Ricci Flow in Dimension 4

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In this talk I want to review R. Hamilton's papers [2] [4] on the Ricci flow in dimension 4, and hope to discuss some general (unstable) features of the Ricci flow in this dimension.

Dedicated to Professor Yukio Matsumoto for his 60th birthday

1. The Ricci flow

The Ricci flow on a smooth n-manifold M evolves a Riemannian metric $g = \{g_{ij}(t)\}$ on M by the the equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$$

where R_{ij} is the Ricci curvature tensor of the metric $g_{ij}(t)$. Written in geodesic coordinates, the right-hand-side $-2R_{ij}$ at the origin equals the ordinary Laplacian $3\sum_{k=1}^{n} \frac{\partial^2}{(\partial x^k)^2} g_{ij}$. Hence the Ricci flow equation is really the natural heat equation for the Riemannian metric g.

(Remark. Normalization of $g(t)$ by rescaling time and space is applied when necessary. The normalized version is $\frac{\partial}{\partial t} g_{ij} = \frac{2}{n} r g_{ij} - 2R_{ij}$ with $r = \int Scal\ d\nu / \int d\nu$)

If $g(t) = g_{ij}(t)$ evolves by the Ricci flow equation, the Riemannian curvature tensor $Rm = \{R_{ijk\ell}\}$ of $g(t)$ evolves by a diffusion-reaction equation which reads

$$\frac{\partial}{\partial t} R_{abcd} = \Delta R_{abcd} + Rm^2_{abcd} + Rm^{\#}_{abcd}$$

in an appropriately evolving orthonormal frame $f = \{f_a(t)\}$. Here Rm^2 is the square of Rm regarded as the operator on the 2-forms $\Lambda^2 \to \Lambda^2$ and $Rm^{\#}$ is the "Lie algebra square" in $\mathfrak{so}(n) \otimes \mathfrak{so}(n)$ with $\mathfrak{so}(n) = \Lambda^2$. (The derivation is non-trivial, and requires 'Uhlenbeck's trick'.)

The Ricci flow was introduced in R. Hamilton's paper [1], where it is shown that if (M^3, g_0) is a closed 3-manifold of positive Ricci curvature and $g(t)$ with $g(0) = g_0$ evolves by the normalized Ricci flow, then $g(t)$ converges as $t \to \infty$to a metric of positive constant Riemannian curvature. In particular, a homotopy 3-sphere with Riemannian metric of positive Ricci curvature is diffeomoriphic to the 3-sphere.

If one can remove the Ricci positivity assumption on the homotopy 3-sphere, then the Poincaré Conjecture follows. We now know that G. Perelman's recent papers [7] [8] almost succeeded in removing the assumption on the metric on general compact 3-manifolds. Specifically, if one can show that finitely many 'surgeries' suffice to improve the metric, then the 3-manifold has the goemetric decomposition in the sense of Thurston.

2. 4-manifolds and 4-spheres

According to Hamilton [3], the Ricci flow may work in dimensions three and four, but not in dimensions ≥ 5. This is because the neck pinches (surgeries) have preferred directions, e.g.

$$\partial D^3 \times D^1 \to D^3 \times \partial D^1 \text{ or } \partial D^4 \times D^1 \to D^4 \times \partial D^1,$$

only in dimensions ≤ 4.

In dimension 4, however, the situations seem quite different from those for 3-manifolds and the direct 4-dimensional analogue of the Hamilton's program for the Thurston's Geomatrization Conjecture for 3-manifolds fails from the beginning.

First of all, the prime (irreducible) decompositions of four-manifolds are not unique. In fact, any simply-connected smooth 4-manifold can be 'stabilized' to a connected sum of some $\mathbb{C}P^2$'s and $\overline{\mathbb{C}P}^2$'s. In other words, one can run the Ricci flow on $k\mathbb{C}P^2 \# \ell\overline{\mathbb{C}P}^2$ for sufficiently large k and ℓ with some initial metric and obtain any (simply-connected) 4-manifold after finitely many necks are pinched off.

Even assuming the irreducibility of 4-manifolds, it seems pinching and collapsing occur rather unstably and irregularly in dimension 4, basically because we can have two orthogonal surfeses of arbitrary metrics locally which evolve by the Ricci flow independently at least for some time. (Consider, for example, the product $S^2 \times S^2$ of 2-spheres of different radii.)

Then, consider homotopy 4-spheres. Optimistically, topological simplicity may help prevent its bad collapsing. However, the issue of differentialbe structures in dimension 4 is unavoidable here. In fact, although any homotopy 4-sphere is homeomorphic to the 4-sphere by the Freedman's classification of simply-connected 4-manifolds, there is no reason to believe that the smooth structure on the 4-sphere is unique. Hence it is fairly important to have (possibly weak or null) sufficient conditions for a homotopy 4-spheres to be diffeomorphic to the standard 4-sphere.

As a natural generalization of the above mentioned result on the homotopy 3-spheres to dimension 4, R. Hamilton showed that if (M^4, g_0) is a 4-manifold with a positive curvature operator and $g(t)$ with $g(0)$ evolves by the normalized Ricci flow equation, then $g(t)$ converges to a metric of constant positive Riemannian curvature. In particular, if M is simply-connected then M^4 is diffeomorphic to the standard 4-sphere [4]. Here the metric has a positive curvature operator if the Riemannian curvature regarded as the symmetric bilinear form on the 2-forms Λ^2 is strictly positive.

Technically, dimension four is accesible partly because we have the decomposition $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$. Then a bilinear form B on $\Lambda^2 = \mathfrak{so}(4)$ can be expressed as the 2 by 2 block matrix (of 3 by 3 blocks) according to the orthogonal decomposition

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-.$$

The main tool here is the maximum principles with which one can control the partial differential equation for the curvature evolution by the accociated ODE:

$$\frac{d}{dt}B = B^2 + B^\#$$

In dimension 3 this ODE is rather simple (compared to that in dimension 4). Using this, Hamilton and Ivey independently observed that for any solution of the Ricci flow (with any initial metric) on a closed 3-manifold, large negative sectional curvature can occur only when there occurs much larger positive sectional curvatures. [6] [5] This provides the starting point for the recent works of Perelman on the Geometrization Conjecture. In dimension 4, some observation of same general sort is lacking and desirable.

Even so, the above theorem for 4-manifolds is improved in the Hamilton's paper "Four-manifolds with positive isotropic curvature" [4]. It is shown in this paper (among other general statements) that if M is a simply-connected closed 4-manifold with positive isotropic curvature, then M is diffeomorphic to the standard 4-sphere. Here a metric has positive isotropic curvature if the sectional curvature $K(P) := R(Z, W, \bar{Z}, \bar{W})$ is positive for all complex isotropic 2-plane P in the complexified tangent space. (Here $Z = X + iY, W = U + iV$ span P with X, Y, U, V orthonormal and the Riemannian curvature is complex multilinear)

The proof requires surgical modifications of the Ricci flow, which is the delicate part of the argument (in Perelman's works for 3-manifolds too) and some details will be reviewed in the talk.

References

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