A CHARACTERIZATION OF BRAID FOLIATIONS ON SEIFERT SURFACES OF GENUS ONE

HIROSHI MATSUDA

Dedicated to Professor Yukio Matsumoto on his 60th birthday

Let A denote an oriented trivial knot in S^3. There is an open book decomposition $\{H_\theta | \theta \in [0, 2\pi]\}$ of S^3, where A is the axis, and H_θ is a disc for every $\theta \in [0, 2\pi]$. The orientation of A induces an orientation of H_θ for every θ, and induces a positive direction of the fibration $\{H_\theta\}$. An oriented link K in S^3 is said to be represented as a *closed n-braid* if $K \cap A = \emptyset$, if K intersects each fiber H_θ transversely in exactly n points, and if the orientation of K agrees with the positive direction of the fibration $\{H_\theta\}$ at every point of the intersection $K \cap H_\theta$. The minimal number of n among all closed n-braid representatives K of a knot type $\mathbf{K}$ is called the *braid index* of $\mathbf{K}$. Let F be a Seifert surface of a knot K which is represented as a closed braid. The surface F has an orientation which is induced from that of K. The intersection $F \cap \{H_\theta\}$ induces a vector field on F. Regarding this vector field as a line field, we obtain a *braid foliation* on F. Braid foliations on a disc bounded by the trivial knot are studied in [7], [3], [4], [2]. In this talk, we study braid foliations on Seifert surfaces of genus one bounded by knots of genus one.

From a braid foliation on F, we obtain a tiling of F. See §1 of [3]. Each source (respectively sink) in the induced vector field on F corresponds to one positive (resp. negative) vertex in the tiling. It is sometimes reasonable to regard ∂F as a "huge" vertex of negative sign. Each hyperbolic singularity in the induced vector field corresponds to one tile in the tiling. Each tile is assigned its sign positive or negative according as the positive direction of the fibration $\{H_\theta\}$ agrees or disagrees with the positive normal to F at the hyperbolic singularity in the tile. We notice that the number of sources and sinks in the induced vector field on $int\, F$ is equal to the number $|F \cap A|$ of intersections of F with A, and that the number $|F \cdot H_\theta|$ of hyperbolic singularities in the induced vector field on F is equal to the number of tiles on F. When $\partial F = K$ is represented as a closed braid, the *complexity* $C(F, H)$ of a Seifert surface F of K, with respect to the fibration $H = \{H_\theta\}$, is the triple $(|K \cap H_\theta|, |F \cap A|, |F \cdot H_\theta|)$, which is ordered lexicographically.

Four graphs $G_{+,+}$, $G_{+,-}$, $G_{-,+}$ and $G_{-,-}$ on F are extracted from a braid foliation as follows. A vertex of $G_{+,+}$ and $G_{+,-}$ is a positive vertex in the tiling, and an edge of $G_{+,+}$ (resp. $G_{+,-}$) is an arc which is contained in the singular leaf on a tile of positive (resp. negative) sign, and which connects the two positive vertices in the tile. An edge of $G_{-,+}$ (resp. $G_{-,-}$) is an arc which is contained in the singular leaf on a tile of positive (resp. negative) sign, and which connects either the two negative vertices in the tile, or one negative vertex and a point on ∂F in the tile, or two points on ∂F in the tile. Vertices of $G_{-,+}$ (resp. $G_{-,-}$) are endpoints of edges of $G_{-,+}$ (resp. $G_{-,-}$) on F. According to [9], Bennequin [1] introduced these four graphs when he used a Reeb foliation of S^3 instead of an open book decomposition H. Menasco [11] studied these graphs when he used an open book decomposition H. From a configuration of $G_{+,+} \cup G_{+,-} \cup G_{-,+} \cup G_{-,-}$ on F, it is easy to construct the underlying braid foliation on F. Therefore studying braid foliations on F is equivalent to studying configurations of $G_{+,+} \cup G_{+,-} \cup G_{-,+} \cup G_{-,-}$ on F. The following proposition shows that studying braid foliations on F is equivalent to studying configurations of $G_{-,+} \cup G_{-,-}$ on F.

Department of Mathematics, Graduate School of Science, Hiroshima University, Hiroshima 739-8526, JAPAN.
e-mail: matsuda@math.sci.hiroshima-u.ac.jp .

Proposition. [10, Proposition 3.3] *From a configuration of $G_{-,+} \cup G_{-,-}$ on F, we can construct the underlying configuration of $G_{+,+} \cup G_{+,-} \cup G_{-,+} \cup G_{-,-}$ on F; therefore we can construct the underlying braid foliation on F.*

It is shown in [10, Proposition 4.1] that if $C(F, H)$ is minimal, and if the braid index of $\mathbf{K}$ representing $K = \partial F$ is at least four, then there is a properly embedded arc on F which consists of negative vertices and edges of $G_{-,\delta}$, where $\delta = +$ or $-$. An arc consisting of negative vertices and edges of $G_{-,\delta}$ is called an *edge-path* of $G_{-,\delta}$. Suppose first that there is such an arc γ on F. Cut F along γ, and we obtain an annulus. Figure 1 illustrates a configuration of $G_{-,+}$ and $G_{-,-}$ on the annulus, where dark thick arcs represent edge-paths of $G_{-,+}$, say, and light thick arcs represent edge-paths of $G_{-,-}$. The number n next to these arcs indicates that there are n arcs which are parallel to the arc. We notice that the intersection points of dark thick arcs and light thick arcs correspond to negative vertices on F. If a dark or light thick arc which is properly embedded on F is disjoint from other dark or light thick arcs, then we assume that the arc assigned with a number n represents n edges of $G_{-,+}$ and $G_{-,-}$. Suppose next that there are at least two arcs γ_1 and γ_2 on F such that each of γ_1 and γ_2 represents an edge-path of $G_{-,+}$ or $G_{-,-}$, and that γ_1 and γ_2 do not cobound a disc on F together with two subarcs of ∂F. Cut F along γ_1 and γ_2, and we obtain a disc. Figures 2 – 6 illustrate configurations of $G_{-,+}$ and $G_{-,-}$ on the disc.

In the configuration of Figure 1, we assume $x \geq 1$, $y, a, b, c \geq 0$ and $a + b + c \geq 1$. We assume $x, y, a \geq 1$ and $z, b, c \geq 0$ in the configuration of Figure 2, $x, y \geq 1$ and $z, a, b, c \geq 0$ in that of Figure 3 (1), and $x, y \geq 1$ and $a, b \geq 0$ in that of Figure 3 (2). We assume $x, y, z, a \geq 0$ in that of Figure 4 (1), $y \geq 1$, $x, z, a \geq 0$ and $x + z + a \geq 1$ in that of Figure 4 (3), $y \geq 1$ and $x, a \geq 0$ in that of Figure 5, and $x, a \geq 1$ in that of Figure 6. In the configuration of Figure 4 (2), we assume $x, y, a \geq 0$, and that if x or y, say x, is 0, then both y and a are at least 1. In the configuration of Figure 4 (4), we assume either $y, z \geq 1$, $x, a \geq 0$ and $x + a \geq 1$, or $x, y \geq 2$ and $z = a = 0$. The following is our main theorem.

Theorem. [10, Theorem 1.3] *Let K be an arbitrary closed n-braid representative of a knot $\mathbf{K}$ of genus one. Suppose that the braid index of $\mathbf{K}$ is at least four. Let F be a Seifert surface of genus one bounded by K. Then there exists a finite sequence of Seifert surfaces of genus one: $F = F_0 \to F_1 \to \cdots \to F_m$ satisfying the following properties;*

(a) *the pair $(F_{i+1}, \partial F_{i+1})$ is ambient isotopic to $(F_i, \partial F_i)$,*
(b) *each $K_i = \partial F_i$ is represented as a closed braid with respect to the same braid axis A,*
(c) *$C(F_{i+1}, H) \leq C(F_i, H)$, and*
(d) *the braid foliation on F_m is constructed from one of the configurations of $G_{-,+} \cup G_{-,-}$ illustrated in Figures 1, 2, 3 (1), 3 (2), 4 (1), 4 (2), 4 (3), 4 (4), 5 and 6.*

Moreover, the associated sequence of closed braids: $K = \partial F = \partial F_0 \to \partial F_1 \to \cdots \to \partial F_m$ satisfies the property that each $K_{i+1} = \partial F_{i+1}$ is obtained from $K_i = \partial F_i$ by one of the following moves;

(i) *an isotopy in the complement of the braid axis A,*
(ii) *a destabilization move,*
(iii) *an exchange move, which is introduced in* [6], *or*
(iv) *a cyclic move, which is introduced in* [8].

Remark. Seifert surfaces bounded by closed 3-braids are studied in [5].

REFERENCES

[1] D. Bennequin, *Entrelaccements et équations de Pfaff*, Thèse de Doctorat d'Etat, Université de Paris VII, 24 novembre 1982.

[2] J. Birman, P. Boldi, M. Rampichini, S. Vigna, *Towards an implementation of the B-H algorithm for recognizing the unknot*, J. Knot Theory Ramifications **11** (2002), 601–645.

[3] J. Birman, E. Finkelstein, *Studying surfaces via closed braids*, J. Knot Theory Ramifications **7** (1998), 267–334.

[4] J. Birman, M. Hirsch, *A new algorithm for recognizing the unknot*, Geom. Topol. **2** (1998), 175–220.

[5] J. Birman, W. Menasco, *Studying links via closed braids II: On a theorem of Bennequin*, Topology Appl. **40** (1991), 71–82.

[6] J. Birman, W. Menasco, *Studying links via closed braids IV: Split and composite links*, Invent. Math. **102** (1990), 115–139.

[7] J. Birman, W. Menasco, *Studying links via closed braids V: The unlink*, Trans. Amer. Math. Soc. **329** (1992), 585–606.

[8] J. Birman, W. Menasco, *Stabilization in the braid groups-I: MTWS*, available at http://xxx.lanl.gov/math.GT/0310279.

[9] A. Douady, *Nœuds et structures de contact (d'après D. Bennequin)*, Sém. Bourbaki exposé 604, Astérisque **105-106** (1983), 129–148.

[10] H. Matsuda, *A characterization of braid foliations on Seifert surfaces of genus one*, preprint.

[11] W. Menasco, *The Bennequin-Milnor unknotting conjecture*, C. R. Acad. Sc. Paris **318** Serie I (1994), 831–836.

[12] D. Rolfsen, *Knots and Links*, (Corrected reprint of the 1976 original.) Mathematics Lecture Series, **7**, Publish or Perish Inc., Houston, TX, 1990.

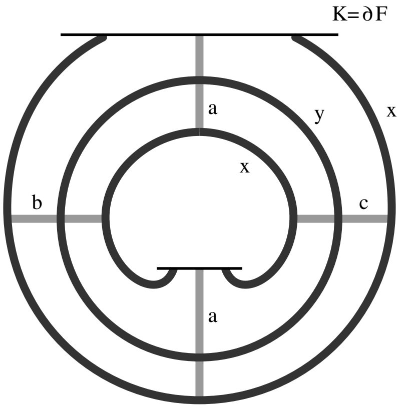


Figure 1

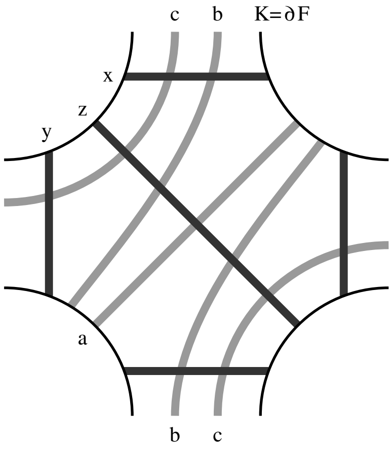
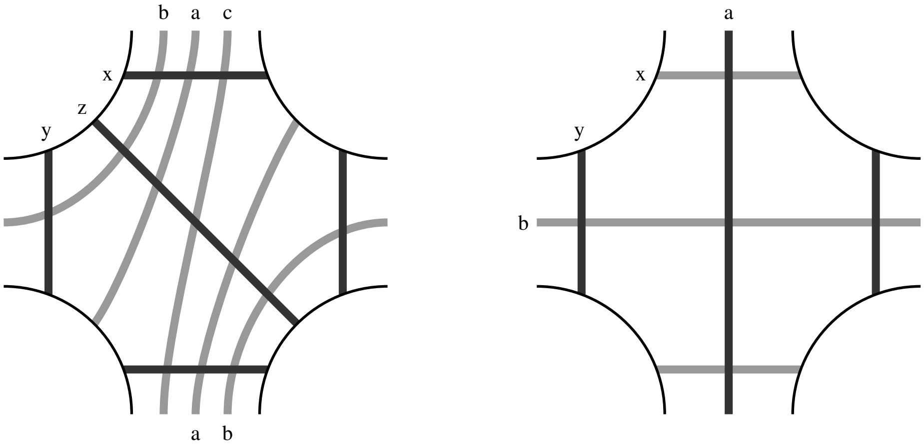


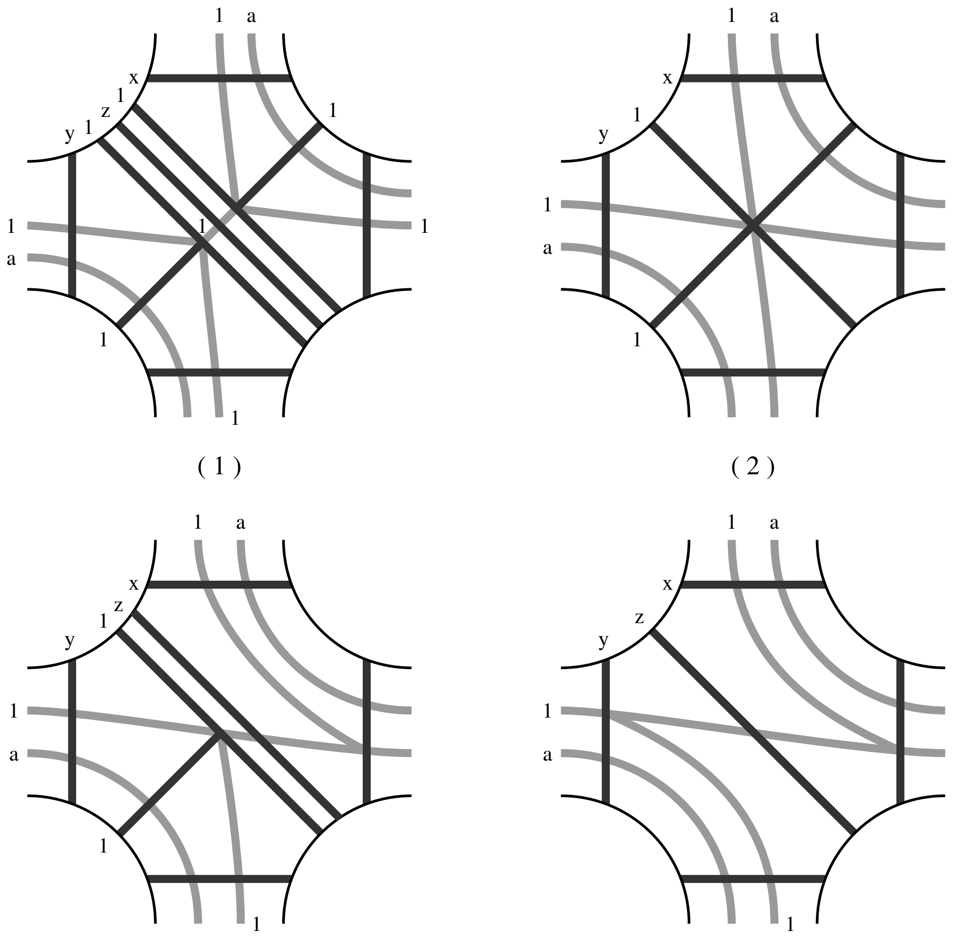
Figure 2



(1)

Figure 3

(2)



(1) (2)

(3) Figure 4 (4)

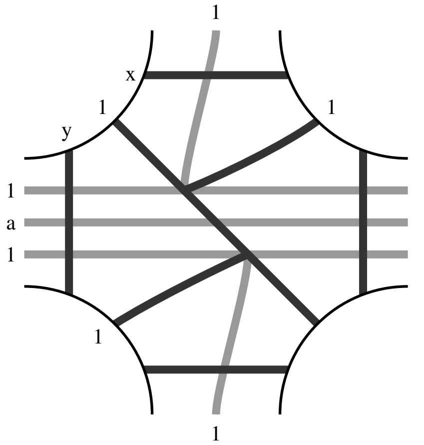


Figure 5

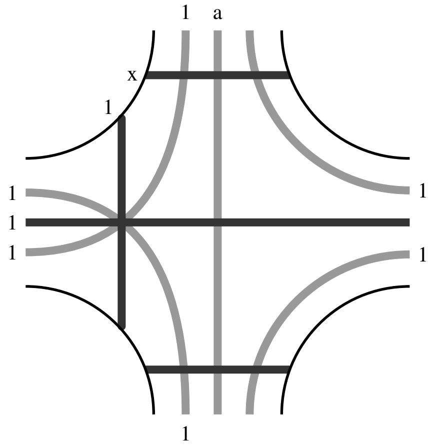


Figure 6