

Finite Group Actions and 3-Manifold Topology

TORU IKEDA

Dedicated to Professor Yukio Matsumoto on his 60th birthday

ABSTRACT. This article provides a brief sketch of the theory of finite group actions studied in terms of 3-manifold topology. We will survey the results in the geometric case and equivariant techniques in 3-manifold topology, and then introduce some results in the non-geometric case.

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1. GEOMETRIC CASE

Let G be a finite group acting on a manifold M as a group of homeomorphisms. In this paper, G is said to be a finite group action on M and regarded as the group of homeomorphisms. Let G_1 and G_2 be finite group actions on a manifold M . If there is a homeomorphism h (which restricts to the identity on ∂M) such that $G_2 = \{h \circ g \circ h^{-1} | g \in G_1\}$, we say G_1 and G_2 are equivalent (relative to ∂M). In particular, according as h is topological or PL (piecewise linear), G_1 and G_2 are said to be topologically equivalent or PL equivalent (relative to ∂M).

Finite group actions on manifolds arise in many questions of topology and geometry. The case of geometric manifolds has been studied from the beginning of this theory. In 1919, Kerékjártó [9] studied that periodic homeomorphisms of D^2 and S^2 are conjugate to orthogonal maps. The Smith conjecture [12] had been one of the main topics in topology. The affirmative answer to this conjecture was reported in 1978.

Theorem 1.1 (The solution to the Smith conjecture). *Any cyclic group action on S^3 generated by an orientation-preserving, periodic PL homeomorphism with nonempty fixed point set is equivalent to an orthogonal action.*

Finite group actions on other geometric 3-manifolds are studied by Meeks and Scott [10] in 1986 as follows.

Theorem 1.2 (Meeks and Scott). *Let M be a closed 3-manifold with a geometric structure modelled on one of $\mathbb{H}^2 \times \mathbb{R}$, $SL_2\mathbb{R}$, Nil , \mathbb{E}^3 or Sol . Then any PL finite group action on M preserves the geometric structure on M .*

Moreover, Thurston announced the orbifold theorem [1, 3], which shows the existence of geometric structures on many 3-orbifolds. As a consequence of this theorem, an irreducible, atoroidal, closed orientable 3-manifold M , which admits a finite group action G with 1-dimensional singular set, can be geometrized so that G is a group of isometries.

2. SOME TECHNIQUES IN 3-MANIFOLD TOPOLOGY

By the affirmative answer to the Hauptvermutung given by Moise [11], the 3-manifold topology is often studied in the PL category.

Theorem 2.1 (Moise). *Let K_1 and K_2 be simplicial complexes such that each $|K_i|$ is a 3-manifold. If there is a topological homeomorphism $f: |K_1| \rightarrow |K_2|$, then there is a PL homeomorphism $g: |K_1| \rightarrow |K_2|$.*

However, the above result does not work for the studies of equivalences of finite group actions on 3-manifolds. The author extended the above result to the case of 3-orbifolds [5].

Theorem 2.2. *Let K_1 and K_2 be simplicial complexes such that each $|K_i|$ is a 3-orbifold with a polyhedral singularity. If there is a topological orbifold isomorphism $f: |K_1| \rightarrow |K_2|$, then there is a PL orbifold isomorphism $g: |K_1| \rightarrow |K_2|$.*

Corollary 2.3. *Any two PL finite group actions on a compact 3-manifold are PL equivalent if and only if they are topologically equivalent.*

On the other hand, compact orientable 3-manifolds are decomposed into manifolds with some geometrical properties by splitting along essential spheres, disks, annuli or tori. These techniques depend on the loop theorem [4], the sphere theorem [4] and the JSJ decomposition theorem [7, 8].

For the study of finite group actions on 3-manifolds, some equivariant techniques are available. The equivariant loop theorem is given by Meeks and Yau in 1982 as a result in the minimal surface theory. The equivariant sphere theorem is given by Plotnick [13] in 1984. The equivariant JSJ decomposition theorem

is proved by Meeks and Scott [10] in 1986 in the case of Haken manifolds other than torus bundles over S^1 .

3. NON-GEOMETRIC CASE

In this section, we work in the PL category. It has been studied much about involutions on non-geometric 3-manifolds. However, the results introduced below admit other possibilities. In the case of Haken 3-manifolds, Zimmermann proved the following result in 1986.

Theorem 3.1 (Zimmermann). *Let M be a Haken 3-manifold which is not Seifert fibered. Let G_1 and G_2 be mutually isomorphic finite group actions on M which preserve an JSJ decomposition of M . Suppose that each G_i operates as a group of isometries with respect to some geometric structure of decomposing pieces and that G_1 and G_2 induce the same subgroup $G \subset \text{Out}(\pi_1 M)$. Then G_1 and G_2 are equivalent.*

In the viewpoint of equivariant Heegaard splitting, Zimmermann [15] extended in 1996 the theory of finite group actions on 3-dimensional handlebodies to the case of closed orientable 3-manifolds.

A connected 3-manifold M is said to be *peripheral* if there is a compact, connected surface F in ∂M such that an inclusion induced map $\pi_1 F \rightarrow \pi_1 M$ is an epimorphism. The equivariant loop theorem implies that finite group actions on peripheral manifolds are obtained from actions on pieces such as balls and I -bundles. We are concerned with 3-manifolds with more general properties.

If every loop in a 3-manifold M is freely homotopic into the boundary, M is said to be *totally peripheral*. Brin, Johannson and Scott [2] showed that an orientable, compact, connected, totally peripheral 3-manifold is peripheral.

Let l_1 and l_2 be loops in a 3-manifold M such that $l_1 = l_2$ or $l_1 \cap l_2 = \phi$. A disk D embedded in M is said to be a *band* joining l_1 and l_2 if $a_1 = l_1 \cap D$ and $a_2 = l_2 \cap D$ are disjoint arcs in ∂D and the orientation of ∂D is equal to the orientation of each l_i . In case of $l_1 \cap l_2 = \phi$, $(l_1 \cup l_2 - a_1 \cup a_2) \cup \text{cl}(\partial D - a_1 \cup a_2)$ is said to be the *band sum* of l_1 and l_2 along D . We say a 3-manifold M is *partially peripheral* if every loop l in M is a band sum of finite number of loops freely homotopic into ∂M . In particular, any totally peripheral 3-manifold is partially peripheral. The author proved the following theorem in 2003.

Theorem 4.2. *Let M be a compact, orientable, partially peripheral 3-manifold with no fake 3-ball. Then the following holds.*

- (1) *M admits a prime factorization $M = M_1 \# \cdots \# M_n$ where each M_i is an irreducible, partially peripheral 3-manifold.*
- (2) *Let G_1 and G_2 be orientation-preserving finite group actions on M such that*
 - (a) *G_1 preserves any ∂M_i and that*
 - (b) *for any essential sphere or torus F in M , the setwise stabilizer of F in each G_i is free on F .*

If G_1 and G_2 restrict to the same free action on ∂M , they are equivalent relative to ∂M .

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Department of Mathematics and Information Science, Faculty of Science, Kochi University,
2-5-1 Akebono-cho, Kochi 780-8520, Japan
E-mail address: ikeda@math.kochi-u.ac.jp