t}\,\varphi_1(\hat{t})\,\psi(\hat{t})\prod_{i\neq 1} 2\,\varphi_i(\hat{t}) + 8\,\hat{t}\,\varphi_2(\hat{t})\,\psi(\hat{t})\prod_{i\neq 2} 2\,\varphi_i(\hat{t})
$$

$$
+\,\cdots\,+ 8\,\hat{t}\,\varphi_n(\hat{t})\,\psi(\hat{t})\prod_{i\neq n} 2\,\varphi_i(\hat{t}) + 4\Big(\hat{t}\,\psi'(\hat{t}) + \frac{\hat{t}+1}{\hat{t}-1}\psi(\hat{t})\Big)\prod_i 2\,\varphi_i(\hat{t})
$$

$$
=\; 4\,\hat{t}\Big(\psi(\hat{t})\prod_i 2\,\varphi_i(\hat{t})\Big)' + 4\,\frac{\hat{t}+1}{\hat{t}-1}\psi(\hat{t})\prod_i 2\,\varphi_i(\hat{t}),
$$

Hence, by putting $\varphi(t) = 4\,\psi(t)\prod_i 2\,\varphi_i(t)$,

$$
-\frac{1}{2}\overset{t\,\Delta'/\Delta}{\bigcirc}\!\!-\!\!\underset{D}{\bigcirc}\;+\;\overset{*}{\underset{D}{\bigcirc}}\;\underset{(\mathfrak{sl}_2)}{\equiv}\;-\frac{\hat{t}\,\Delta'(\hat{t})\,\varphi(\hat{t})}{\Delta(\hat{t})} + \hat{t}\,\varphi'(\hat{t}) + \frac{\hat{t}+1}{\hat{t}-1}\varphi(\hat{t}).
$$

Therefore, the change of $\sum_\ell \big(\hat{P}_\ell(t)/\Delta(t)^{2\ell+1}\big)(q-1)^\ell$ is given by

$$
\frac{\Delta(1)}{\Delta(t)}\Big(-\frac{t\,\Delta'(t)\,\varphi(t)}{\Delta(t)} + t\,\varphi'(t) + \frac{t+1}{t-1}\varphi(t)\Big) = \Delta(1)\Big(t\big(\frac{\varphi(t)}{\Delta(t)}\big)' + \frac{t+1}{t-1}\cdot\frac{\varphi(t)}{\Delta(t)}\Big).
$$

Further, the change of the defining formula of $\tau(M;c)$ is given by

$$-\frac{1}{2}\cdot\frac{1+c}{1-c}\operatorname*{Res}_{t=c}(1-t^{-1})^2\Delta(1)\Big(t\big(\frac{\varphi(t)}{\Delta(t)}\big)'+\frac{t+1}{t-1}\cdot\frac{\varphi(t)}{\Delta(t)}\Big)$$

$$=-\frac{\Delta(1)}{2}\cdot\frac{1+c}{1-c}\operatorname*{Res}_{t=c}\Big(\frac{(t+t^{-1}-2)\varphi(t)}{\Delta(t)}\Big)'=0,$$

since the residue of the differential of some function is always equal to 0. Hence, we obtain the invariance of $\tau(M;c)$ under the KII move over K_0. $\qquad\square$

In order to show Lemma 5.7 below, we extend the definition of the t-through relation to Jacobi diagrams with labels of rational functions. To extend it, we show some properties of the marking connected to a label of a rational function, as follows. We define the marking connected to a label of t^k by

$$\underset{t^k}{\overset{*}{\frown}}\ =\ \underset{t^k}{\frown}\ +\ \underset{t^{k-1}\ \ t}{\frown}\ +\cdots+\ \underset{t\ \ t^{k-1}}{\frown}$$

$$=\ \underset{t^k\ \ 1+t+\cdots+t^{k-1}}{\frown}\ =\ \underset{t^k\ \ (t^k-1)/(t-1)}{\frown} \tag{24}$$

for $k\in\mathbb{Z}$. By extending this definition linearly, we define the marking connected to a label of a polynomial $f\in\mathbb{Q}[t^{\pm1}]$. Then, we can show, for polynomials $f,g\in\mathbb{Q}[t^{\pm1}]$, that

$$\underset{fg}{\overset{*}{\frown}}\ =\ \underset{f\ \ g}{\overset{*}{\frown}}\ +\ \underset{f\ \ g}{\overset{\ \ *}{\frown}}\ , \tag{25}$$

by reducing the proof to the case where $f=t^i$ and $g=t^j$. Further, we define the marking connected to a label of a rational function f/g (for $f,g\in\mathbb{Q}[t^{\pm1}]$ with $g(1)\neq0$) by

$$\underset{f/g}{\overset{*}{\frown}}\ =\ \underset{f\ \ 1/g}{\overset{*}{\frown}}\ -\ \underset{f/g\ \ g\ \ 1/g}{\frown}\ .$$

In particular,

$$\underset{1/g}{\overset{*}{\frown}}\ =\ -\ \underset{1/g\ \ g\ \ 1/g}{\overset{\ \ \ *}{\frown}}\ .$$

Note that these two formulas imply that

$$\underset{f/g}{\overset{*}{\frown}}\ =\ \underset{f\ \ 1/g}{\overset{*}{\frown}}\ +\ \underset{f\ \ 1/g}{\overset{\ \ *}{\frown}}\ . \tag{26}$$

Lemma 5.3. *For polynomials $f,g\in\mathbb{Q}[t^{\pm1}]$ with $g(1)\neq0$,*

$$\underset{f/g}{\overset{*}{\frown}}\ =\ \underset{1/g\ \ f}{\overset{*}{\frown}}\ +\ \underset{1/g\ \ f}{\overset{\ \ *}{\frown}}\ .$$

Proof. By definition, it is sufficient to show that

$$\underset{\overset{*}{f}\quad 1/g}{\frown} \;-\; \underset{f/g\quad \overset{*}{g}\quad 1/g}{\frown} \;=\; \underset{1/g\quad \overset{*}{f}}{\frown} \;-\; \underset{1/g\quad \overset{*}{g}\quad f/g}{\frown}\;.$$

By multiplying g from both sides of each term, this formula is equivalent to the formula

$$\underset{g\quad \overset{*}{f}}{\frown} \;-\; \underset{f\quad \overset{*}{g}}{\frown} \;=\; \underset{\overset{*}{f}\quad g}{\frown} \;-\; \underset{\overset{*}{g}\quad f}{\frown}\;.$$

This is obtained by applying (25) to diagrams with labels of fg and gf. $\qquad\square$

Lemma 5.4. *For polynomials $f, g \in \mathbb{Q}[t^{\pm 1}]$ with $f(1), g(1) \neq 0$,*

$$\underset{1/\overset{*}{f}g}{\frown} \;=\; \underset{1/\overset{*}{f}\quad 1/g}{\frown} \;+\; \underset{1/f\quad 1/\overset{*}{g}}{\frown}\;.$$

Proof. By definition, it is sufficient to show that

$$-\;\underset{1/fg\quad \overset{*}{fg}\quad 1/fg}{\frown} \;=\; -\;\underset{1/f\quad \overset{*}{f}\quad 1/fg}{\frown} \;-\; \underset{1/fg\quad \overset{*}{g}\quad 1/g}{\frown}\;.$$

This is obtained by applying (25) to the label of fg. $\qquad\square$

Lemma 5.5. *For rational functions φ, ψ of t,*

$$\underset{\overset{*}{\varphi\psi}}{\frown} \;=\; \underset{\overset{*}{\varphi}\quad \psi}{\frown} \;+\; \underset{\varphi\quad \overset{*}{\psi}}{\frown}\;.$$

Proof. We put $\varphi = f_1/g_1$ and $\psi = f_2/g_2$ for polynomials $f_i, g_i \in \mathbb{Q}[t^{\pm 1}]$ with $g_i(1) \neq 0$. Then, the left-hand side of the required formula is equal to

$$\underset{f_1 f_2\quad 1/\overset{*}{g_1 g_2}}{\frown} \;+\; \underset{f_1 f_2\quad 1/\overset{*}{g_1 g_2}}{\frown}$$

$$=\; \underset{\overset{*}{f_1}\quad f_2/g_1 g_2}{\frown} \;+\; \underset{f_1\quad \overset{*}{f_2}\quad 1/g_1 g_2}{\frown} \;+\; \underset{f_1 f_2\quad 1/\overset{*}{g_1}\quad 1/g_2}{\frown} \;+\; \underset{f_1 f_2/g_1\quad 1/\overset{*}{g_2}}{\frown}\;,$$

where the equality is obtained from (25) and Lemma 5.4. Further, since we can exchange the order of f_2 and $1/g_1$ in the second and third terms by (26) and Lemma 5.3, the above formula is equal to

$$\underset{\overset{*}{f_1}\quad f_2/g_1 g_2}{\frown} \;+\; \underset{f_1\quad 1/\overset{*}{g_1}\quad f_2/g_2}{\frown} \;+\; \underset{f_1/g_1\quad \overset{*}{f_2}\quad 1/g_2}{\frown} \;+\; \underset{f_1 f_2/g_1\quad 1/\overset{*}{g_2}}{\frown}\;.$$

By definition, the right-hand side of the required formula is equal to this formula. $\qquad\square$

Extending the previous definition, we redefine the *t-through relation* to be the relation generated by

$$\begin{array}{c} \boxed{D}^{\ *} \ \sim \ 0, \end{array}$$

where D is a Jacobi diagram with labels of rational functions of t, and we define a diagram with the marking to be the sum of diagrams obtained from the original diagram by connecting the marking at each label; for example,

$$\varphi \bigcirc^{\psi\ *} \ = \ \overset{*}{\varphi} \bigcirc^{\psi} \ + \ \varphi \bigcirc^{\overset{*}{\psi}} \ .$$

Lemma 5.6. *For a rational function φ of t,*

$$\overset{*}{\varphi}\bigcirc \ = \ t\,\varphi'\bigcirc\ .$$

Proof. When $\varphi = t^k$, we obtain the lemma,

$$\overset{*}{t^k}\bigcirc \ = \ k\,t^k\bigcirc\ ,$$

since this equality is obtained from (24) and the property

$$f\bigcirc\!\!-\!\!-\,t \ = \ f\bigcirc\!\!-\!\!-\ .$$

In general, putting $\varphi = f/g$ for polynomials f and g in $t^{\pm 1}$, we obtain the lemma,

$$\overset{*}{f/g}\bigcirc \ = \ \overset{*}{f}\ 1/g\bigcirc \ - \ f/g\ \overset{*}{g}\ 1/g\bigcirc \ = \ t\,(f/g)'\bigcirc\ ,$$

where we obtain the second equality in the same way as above. $\qquad\square$

Extending Lemma 5.6, we define the marking connected to a label of $\log f$ on a loop for a polynomial f in $t^{\pm 1}$ by

$$\overset{*}{\log f}\bigcirc \ = \ t\,f'/f\bigcirc\ .$$

Lemma 5.7. *If $\chi^{-1}Z(K_0 \cup L)$ and $\chi^{-1}Z(K_0 \cup L')$ are related by the t-through relation, then $\int^{\mathrm{rat}}\chi^{-1}Z(K_0 \cup L)dY$ and $\int^{\mathrm{rat}}\chi^{-1}Z(K_0 \cup L')dY$ are related by the t-through relation on Jacobi diagrams with labels of rational functions.*

Proof. When L is of 1 component, we show the lemma, as follows. In this case, the difference of the t-through relation is generated by the form,

$$\exp\left(\frac{1}{2}\,\overset{\Delta}{\underset{y\qquad y}{\frown}}\right)\overset{*}{\underset{y\ y\quad y}{D}} = \exp\left(\frac{1}{2}\,\overset{\Delta}{\underset{y\qquad y}{\frown}}\right)\left(\overset{*}{\underset{y\ y\quad y}{D}} + \frac{1}{2}\,\overset{*}{\underset{y\ y\ y\quad y}{\overset{\Delta}{\frown}\ D}}\right), \quad (27)$$

where D is a diagram with $2k$ legs. By the rational version of the Aarhus integral, this is taken to

$$\exp_{\sqcup}\left(\overset{-\frac{1}{2}\log(\Delta/\Delta_1)}{\bigcirc}\right)\left\langle\exp\left(-\frac{1}{2}\overset{\partial y\quad\partial y}{\underset{1/\Delta}{\smile}}\right),\ \overset{*}{\underset{y\ y\quad y}{D}} + \frac{1}{2}\overset{*}{\underset{y\ y\ y\quad y}{\overset{\Delta}{\frown}\ D}}\right\rangle. \quad (28)$$

Further, the part of the bracket is calculated as

$$\frac{1}{(-2)^k\,k!}\left(\underset{1/\Delta\ \ 1/\Delta\ \ 1/\Delta}{\overset{*}{D}} - \frac{1}{2}\underset{1/\Delta\ 1/\Delta\ \ 1/\Delta}{\overset{*\ \overset{\Delta}{D}}{\bigcirc}} - k\ \underset{1/\Delta\ \ \ \ 1/\Delta\ \ 1/\Delta}{\overset{*\ \overset{\Delta}{D}}{}}\right).$$

Hence, (28) is equal to

$$\exp_{\sqcup}\left(\overset{-\frac{1}{2}\log(\Delta/\Delta_1)}{\bigcirc}\right)\ \sqcup\ \frac{1}{(-2)^k\,k!}\ \underset{1/\Delta\ 1/\Delta\ \ 1/\Delta}{\overset{*}{D}}\ , \quad (29)$$

where we use

$$\overset{\log(\overset{*}{\Delta}/\Delta_1)}{\bigcirc} = \overset{t\,\Delta'/\Delta}{\bigcirc} = \overset{\overset{*}{\Delta}\ \ 1/\Delta}{\bigcirc}. \quad (30)$$

Therefore, the Aarhus integral of (27) is presented by the form (29), and this implies the lemma. Note that (29) can also be presented by

$$\exp_{\sqcup}\left(\overset{-\frac{1}{2}\log(\Delta/\Delta_1)}{\bigcirc}\right)\left\langle\exp\left(-\frac{1}{2}\overset{\partial y\quad\partial y}{\underset{1/\Delta}{\smile}}\right),\ \overset{*}{\underset{y\ y\quad y}{D}}\right\rangle.$$

In general, the difference of the t-through relation is generated by the form,

$$\exp\left(\frac{1}{2}\sum_{i,j}\overset{A_{ij}}{\underset{y_i\qquad y_j}{\frown}}\right)\overset{*}{\underset{y_{i_1}y_{i_2}\quad y_{i_{2k}}}{D}} = \exp\left(\frac{1}{2}\sum_{i,j}\overset{A_{ij}}{\underset{y_i\qquad y_j}{\frown}}\right)\left(\overset{*}{\underset{y_{i_1}y_{i_2}\quad y_{i_{2k}}}{D}} + \frac{1}{2}\sum_{i,j}\overset{*}{\underset{y_i\quad y_j y_{i_1}y_{i_2}\quad y_{i_{2k}}}{\overset{A_{ij}}{\frown}\ D}}\right).$$

Similarly as the above case, we can show, by replacing Δ and $1/\Delta$ with A_{ij} and $(A^{-1})_{ij}$, that the Aarhus integral takes the above formula to

$$\exp_{\sqcup}\Big(\underbrace{}_{-\frac{1}{2}\log\,(\Delta/\Delta_1)}\Big)\Big\langle\exp\Big(-\frac{1}{2}\sum_{i,j}\underset{(A^{-1})_{ij}}{\overset{\partial y_j\ \ \partial y_i}{\bigcup}}\Big),\ \overset{*}{\underset{y_{i_1}\,y_{i_2}\ \cdots\ y_{i_{2k}}}{\boxed{D}}}\Big\rangle,$$

where, instead of the second equality of (30), we use

$$\underset{t\,\Delta'/\Delta}{\bigcirc}\ =\ \sum_{i,j}\ \overset{\overset{*}{A_{ij}}\ (A^{-1})_{ji}}{\bigcirc},$$

since $\Delta'/\Delta = \mathrm{trace}\,(A'\cdot A^{-1}) = \sum_{i,j} A'_{ij}\cdot(A^{-1})_{ji}$. Therefore, we obtain the lemma. $\quad\square$

Lemma 5.8. *Let D_1 and D_2 be Jacobi diagrams with a marking such that they are related by the procedure of $\mathfrak{sl}_2$ reduction explained in Section 4.2. Then, the diagrams obtained from D_1 and D_2 by applying the marking on themselves are related by $\mathfrak{sl}_2$ reduction.*

The lemma implies that the operation of the marking $*$ commutes with $\mathfrak{sl}_2$ reduction. In other words, it implies that, if $\overset{*}{\boxed{D_1}}\underset{(\mathfrak{sl}_2)}{\equiv}\overset{*}{\boxed{D_2}}$, then $\overset{*}{\boxed{D_1}}\underset{(\mathfrak{sl}_2)}{\equiv}\overset{*}{\boxed{D_2}}$.

Proof. The procedure of $\mathfrak{sl}_2$ reduction mainly consists of the application of (17), and this commutes with the operation of the marking $*$. The non-trivial part is the procedure of ignoring a trivalent vertex whose adjacent edges have labels $\varphi_i(t)$ $(i = 1, 2, 3)$ with $\varphi_i(1) = 0$. Hence, it is sufficient to show that

$$\overset{\overset{*}{\varphi_1}}{\underset{\varphi_3}{\overset{\varphi_2}{\rangle}}}\ +\ \overset{\varphi_1}{\underset{\varphi_3}{\overset{\overset{*}{\varphi_2}}{\rangle}}}\ +\ \overset{\varphi_1}{\underset{\overset{*}{\varphi_3}}{\overset{\varphi_2}{\rangle}}}\ \underset{(\mathfrak{sl}_2)}{\equiv}\ 0 \tag{31}$$

for rational functions $\varphi_i(t)$ with $\varphi_i(1) = 0$. We put $\varphi_i = f_i/g_i$ for polynomials f_i and g_i with $f_i(1) = 0$. Then, the left-hand side of the above formula is equal to

$$\overset{\overset{*}{1/g_1}\ f_1}{\underset{1/g_3\ f_3}{\overset{1/g_2\ f_2}{\rangle}}}+\overset{1/g_1\ f_1}{\underset{1/g_3\ f_3}{\overset{\overset{*}{1/g_2}\ f_2}{\rangle}}}+\overset{1/g_1\ f_1}{\underset{\overset{*}{1/g_3}\ f_3}{\overset{1/g_2\ f_2}{\rangle}}}+\overset{1/g_1\ \overset{*}{f_1}}{\underset{1/g_3\ f_3}{\overset{1/g_2\ f_2}{\rangle}}}+\overset{1/g_1\ f_1}{\underset{1/g_3\ f_3}{\overset{1/g_2\ \overset{*}{f_2}}{\rangle}}}+\overset{1/g_1\ f_1}{\underset{1/g_3\ \overset{*}{f_3}}{\overset{1/g_2\ f_2}{\rangle}}}$$

Further, the sum of the first three terms is equivalent to 0 by $\mathfrak{sl}_2$ reduction, since $f_i(1) = 0$ for all i. Hence, it is sufficient to show (31) when $\varphi_i = f_i$. Further, the proof can be reduced to the case where $f_i = t^{k_i} - 1$. In this case, we can check (31) concretely by (24) and (17). $\quad\square$

Lemma 5.9. *For a rational function $\varphi(t)$ with $\varphi(t) = -\varphi(t^{-1})$,*

$$\left(\,\overset{\ast}{\varphi(t)}\,\bigcirc\right) \underset{(\mathfrak{sl}_2)}{\equiv} 4\,\hat{t}\,\varphi'(\hat{t}) + 4\,\frac{\hat{t}+1}{\hat{t}-1}\,\varphi(\hat{t}).$$

Proof. We put $\varphi(t) = f(t)/g(t)$ for polynomials $f(t)$ and $g(t)$ with $f(t) = -f(t^{-1})$ and $g(t) = g(t^{-1})$. By definition,

$$\left(\,\overset{\ast}{\varphi(t)}\,\bigcirc\right) = \left(\,\overset{\ast}{f}\ 1/g\,\bigcirc\right) - \left(\,f/g\ \overset{\ast}{g}\ 1/g\,\bigcirc\right).$$

Further,

$$\left(\,\overset{\ast}{f}\ 1/g\,\bigcirc\right) \underset{(\mathfrak{sl}_2)}{\equiv} 2\,f(\hat{t})\Big(\frac{1}{g(1)} + \frac{1}{g(\hat{t})}\Big)\frac{\hat{t}+1}{\hat{t}-1} + \frac{4\,\hat{t}\,f'(\hat{t})}{g(\hat{t})},$$

where we can show this equivalence by (18) and (24) putting $f(t) = t^k + t^{-k}$. In a similar way, we can show that

$$\left(\,f/g\ \overset{\ast}{g}\ 1/g\,\bigcirc\right) \underset{(\mathfrak{sl}_2)}{\equiv} 2\,f(\hat{t})\Big(\frac{1}{g(1)} - \frac{1}{g(\hat{t})}\Big)\frac{\hat{t}+1}{\hat{t}-1} + \frac{4\,\hat{t}\,f(\hat{t})\,g'(\hat{t})}{g(\hat{t})^2}.$$

Hence,

$$\left(\,\overset{\ast}{\varphi(t)}\,\bigcirc\right) \underset{(\mathfrak{sl}_2)}{\equiv} 4\,\hat{t}\,\Big(\frac{f'(\hat{t})}{g(\hat{t})} - \frac{f(\hat{t})\,g'(\hat{t})}{g(\hat{t})^2}\Big) + 4\,\frac{\hat{t}+1}{\hat{t}-1}\cdot\frac{f(\hat{t})}{g(\hat{t})},$$

which implies the required formula. $\qquad\square$

6 Arithmetic limits of rational functions of roots of unity

The aim of this section is to show Proposition 6.1, which is used when we calculate arithmetic expansion of quantum invariants in Section 1.

Let p be an odd prime. We put $\zeta = \exp(2\pi\sqrt{-1}/p)$, and denote by $\mathbb{Z}_{(p)}$ the subring of $\mathbb{Q}$ consisting of rational numbers whose denominators are not divisible by p.

Proposition 6.1. *We set $\varphi(t)$ by $\varphi(t) = f(t)/g(t)$ where $f(t), g(t) \in \mathbb{Z}_{(p)}[t, t^{-1}]$ with $f(t) = f(t^{-1})$, $g(t) = g(t^{-1})$ and $g(1)$ is not divisible by p. Then,*

$$\frac{1}{p}\sum_{n \in \mathbb{Z}/p\mathbb{Z}} \varphi(\zeta^n) \ \in \mathbb{Z}_{(p)}[\zeta].$$

Further, if $p > \deg f(t) - \deg g(t)$, then,

$$\frac{1}{p}\sum_{n \in \mathbb{Z}/p\mathbb{Z}} \varphi(\zeta^n) = \sum_{c=0,c_1,\cdots,c_n} \Big(\frac{1+c}{1-c}\Big)^p \operatorname*{Res}_{t=c} \frac{\varphi(t)}{t} \ + O\big((\zeta-1)^{p-1}; \mathbb{Z}_{(p)}[\zeta]\big),$$

where $c_1, c_1^{-1}, \cdots, c_n, c_n^{-1}$ are the zeros of $g(t)$.

To prove Proposition 6.1, we show some lemmas, which give particular cases of Proposition 6.1.

Lemma 6.2. *Let $\varphi(t)$ be a polynomial in $\mathbb{Q}[t, t^{-1}]$ with $\varphi(t) = \varphi(t^{-1})$. If $p > \deg \varphi(t)$, then*

$$\frac{1}{p} \sum_{n \in \mathbb{Z}/p\mathbb{Z}} \varphi(\zeta^n) = \operatorname*{Res}_{t=0} \frac{\varphi(t)}{t}.$$

Proof. It is sufficient to show the lemma when $\varphi(t) = t^i$ for $|i| < p$. Then, both sides of the required formula are equal to 1 if $i = 0$, 0 otherwise. $\square$

Lemma 6.3. *We set $\varphi(t)$ by $\varphi(t) = f(t)/g(t)$ where $g(t) = 1 - b(t + t^{-1} - 2)$ with $b \in \mathbb{Z}_{(p)}$ and $f(t) \in \mathbb{Z}_{(p)}[t, t^{-1}]$ with $f(t) = f(t^{-1})$. If $p > \deg f(t) - \deg g(t)$, then*

$$\frac{1}{p} \sum_{n \in \mathbb{Z}/p\mathbb{Z}} \varphi(\zeta^n) = \sum_{c=0,c_1} \left(\frac{1+c}{1-c}\right)^p \operatorname*{Res}_{t=c} \frac{\varphi(t)}{t} + O\left((\zeta - 1)^{p-1}; \mathbb{Z}_{(p)}[\zeta]\right),$$

where c_1 is a zero of $g(t)$.

Proof. We assume that b is not divisible by p (otherwise, the lemma is reduced to Lemma 6.2). Further, by Lemma 6.2, we can reduce the lemma by subtracting a polynomial from $\varphi(t)$. Hence, it is sufficient to show the lemma when $f(t) = 1$. Putting $\eta = \zeta^n + \zeta^{-n} - 2$, we expand

$$\frac{1}{1 - b\eta} = \sum_{0 \leq i < p} b^i \eta^i + \frac{\eta^p}{1 - b\eta} \equiv \sum_{0 \leq i < p} b^i \eta^i,$$

since $\eta^p \in O\left((\zeta - 1)^{2p}\right)$ and $1 - b\eta$ is invertible in $\mathbb{Z}_{(p)}[\zeta]$, where the equivalence in the formula means the equivalence modulo $O\left((\zeta - 1)^{2p}\right)$. Further, for $i < p$,

$$
\begin{aligned}
\frac{1}{p} \sum_{n \in \mathbb{Z}/p\mathbb{Z}} \eta^i &= \frac{1}{p} \sum_{n \in \mathbb{Z}/p\mathbb{Z}} (\zeta^n + \zeta^{-n} - 2)^i \\
&= \left(\text{constant term of } (t + t^{-1} - 2)^i = (t^{1/2} - t^{-1/2})^{2i}\right) \\
&= (-1)^i \binom{2i}{i} = 4^i \binom{-1/2}{i}.
\end{aligned}
$$

Hence,

$$
\frac{1}{p} \sum_{n \in \mathbb{Z}/p\mathbb{Z}} \frac{1}{1 - b\eta} \equiv \sum_{0 \leq i < p} b^i 4^i \binom{-1/2}{i} \underset{(p)}{\equiv} \sum_{i=0}^{(p-1)/2} (4b)^i \binom{(p-1)/2}{i}
$$

$$
\underset{(p)}{\equiv} (4b + 1)^{(p-1)/2} = \left(\frac{1 + c_1}{1 - c_1}\right)^{p-1}, \tag{32}
$$

since $b^{-1} = c_1 + c_1^{-1} - 2$. Further,

$$\operatorname*{Res}_{t=0} \frac{1}{t\, g(t)} = 0,$$

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$$\operatorname*{Res}_{t=c_1} \frac{1}{t\, g(t)} = \lim_{t\to c_1} \frac{t-c_1}{t\, g(t)} = \frac{1}{c_1\, g'(c_1)} = -\frac{1}{b(c_1 - c_1^{-1})} = \frac{1-c_1}{1+c_1}.$$

Therefore, noting that $p \in O\big((\zeta-1)^{p-1}\big)$, we obtain the required formula. $\qquad\square$

Lemma 6.4. *For a positive integer ℓ, we set $\varphi(t)$ by $\varphi(t) = f(t)/g(t)^\ell$, where $f(t)$ and $g(t)$ are as in Lemma 6.3. If $p > \deg f(t) - \deg g(t)^\ell$, then*

$$\frac{1}{p}\sum_{n\in\mathbb{Z}/p\mathbb{Z}} \varphi(\zeta^n) \;=\; \sum_{c=0,c_1}\left(\frac{1+c}{1-c}\right)^p \operatorname*{Res}_{t=c}\frac{\varphi(t)}{t} \;+\; O\big((\zeta-1)^{p-1}; \mathbb{Z}_{(p)}[\zeta]\big),$$

where c_1 is a zero of $g(t)$.

Proof. It is sufficient to show the lemma for $\varphi(t) = (t + t^{-1} - 2)^{\ell-1}/(1 - b(t + t^{-1} - 2))^\ell$, because other $\varphi(t)$ can be obtained as a linear sum of a polynomial and such $\varphi(t)$. Similarly as (32), putting $\eta = \zeta^n + \zeta^{-n} - 2$, we have that

$$\frac{1}{p}\sum_{n\in\mathbb{Z}/p\mathbb{Z}} \frac{\eta^{\ell-1}}{(1-b\eta)^\ell} \equiv \sum_{0\le i < p-\ell+1}(-1)^i\binom{-\ell}{i}b^i\eta^{i+\ell-1} = \sum_{0\le i < p-\ell+1}(-1)^i\binom{-\ell}{i}b^i 4^{i+\ell-1}\binom{-1/2}{i+\ell-1}.$$

Further,

$$(-1)^i\binom{-\ell}{i}4^{\ell-1}\binom{-1/2}{i+\ell-1} = 4^{\ell-1}\binom{i+\ell-1}{i}\binom{-1/2}{i+\ell-1} = 4^{\ell-1}\binom{-1/2}{\ell-1}\binom{1/2-\ell}{i},$$

and

$$\begin{aligned}
4^{\ell-1}\binom{-1/2}{\ell-1} &= \frac{4^{\ell-1}}{(\ell-1)!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2\ell-3}{2}\right)\\
&= \frac{2^{\ell-1}}{(\ell-1)!}(-1)^{\ell+1}(2\ell-3)!! = (-1)^{\ell+1}2\binom{2\ell-3}{\ell-1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{1}{p}\sum_{n\in\mathbb{Z}/p\mathbb{Z}}\frac{\eta^{\ell-1}}{(1-b\eta)^\ell} &\equiv (-1)^{\ell+1}2\binom{2\ell-3}{\ell-1}\sum_{0\le i\le(p+1)/2-\ell}\binom{1/2-\ell}{i}(4b)^i\\
&\equiv (-1)^{\ell+1}2\binom{2\ell-3}{\ell-1}(4b+1)^{(p+1)/2-\ell} = (-1)^{\ell+1}2\binom{2\ell-3}{\ell-1}\left(\frac{1+c_1}{1-c_1}\right)^{p-2\ell+1}.
\end{aligned}$$

Further,

$$\operatorname*{Res}_{t=0}\frac{(t+t^{-1}-2)^{\ell-1}}{t\left(1-b(t+t^{-1}-2)\right)^\ell} = 0,$$

$$\operatorname*{Res}_{t=c_1}\frac{(t+t^{-1}-2)^{\ell-1}}{t\left(1-b(t+t^{-1}-2)\right)^\ell} = (-1)^{\ell+1}2\binom{2\ell-3}{\ell-1}\left(\frac{1-c_1}{1+c_1}\right)^{2\ell-1},$$

where the second formula is obtained by Lemma 6.5 below. Hence, we obtain the required formula for $\varphi(t) = (t + t^{-1} - 2)^{\ell-1}/(1 - b(t + t^{-1} - 2))^\ell$. $\qquad\square$

Lemma 6.5. *Put $z = t + t^{-1} - 2$, and let $t = c$ be a zero of $1 - bz$. Then,*

$(1)_\ell$
$$\operatorname*{Res}_{t=c} \frac{z^\ell}{t\,(1-bz)^{\ell+1}} = (-1)^\ell\, 2\binom{2\ell-1}{\ell}\left(\frac{1-c}{1+c}\right)^{2\ell+1},$$

$(2)_\ell$
$$\operatorname*{Res}_{t=c} \frac{(\ell-2b)z^\ell - bz^{\ell+1}}{t\,(1-bz)^\ell} = (-1)^\ell\, 2\ell\binom{2\ell-1}{\ell}\left(\frac{1-c}{1+c}\right)^{2\ell-1}.$$

Proof. We show that "$(1)_\ell \Leftrightarrow (2)_\ell$", as follows. The functions of the residues of $(1)_\ell$ and $(2)_\ell$ are related by

$$\frac{z^\ell}{t\,(1-bz)^{\ell+1}} - \frac{b}{\ell(4b+1)} \cdot \frac{d}{dt}\left(\frac{(t-t^{-1})z^\ell}{(1-bz)^\ell}\right) = \frac{1}{\ell(4b+1)} \cdot \frac{(\ell-2b)z^\ell - bz^{\ell+1}}{t\,(1-bz)^\ell}.$$

Since the differential of a function does not contribute to the residue, $(1)_\ell$ is equivalent to $(2)_\ell$, noting that $4b+1 = (1+c)^2/(1-c)^2$.

We show that "$(2)_\ell + b(2)_{\ell+1} = (\ell - 2b)(1)_\ell$", as follows. We have that

$$\big(\text{LHS of } (2)_\ell\big) + b\big(\text{LHS of } (2)_{\ell+1}\big)$$
$$= \operatorname*{Res}_{t=c} \frac{(1-bz)((\ell-2b)z^\ell - bz^{\ell+1}) + b((\ell+1-2b)z^{\ell+1} - bz^{\ell+2})}{t\,(1-bz)^{\ell+1}}$$
$$= \operatorname*{Res}_{t=c} \frac{(\ell-2b)z^\ell}{t\,(1-bz)^{\ell+1}} = (\ell - 2b)\big(\text{LHS of } (1)_\ell\big).$$

Further, it is shown by elementary calculation that

$$\big(\text{RHS of } (2)_\ell\big) + b\big(\text{RHS of } (2)_{\ell+1}\big) = (\ell - 2b)\big(\text{RHS of } (1)_\ell\big).$$

Hence, if $(1)_\ell$ and $(2)_\ell$ hold, then $(2)_{\ell+1}$ holds.

Therefore, by induction, $(1)_\ell$ and $(2)_\ell$ hold for all ℓ. $\qquad\square$

Proof of Proposition 6.1. We can assume that $g(1) = 1$ by replacing $g(t)$ with its quotient by $g(1)$.

If $g(t)$ is of degree 2, we put $g(t) = \big(1 - b_1(t+t^{-1}-2)\big)\big(1 - b_2(t+t^{-1}-2)\big)$ where b_1 and b_2 are complex number in general, but $b_1 + b_2,\ b_1 b_2 \in \mathbb{Z}_{(p)}$. If $(b_1 - b_2)^2$ is divisible by p, the proposition is reduced to Lemma 6.4. We assume that $(b_1 - b_2)^2$ is not divisible by p. We can put $f(t) = f_0 + f_1(t+t^{-1}-2)$ by subtracting a polynomial from $\varphi(t)$. Then,

$$\varphi(t) = \frac{f(t)}{g(t)} = \frac{1}{b_1-b_2}\left(\frac{b_1 f_0 + f_1}{1 - b_1(t+t^{-1}-2)} - \frac{b_2 f_0 + f_1}{1 - b_2(t+t^{-1}-2)}\right).$$

Further, in the same way as the proof of (32),

$$\frac{1}{p}\sum_{n\in\mathbb{Z}/p\mathbb{Z}}\varphi(\zeta^n) \equiv \frac{1}{b_1-b_2}\left((b_1 f_0 + f_1)\left(\frac{1+c_1}{1-c_1}\right)^{p-1} - (b_2 f_0 + f_1)\left(\frac{1+c_2}{1-c_2}\right)^{p-1}\right),$$

where c_i is a zero of $1 - b_i(t+t^{-1}-2)$. Since

$$\operatorname*{Res}_{t=0} \frac{\varphi(t)}{t} = 0,$$

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$$\operatorname*{Res}_{t=c_i} \frac{\varphi(t)}{t} = \lim_{t \to c_i} \frac{(t - c_i)f(t)}{t\, g(t)} = \frac{f(c_i)}{c_i\, g'(c_i)} = \frac{b_i f_0 + f_1}{b_i - b_{3-i}} \cdot \frac{1 - c_i}{1 + c_i},$$

we obtain the required formula in this case.

For a general $g(t)$, we can present $\varphi(t)$ as a linear sum of the $\varphi(t)$ of Lemmas 6.3 and 6.4. Hence, we can show the required formula in the same way as above. $\qquad\square$

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