ON A FIBERING STRUCTURE OF THE FERMAT SURFACE

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ABSTRACT. A fibering structure over $\mathbb{C}P^1$ will be introduced to the degree n Fermat surface V_n. The exact positions and the types of all the sigular fibers of this fibration will be determined on the assumption that $n \not\equiv 1 \pmod 6$. In the excluded case $n \equiv 1 \pmod 6$, we have no general result. We have in this case, however, a recipe for computing the types and the positions of all the singular fibers in the fibration $V_n \to \mathbb{C}P^1$ each time a concrete number n is given.

1. INTRODUCTION

How can we "see" the shape of a 4-dimensional manifold? This is our general problem.

Probably a promising method would be to introduce a fibering structure to a given manifold with surface fibers and a surface base space. Of course, it is too optimistic to expect always a fiber bundle, but if singular fibers are allowed to exist, we have more chance to construct a fibration on the manifold. Assume that we have such a fibering structure. If we know the positions and the types of all the singular fibers in this fibration, and thoroughly understand the global monodromy, then it might be said that we see the shape of the 4-manifold.

In this talk, we will take up a standard family of 4-manifolds, and will study their topological structure via certain fibrations. The manifolds which we want to study are the Fermat surfaces.

The *Fermat surface of degree* n, denoted by V_n, is a nonsingular complex hypersurface of $\mathbb{C}P^3$ defined by

$$z_0^n + z_1^n + z_2^n + z_3^n = 0 \tag{1}$$

where z_0, z_1, z_2, z_3 are homogeneous coordinates of $\mathbb{C}P^3$. For later convenience, we will make obvious coordinate change and will consider that V_n is defined by

$$z_0^n - z_1^n = z_2^n - z_3^n. \tag{2}$$

From topological viewpoint, V_n is a compact, simply connected, smooth 4-manifold.

The Chern numbers of V_n are known (see §1.3 of [2]):

$$c_1^2(V_n) = n(n-4)^2 \text{ and } c_2(V_n) = n(n^2 - 4n + 6)\ . \tag{3}$$

The Euler number χ and the signature of V_n are calculated as follows:

$$\chi(V_n) = c_2(V_n) = n(n^2 - 4n + 6) \tag{4}$$

$$\mathrm{Sign}(V_n) = \frac{1}{3}(c_1^2(V_n) - 2c_2(V_n)) = -\frac{1}{3}n(n^2 - 4)\ . \tag{5}$$

Also the following proposition is well-known:

Proposition 1. (*see Lemma 1.3.9 of* [2]) *V_n is a spin manifold if and only if $n \equiv 0 \pmod 2$.*

The degree 1 surface V_1 is nothing but the complex projective plane $\mathbb{C}P^2$. Thus from now on we will assume $n \geqq 2$.

Now we will introduce a (holomorphic) fibering structure

$$f : V_n \to \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\} \tag{6}$$

by setting

$$f = \frac{z_0 - z_1}{z_2 - z_3} = \frac{\sum_{k=0}^{n-1} z_2^{n-1-k} z_3^k}{\sum_{k=0}^{n-1} z_0^{n-1-k} z_1^k} . \tag{7}$$

We may write in a simpler form

$$f = \begin{cases} z_2^{n-1}/z_0^{n-1} & \text{if } z_0 = z_1 \text{ and } z_2 = z_3 \\ (z_0 - z_1)/(z_2 - z_3) & \text{otherwise.} \end{cases} \tag{8}$$

Note that the first expression of (8) makes sense at the place where the second becomes meaningless. Thus the projection f is globally well-defined over V_n.

This fibration was suggested by Shioda's paper [5], in §4 of which a closely related but slightly different fibration of the Fermat surface of degree 4 was constructed.

Our main results will give the types and positions of all the singular fibers appearing in the above fibration. To understand the full topology of the fibration $V_n \to \hat{\mathbb{C}}$ we have to determine the *global* monodromy, which we know, however, only for $n \leqq 5$. See [1], [3].

2. Preliminary Observations

The fibration $f : V_n \to \mathbb{C}P^1$ defined by (7) has rich symmetry. In what follows, we will identify $\mathbb{C}P^1$ with the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Proposition 2. *Let* inv : $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *denote the biholomorphic mapping $\sigma \mapsto 1/\sigma$. Then there exists a biholomorphic mapping $H : V_n \to V_n$ which commutes the following diagram*

$$\begin{array}{ccc} V_n & \xrightarrow{H} & V_n \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \hat{\mathbb{C}} & \xrightarrow[\text{inv}]{} & \hat{\mathbb{C}} \end{array}$$

Proof. We have only to define

$$H(z_0 : z_1 : z_2 : z_3) = (z_2 : z_3 : z_0 : z_1) . \tag{9}$$

□

A cyclic group of order n, $G_n = \{\xi \in \mathbb{C} \mid \xi^n = 1\}$, acts on V_n and on $\hat{\mathbb{C}}$ via

$$\xi \cdot (z_0 : z_1 : z_2 : z_3) = (\xi z_0 : \xi z_1 : z_2 : z_3) \tag{10}$$

and

$$\xi : \sigma \mapsto \xi\sigma . \tag{11}$$

Then the following proposition is immediate.

Proposition 3. *The actions of G_n on V_n and on $\hat{\mathbb{C}}$ are equivariant, i.e., the following diagram commutes for each $\xi \in G_n$:*

$$\begin{array}{ccc} V_n & \xrightarrow{\xi \cdot} & V_n \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \hat{\mathbb{C}} & \xrightarrow[\xi]{} & \hat{\mathbb{C}} \end{array}$$

If n is even, then a cyclic group of order $2n$, $G_{2n} = \{\xi \in \mathbb{C} \mid \xi^{2n} = 1\}$, acts on V_n similarly.

It is easy to see that for $n = 2$ the fiberation $f : V_2 \to \hat{\mathbb{C}}$ has no critical points, thus it is a fiber bundle. The fiber is a complex projective line. Since V_2 is a spin manifold (by Proposition 1), it is diffeomorphic to $S^2 \times S^2$. (In fact, V_2 is biholomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$.)

In what follows, we will assume $n \geqq 3$.

Let us study the fibers of the fibration $f : V_n \to \hat{\mathbb{C}}$ over 0 and $\infty \in \hat{\mathbb{C}}$.

Proposition 4. *The fibers $f^{-1}(0)$ and $f^{-1}(\infty)$ each consist of $n-1$ complex projective lines intersecting each other transversely at a point. (Thus if $n \geqq 3$, they are singular fibers.)*

Proof. By Proposition 2, we have only to examine the fiber $f^{-1}(0)$. By (7), the fiber $f^{-1}(0)$ is given by

$$z_0 = z_1 \quad \text{and} \quad \sum_{k=0}^{n-1} z_2^{n-k-1} z_3^k = \prod_{i=1}^{n-1} (z_2 - \xi^i z_3) = 0 \tag{12}$$

where ξ is a primitive n-th root of unity. Thus this fiber consists of $n-1$ complex projective lines intersecting transversely at a point, $(1 : 1 : 0 : 0)$. □

Proposition 5. *A general fiber of $f : V_n \to \hat{\mathbb{C}}$ has genus $\frac{1}{2}(n-2)(n-3)$.*

Proof. By (7), the fiber $f^{-1}(\sigma)$ over $\sigma \in \hat{\mathbb{C}}$ is given by

$$z_0 - z_1 = \sigma(z_2 - z_3) \quad \text{and} \quad \sum_{k=0}^{n-1} z_0^{n-k-1} z_1^k = \sigma\Big(\sum_{k=0}^{n-1} z_2^{n-k-1} z_3^k\Big) \ . \tag{13}$$

If σ is generic, this is a nonsingular plane curve of degree $n-1$. Thus it has genus $\frac{1}{2}(n-2)(n-3)$ by the Plücker formula. □

Remark. The topological monodromy around the singular fiber $f^{-1}(0)$ is a periodic map of order $n-1$. See [4]. The singular fiber $f^{-1}(\infty)$ has, of course, the same monodromy.

Let C_0 denote the comlex projective line in $\mathbb{C}P^3$ defined by

$$z_0 = z_1 \quad \text{and} \quad z_2 = z_3 \ . \tag{14}$$

This is contained in V_n, and is a common "binding"of the planes $z_0 - z_1 = \sigma(z_2 - z_3)$, $\sigma \in \mathbb{C} \setminus \{0\}$. By (8), the projection f restricted to C_0 is given by

$$f(z_0 : z_0 : z_2 : z_2) = \left(\frac{z_2}{z_0}\right)^{n-1} \ . \tag{15}$$

Thus $f|C_0 : C_0 \to \hat{\mathbb{C}}$ is an $(n-1)$-fold branched covering, branched at $\{0, \infty\} \subset \hat{\mathbb{C}}$. If $\sigma \neq 0, \infty$, the (singular or nonsingular) fiber $f^{-1}(\sigma)$ is a curve of degree $n-1$ on the plane $z_0 - z_1 = \sigma(z_2 - z_3)$. The line C_0 is on the same plane, and intersects

$f^{-1}(\sigma)$ in $n-1$ points, $\{(z_0 : z_0 : z_2 : z_2) \,|(z_2/z_0)^{n-1} = \sigma\}$. The number $n-1$ is the maximum number of the geometric intersection points between a complex line and a plane curve of degree $n-1$. Thus the intersections are necessarily transverse. This proves the following:

Proposition 6. *If $\sigma \neq 0, \infty$, the projection f has no critical points on $f^{-1}(\sigma) \cap C_0$.*

Corollary 6.1. *The fibration $f : V_n \to \hat{\mathbb{C}}$ has no multiple fibers.*

This corollary follows from Propositions 6 and 4.

3. Main results

We come to a position to state our main results. For n with $3 \leqq n \leqq 5$, the fibering structure needs a special description, because n is still rather small. In this section, we will assume $n \geqq 6$.

Our main theorem is the following:

Theorem 1. *If n $(\geqq 6)$ is even, the fibration $f : V_n \to \hat{\mathbb{C}}$ has three types of singular fibers:*
(I) *two singular fibers over $\{0, \infty\}$, each consisting of $n-1$ complex projective lines intersecting each other in a common point,*
(II) *$2n$ singular fibers over $2n$-th roots of 1, each consisting of a smooth irreducible plane curve of degree $n-2$ and a complex projective line which intersects the degree $n-2$ curve in $n-2$ points, and*
(III) *for each ordered pair (k, l) of distinct integers with $1 \leqq k,\ l \leqq \frac{n-2}{2}$, there are $2n$ singular fibers over $2n$-th roots of*

$$\left(\frac{\sin \frac{k\pi}{n-1}}{\sin \frac{l\pi}{n-1}} \right)^{2(n-1)}$$

each being an irreducible plane curve of degree $n-1$ which has two nodes. The vanishing cycles corresponding to the two nodes are nonseparating, mutually non-homologous simple closed curves.

The above list of singular fibers is complete.

If n $(\geqq 7)$ is odd, we additionally assume $n \not\equiv 1 \pmod 6$ if $n \geqq 13$. Then the fibration $V_n \to \hat{\mathbb{C}}$ has four types of singular fibers:
(I) *two singular fibers over $\{0, \infty\}$, each consisting $n-1$ complex projective lines intersecting each other in a common point,*
(II) *n singular fibers over n-th roots of 1, each consisting of a smooth irreducible plane curve of degree $n-3$ and two complex projective lines each of which intersects the plane curve of degree $n-3$ in $n-3$ points, and intersects the other line in a point, and*
(III) *for each integer k with $1 \leqq k < \frac{n-1}{2}$, there are $2n$ singular fibers over the union of n-th roots of*

$$(-1)^{\frac{n-1}{2}+k} \left(\sin \frac{k\pi}{n-1} \right)^{n-1}$$

and n-th roots of

$$(-1)^{\frac{n-1}{2}+k} \left(\frac{1}{\sin \frac{k\pi}{n-1}} \right)^{n-1}$$

each being an irreducible plane curve of degree $n-1$ which has two nodes. The vanishing cycles corresponding to the two nodes are nonsparating, mutually non-homologous simple closed curves, and
(IV) *for each orderd pair (k,l) of distinct integers with $1 \leqq k,l < \frac{n-1}{2}$, there are n singular fibers over n-th roots of*

$$(-1)^{k-l}\left(\frac{\sin\frac{k\pi}{n-1}}{\sin\frac{l\pi}{n-1}}\right)^{n-1}$$

each being an irreducible plane curve of degree $n-1$ which has four nodes. The vanishing cycles corresponding to the four nodes are nonseparating, mutually non-homologous simple closed curves.

The above list of singular fibers is complete.

Remark. In the case where $n \equiv 1 \pmod 6$ and $n \geqq 13$, certain "confluence"of some singular fibers occurs to produce more complicated singular fibers. This confluence is controlled by the cyclotomic polynomial.

4. Critical points of f

In this section, we will assume $\sigma \neq 0,\ \infty$, and will find all the critical points of f over σ.

We can prove that when seeking the critical points we may assume

$$z_0 \neq 0 \tag{16}$$

and look for the critical points in this affine space $\mathbb{A}_0 := \mathbb{C}P^3 \setminus \{z_0 = 0\}$.

Let us introduce the affine coordinates (x_1, x_2, x_3) to the space $\mathbb{A}_0$ by

$$x_i = \frac{z_i}{z_0}, \quad i = 1, 2, 3\ . \tag{17}$$

We fix a complex number $\sigma \in \mathbb{C} \setminus \{0\}$, and examine whether there exists a critical point on the fiber $f^{-1}(\sigma)$.

For convenience, we set

$$g = 1 - x_1^n - x_2^n + x_3^n\ . \tag{18}$$

Then $f^{-1}(\sigma) \cap \mathbb{A}_0$ is given by two equations:

$$x_3 = x_2 + \frac{1}{\sigma}(x_1 - 1) \tag{19}$$

$$g = 1 - x_1^n - x_2^n + x_3^n = 0\ . \tag{20}$$

The first equation (19) comes from

$$f = \frac{1 - x_1}{x_2 - x_3} = \sigma \tag{21}$$

and the second one (20) is the defining equation of the affine Fermat surface $V_n \cap \mathbb{A}_0$.

Fix a number σ ($\in \mathbb{C} \setminus \{0\}$). We will make our computation always on the "σ-plane "(19), to which we will introduce (x, y) coordinates by setting

$$x = x_1, \quad \text{and} \quad y = x_2\ . \tag{22}$$

Then we substitute (19) into (20), and rewrite the latter in (x, y) coordinates:

$$g = 1 - x^n - y^n + (y + \frac{1}{\sigma}(x-1))^n = 0\ . \tag{23}$$

Note that $x = 1$ satisfies the equation (23), but this line is not a component of $f^{-1}(\sigma) \cap \mathbb{A}_0$. In fact, the line $x = 1$ is nothing but the line C_0 of Proposition 6. By the same proposition, the fiber $f^{-1}(\sigma)$ intersects this line transversely, and has no singular points there. Thus in what follows, we may assume

$$x \neq 1 \ . \tag{24}$$

In search of critical points of f, we differentiate:

$$\frac{\partial g}{\partial x} = -nx^{n-1} + \frac{n}{\sigma}(y + \frac{1}{\sigma}(x-1))^{n-1} = 0 \ . \tag{25}$$

Now let us introduce a complex number τ satisfying

$$\sigma = \tau^{n-1} \ . \tag{26}$$

The number τ has ambiguity of $(n-1)$-th roots of unity, but this will affect nothing in the final results.

From (25), we have

$$y + \frac{1}{\sigma}(x-1) = \tau x \tag{27}$$

or equivalently

$$(\frac{1}{\tau^n} - 1)x + \frac{1}{\tau}y = \frac{1}{\tau^n} \ . \tag{28}$$

Also differentiating:

$$\frac{\partial g}{\partial y} = -ny^{n-1} + n(y + \frac{1}{\sigma}(x-1))^{n-1} = 0 \tag{29}$$

we have

$$y = \zeta(y + \frac{1}{\sigma}(x-1)) \tag{30}$$

or equivalently,

$$\frac{\zeta}{\tau^{n-1}}x + (\zeta - 1)y = \frac{\zeta}{\tau^{n-1}} \ , \tag{31}$$

where ζ may be any $(n-1)$-th root of unity other than 1. (If $\zeta = 1$, the equation (30) would imply $x = 1$, which would violate our assumption (24).)

We will solve the simultaneous equation (28) and (31). For this purpose, we calculate the determinant of the coefficient matrix:

$$\begin{vmatrix} \frac{1}{\tau^n} - 1 & \frac{1}{\tau} \\ \frac{\zeta}{\tau^{n-1}} & \zeta - 1 \end{vmatrix} = 1 - \frac{1}{\tau^n} - \zeta \ . \tag{32}$$

Lemma 1.

$$1 - \frac{1}{\tau^n} - \zeta \neq 0$$

Proof. Sustituting (27) into (23), we have

$$g = 1 - x^n - y^n + \tau^n x^n = 0 \ . \tag{33}$$

From (27) and (30), we have

$$y = \zeta\tau x \ . \tag{34}$$

Substituting (34) into (33), we obtain

$$g = 1 - x^n - \zeta^n \tau^n x^n + \tau^n x^n \tag{35}$$
$$= 1 - \tau^n x^n \left(\frac{1}{\tau^n} + \zeta - 1\right) \qquad (\because \zeta^n = \zeta\,)$$
$$= 0\,.$$

Thus

$$\tau^n x^n \left(\frac{1}{\tau^n} + \zeta - 1\right) = 1\,. \tag{36}$$

This proves Lemma 1. □

We solve the simultaneous equation (28) and (31):

$$x = \frac{\begin{vmatrix} \frac{1}{\tau^n} & \frac{1}{\tau} \\ \frac{\zeta}{\tau^{n-1}} & \zeta - 1 \end{vmatrix}}{1 - \frac{1}{\tau^n} - \zeta} = \frac{\frac{1}{\tau^n}(\zeta - 1) - \frac{\zeta}{\tau^n}}{1 - \frac{1}{\tau^n} - \zeta} = \frac{1}{1 - \tau^n(1 - \zeta)} \tag{37}$$

$$y = \frac{\begin{vmatrix} \frac{1}{\tau^n} - 1 & \frac{1}{\tau^n} \\ \frac{\zeta}{\tau^{n-1}} & \frac{\zeta}{\tau^{n-1}} \end{vmatrix}}{1 - \frac{1}{\tau^n} - \zeta} = \frac{(\frac{1}{\tau^n} - 1)\frac{\zeta}{\tau^{n-1}} - \frac{1}{\tau^n}\cdot\frac{\zeta}{\tau^{n-1}}}{1 - \frac{1}{\tau^n} - \zeta} = \frac{\zeta\tau}{1 - \tau^n(1 - \zeta)}\,. \tag{38}$$

Substituting (37) into (36), we have

$$\left(\frac{1}{1 - \tau^n(1 - \zeta)}\right)^{n-1} = 1\,. \tag{39}$$

Thus

$$1 - \tau^n(1 - \zeta) = \zeta'\,, \tag{40}$$

where ζ' is an $(n-1)$-th root of unity, which is not 1 because $\zeta \neq 1$.

Substituting (40) into (37) and (38), we have shown that a critical point of f over σ must satisfy

$$(x,\ y) \ = \ \left(\frac{1}{\zeta'}\,,\ \frac{\zeta\tau}{\zeta'}\right)\,, \tag{41}$$

where the numbers σ, τ, ζ, and ζ' are related as follows:

$$\sigma = \tau^{n-1}\,, \qquad \text{and} \tag{42}$$

$$\tau^n = \frac{1 - \zeta'}{1 - \zeta}\,, \qquad \zeta,\ \zeta' \ \in \{(n-1)\text{-th roots of unity}\} \smallsetminus \{1\}\,. \tag{43}$$

Conversely, we can prove that any such point $(x, y) = (\frac{1}{\zeta'}, \frac{\zeta\tau}{\zeta'})$ is a critical point of f over σ, and that it is a node of the fiber $f^{-1}(\sigma)$.

Now we have the following theorem:

Theorem 2. *A point (x,y) on the σ-plane, with $\sigma \neq 0, \infty$, is a critical point of $f : V_n \to \hat{\mathbb{C}}$, if and only if there is a triple of numbers (ζ, ζ', τ) satisfying*

$$\sigma = \tau^{n-1}, \quad \tau^n = \frac{1 - \zeta'}{1 - \zeta} \qquad (\zeta,\ \zeta' \ \in \{(n-1)\text{-}th\ roots\ of\ unity\} \smallsetminus \{1\})\,, \tag{44}$$

and the point is given by

$$(x, y) \ = \ \left(\frac{1}{\zeta'}, \frac{\zeta\tau}{\zeta'}\right)\,. \tag{45}$$

Moreover, this critical point is a node of $f^{-1}(\sigma)$.

In terms of the affine coordinates (x_1, x_2, x_3) of $\mathbb{A}_0$, the critical point (45) is given by

$$(x_1, x_2, x_3) = (\frac{1}{\zeta'}, \frac{\zeta\tau}{\zeta'}, \frac{\tau}{\zeta'}) . \tag{46}$$

One obtains (46) by combining (19) with (27) and (45).

Thus in terms of the homogeneous coordinates $(z_0 : z_1 : z_2 : z_3)$ of V_n, this critical point is given by

$$(z_0 : z_1 : z_2 : z_3) \ = \ (\zeta' : 1 : \zeta\tau : \tau) . \tag{47}$$

Theorem 3. *All the critical points of $f : V_n \to \hat{\mathbb{C}}$ over $\sigma \neq 0, \infty$, are given by*

$$(\zeta' : 1 : \zeta\tau : \tau) , \tag{48}$$

where (ζ, ζ', τ) is a triple satisfying (44). Moreover, all of these critical points are nodes of $f^{-1}(\sigma)$.

Thus all critical points of $f : V_n \to \mathbb{C}P^1$ over $\mathbb{C}P^1 \setminus \{0, \infty\}$ are nodes, and the number of them is $n(n-2)^2$. Our task will be to determine which critical points lie on the same fiber.

Since ζ and ζ' are $(n-1)$-th roots of unity other than 1, we may set

$$\zeta = \exp(\frac{2a\pi\sqrt{-1}}{n-1}), \quad \text{and} \quad \zeta' = \exp(\frac{2b\pi\sqrt{-1}}{n-1}) , \tag{49}$$

where a and b are integers with $1 \leqq a,\ b \leqq n-2$.

Since

$$\sigma^n = \left(\frac{1-\zeta'}{1-\zeta}\right)^{n-1} , \qquad \zeta,\ \zeta' \ \in \{(n-1)\text{-th roots of unity}\} \setminus \{1\} . \tag{50}$$

we have

$$\sigma^n = (-1)^{b-a} \left(\frac{\sin(\frac{b\pi}{n-1})}{\sin(\frac{a\pi}{n-1})}\right)^{n-1} . \tag{51}$$

Therefore, to our mentioned task, the following problem is essential (in which we set $m = n-1$ for notational simplicity):

Problem. Let $a,\ b,\ c,\ d$ are integers satisfying $1 \leqq a,\ b,\ c,\ d \ \leqq [m/2]$ with $a \neq b$ and $c \neq d$. Suppose

$$\frac{\sin(\frac{a\pi}{m})}{\sin(\frac{b\pi}{m})} = \frac{\sin(\frac{c\pi}{m})}{\sin(\frac{d\pi}{m})} , \tag{52}$$

then is it true that $a = c$ and $b = d$?

The author gave a talk at the 12th MSJ-IRI Simposium (Singularity Theory and Its Applications) in Sapporo, September 2003, assuming as a conjecture that the answer is affirmative. Immediately after the talk, Prof.C.T.C.Wall suggested a method to solve the problem using cyclotomic integers. A few weeks later, Dr.Pho Duc Tai at Hokkaido University informed to the author that he made a search using Maple software, and found infinitely many counter-examples in the case where $m \equiv 0 \pmod 6$ and $m \geqq 12$. The author informed of this to Prof.Wall. In his reply, he kindly explained more details of his argument. However, before working out Prof. Wall's argument, the author received the information (in August 2004, through Profs. J.H. Keum and K.Oguiso) that Dr. Sung Woo Choi at KIAS solved the problem affirmatively under the assumtion $m \not\equiv 0 \pmod 6$. His solution uses the Chebyshev polynomials of the second kind, and is very clear and "elementary". Thus at present the only proof of the following theorem which the author has worked out is Dr. Choi's one:

Theorem 4. *If* $m \not\equiv 0 \pmod 6$*, the problem is affirmative.*

Our main Theorem 1 depends on this theorem. In the remainig case where $m \equiv 0 \pmod 6$ and $m \geqq 12$, if a concrete number m is given, we can calculate all the "exceptional" quadruplets (a, b, c, d) for which (52) holds, by using cyclotomic integers as Prof.Wall suggested. However, the author has not yet succeeded in finding out general rules governing the occurence of such exceptional quadruplets.

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