Non-ribbon surface braids whose closures are ribbon

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Dedicated to Professor Yukio Matsumoto on the occasion of his 60th birthday.

Abstract

In two-dimensional knot theory, Alexander's theorem and Markov's theorem were proved (See [6]). The former insists that every oriented surface-link is ambient isotopic to the closure of a surface braid, and the latter gives the necessary and sufficient condition for closures of two surface braids to be ambient isotopic to the same surface-link. At present, however, there exist few results about relation between surface-links and surface braids. In this talk, we show that there exist non-ribbon surface braids whose closures are ribbon surface-links.

A *surface braid* of degree m is an oriented compact piecewise linear surface S locally flatly and properly embedded in the 4-disk $D^4 \simeq D_1^2 \times D_2^2$ such that

i) the restriction map of the second projection

$$pr_2\,|_S : S \to D_2^2$$

is a *simple* branched covering map of degree m, where 'simple' means that the inverse image $pr_2^{-1}(y) \cap S$ for each $y \in D_2^2$ consists of at least $m-1$ points, and

ii) $\partial S = X_m \times \partial D_2^2$.

Here, X_m is a fixed set of m interior points of D_1^2. We denote the branch point set of $pr_2\,|_S$ by $\Sigma(S)$. Note that we sometimes consider *non-simple* surface braids, i.e. surfaces which satisfy the above conditions except 'simple'.

Let y_0 be a fixed point in ∂D_2^2. We consider a surface braid as a branched covering space embedded in $D_1^2 \times D_2^2$. Then we define a *surface braid monodromy* ρ_S, that is, a homomorphism from $\pi_1(D_2^2 \setminus \Sigma(S), y_0)$ to $\mathrm{B}(D_1^2, X_m)$. Here $\mathrm{B}(D_1^2, X_m)$ is a group consisting of isotopy classes of all orientation preserving homeomorphisms of the pair (D_1^2, X_m) which is the identity map on ∂D_1^2. Each representative c of an element of $\pi_1(D_2^2 \setminus \Sigma(S), y_0)$ defines an isotopy of m-points in the interior of D_1^2 as shown in the following:

$$pr_1(pr_2^{-1}(c(t)) \cap S) \subset D_1^2$$

for each t in the interval. This isotopy extends to an orientation preserving homeomorphism of (D_1^2, X_m).

Let n be the number of $\Sigma(S)$. A *Hurwitz generating system* of $\pi_1(D_2^2 \setminus \Sigma(S), y_0)$ is an ordered free generating system consisting of n *lasso*s ξ_j such that the total product $\xi_1 \cdots \xi_n$ is homotopic to the oriented boundary. Here, a lasso is a homotopy class of a simple closed curve which surrounds only one branch point (the left of Figure 1). Although a Hurwitz generating system is not unique, any two of them are equivalent under Hurwitz move (the right of Figure 1).

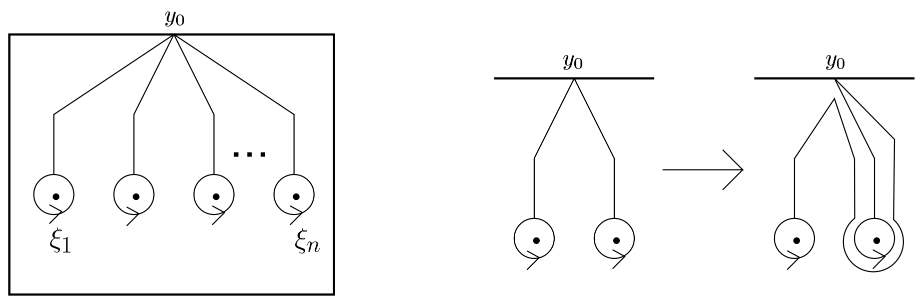

Figure 1: Hurwitz generating system and Hurwitz move

A *monodromy system* $\vec{b} = (b_1, \dots, b_n)$ of the surface braid S is an ordered n-tuple of elements in $\mathrm{B}(D_1^2, X_m)$ defined as

$$\vec{b} = (\rho(\xi_1), \rho(\xi_2), \dots, \rho(\xi_n)).$$

In general, there exists an equivalence relation on n-tuples of elements of a group G called *Hurwitz equivalence*. Let $(g_1, g_2, \dots, g_n)$ be an n-tuple of elements in G. The Hurwitz equivalence is an equivalence relation generated by the relations of the following: for each $j = 1, 2, \dots, n-1$,

$$(g_1, \dots, g_j, g_{j+1}, \dots, g_n) \sim (g_1, \dots, g_{j+1}, g_{j+1}^{-1} g_j g_{j+1}, \dots, g_n).$$

In [5], Kamada proved that two surface braids S and S' are equivalent if and only if their monodromy systems are Hurwitz equivalent.

A *surface-link* is a closed piecewise linear 2-dimensional manifold embedded in the 4-sphere locally flatly. If a surface-link consists of one connected component, it is called a *surface-knot*. We obtain a closed surface from a surface braid S by attaching $(D_1^2 \times \overline{D_2^2}, X_m \times \overline{D_2^2})$ to $(D_1^2 \times D_2^2, S)$, identifying $D_1^2 \times \partial D_2^2$ as $D_1^2 \times \partial \overline{D_2^2}$. Since this closed surface $S \cup X_m \times \overline{D_2^2}$ is in $D_1^2 \times S^2$, we obtain a closed surface in the 4-sphere by identifying $D_1^2 \times S^2$ with the regular neighourhood of an unknotted S^2 in the 4-sphere. We call it a *closure* of S, and denote it by $\widehat{S}$.

A surface-link is a *ribbon surface-link* if it is obtained from a trivial 2-link F_0 by surgery along some mutually disjoint 1-handles attaching to F_0. A surface braid is a *ribbon surface braid* if the monodromy system is equivalent to $(b_1, b_2, \dots, b_n)$ such that $b_{2j-1} b_{2j} = \mathrm{id}$ for each $j = 1, 2, \dots, n/2$. In [4], Kamada showed the following theorem:

Theorem 1 ([4]) *The closure of a ribbon surface braid is a ribbon surface-link and that any ribbon surface-link is ambient isotopic to the closure of a ribbon surface braid.*

We prove the following theorem related to the above fact:

Theorem 2 *There exist non-ribbon surface braids whose closures are ribbon.*

We indentify $\mathrm{B}(D_1^2, X_m)$ as the braid group B_m of m-strings. Artin's presentation of B_m is the following:

$$B_m \simeq \langle \sigma_1, \sigma_2, \dots, \sigma_{m-1} \,|\, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_i \sigma_j = \sigma_j \sigma_i \quad (|i-j| > 1) \rangle.$$

We give examples for Theorem 2. For given integers p and q, we consider the monodromy systems $\vec{b} = (\beta_1, \beta_2, \ldots, \beta_8)$ of surface braids of degree 5 such that

$$\begin{aligned} \beta_1 &= \sigma_1^p \sigma_2 \sigma_3^{-1} \sigma_2^{-1} \sigma_1^{-p}, & \beta_5 &= \sigma_4^q \sigma_2 \sigma_3^{-1} \sigma_2^{-1} \sigma_4^{-q}, \\ \beta_2 &= \sigma_1^p \sigma_3 \sigma_2 \sigma_3^{-1} \sigma_1^{-p}, & \beta_6 &= \sigma_4^q \sigma_3 \sigma_2 \sigma_3^{-1} \sigma_4^{-q}, \\ \beta_3 &= \sigma_3 \sigma_2^{-1} \sigma_3^{-1}, & \beta_7 &= \sigma_4^q \sigma_1^p \sigma_3 \sigma_2^{-1} \sigma_3^{-1} \sigma_1^{-p} \sigma_4^{-q}, \\ \beta_4 &= \sigma_2 \sigma_3 \sigma_2^{-1}, & \beta_8 &= \sigma_4^q \sigma_1^p \sigma_2 \sigma_3 \sigma_2^{-1} \sigma_1^{-p} \sigma_4^{-q}. \end{aligned}$$

We refer the surface braid with the above monodromy system as $S_{p,q}$.

The closure $\widehat{S_{p,q}}$ is a ribbon surface-link. Moreover the closure of the surface braid with the monodromy system

$$\vec{b'} = (\beta_1, \beta_1^{-1}, \beta_2, \beta_2^{-1}, \beta_3, \beta_3^{-1}, \beta_8, \beta_8^{-1})$$

is ambient isotopic to $\widehat{S_{p,q}}$. A *stabilization* for surface braids, one of Markov move, is described by the transformation of monodromy systems as following, by which we obtain an $(m+1)$-degree surface braid from an m-degree surface braid:

$$(b_1, \ldots, b_n) \to (b_1, \ldots, b_n, w\sigma_m w^{-1}, w\sigma_m^{-1} w^{-1}),$$

where b_j and w are in B_m. This transformation does not change the closure of surface braids. We show by a direct calculation that

$$(\beta_1, \beta_2, \ldots, \beta_8, \sigma_4\sigma_5\sigma_4^{-1}, \sigma_4\sigma_5^{-1}\sigma_4^{-1})$$

is Hurwitz equivalent to

$$(\beta_1, \beta_1^{-1}, \beta_2, \beta_2^{-1}, \beta_3, \beta_3^{-1}, \beta_8, \beta_8^{-1}, \sigma_4\sigma_5\sigma_4^{-1}, \sigma_4\sigma_5^{-1}\sigma_4^{-1})$$

Remark: Surface braids $S_{p,q}$ appear in [3] in which they classify surface braids with 4 white vertices. We see from the technique in [7] that the ribbon surface-link $\widehat{S_{p,q}}$ is a connected sum of a spun $(2,2)$-link, a spun $(2,p)$-link and a spun $(2,q)$-link.

One of 'easy' constructions of an invariant of surface braids is to consider a homomorphism from B_m to a finite group G. If we can classify orbits under the Hurwitz equivalence on G^n, we obtain an invariant of surface braids. For example, we choose an associated permutation, from B_m to the permutation group S_m. Then we obtain the genus of each connected component of the closure.

To show that $S_{p,q}$ is non-ribbon surface braid if $p \equiv q \equiv 2$ modulo 4, we use a homomorphism

$$\varphi : B_m \to M_m(\mathbb{Z}[t]/(t^2-1))$$

defined as

$$\sigma_i \mapsto \begin{pmatrix} I_{i-1} & & & \\ & t & 0 & \\ & 0 & 1 & \\ & & & I_{m-i-1} \end{pmatrix} \begin{pmatrix} I_{i-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{m-i-1} \end{pmatrix}.$$

Since the above factorization gives an action of S_m on $\mathbb{Z}_2^m$, this homomorphism is considered to be a homomorphism from B_m to $\mathbb{Z}_2^m \rtimes S_m$, also denoted by φ. In a diagonal matrix, the left factor of the factorization, each entry on the diagonal part is either 1 or t. They correspond respectively to 0 or 1 of $\mathbb{Z}_2^m$.

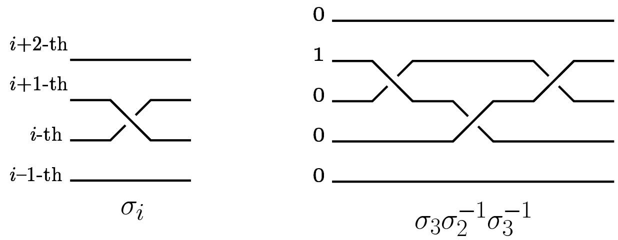

Figure 2: A value of the homomorphism φ

Lemma 3 *If $p \equiv q \equiv 2$ modulo 4, then the surface braid $S_{p,q}$ is non-ribbon.*

Proof. If $p \equiv q \equiv 2$ modulo 4, it holds that

$$(\varphi(\beta_1), \varphi(\beta_2), \dots, \varphi(\beta_8)) = (x, x, \dots, x),$$

where $x = ((0,0,0,1,0),(2,4))$. This 8-tuple is invariant under Hurwitz equivalence. Note that the images through φ of two equivalent monodromy systems are also Hurwitz equivalent. Since x has the order 4 in $\mathbb{Z}_2^m \rtimes S_m$, the 8-tuple $(x, x, \dots, x)$ can not be the image of a monodromy system of a ribbon surface braid. Thus $S_{p,q}$ is non-ribbon. □

Remark: In [6], Kamada proved Markov's theorem for surface braids. There exist three kinds of Markov moves, conjugation, fission/fusion and stabilization/destabilization. For the case $p \equiv q \equiv 2$ modulo 4, the surface braid $S_{p,q}$ nor its conjugate surface braid cannot be applied each of fission/fusion and destabilization. Thus, to relate $S_{p,q}$ and the ribbon surface braid, we need a stabilization.

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