TOPOLOGY OF MANIFOLDS AND SINGULARITIES OF DIFFERENTIABLE MAPS

DEDICATED TO PROFESSOR YUKIO MATSUMOTO ON THE OCCASION OF HIS 60TH BIRTHDAY

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Abstract. This is a survey talk on recent developments in the global theory of singularities of differentiable maps. We give several remarkable results concerning differentiable structures of manifolds and singularities of differentiable maps. A generalization of the theory of Morse functions to generic maps is also discussed.

I would like to thank Professor Yukio Matsumoto for having introduced me, as my supervisor, the wonderful world of Topology and Singularities. His encouragement has always made me happy and given me a lot of energy to pursue my work. It is therefore a great pleasure for me to talk at this conference celebrating his 60th birthday.

1. Introduction

This is a survey talk on recent developments in the global theory of singularities of differentiable maps.[1] In this talk, all manifolds and maps are differentiable of class C^∞: i.e. we work in the smooth category.

The first problem that we consider is the following.

Problem 1.1. Given two smooth manifolds M and N, what is the minimal set of singularities that appear for generic smooth maps $M \to N$?

Here, a *generic* map refers to a C^∞ *stable map.* Recall that C^∞ stable maps are characterized by the following property (for details, see [9, 1, 5]):

(1) any approximation of a C^∞ stable map f coincides with f up to diffeomorphisms of the source and the target.

Furthermore, they satisfy the following:

(2) if the dimension pair $(\dim M, \dim N)$ is in the nice range in the sense of Mather [14] (for example, if $\dim N \leq 5$), then any map $f : M \to N$ can be approximated by a C^∞ stable map.

Let $C^\infty(M, N)$ denote the space of all smooth maps of M into N endowed with the Whitney C^∞ topology. Furthermore, let $S^\infty(M, N)$ be the subspace of $C^\infty(M, N)$ consisting of the C^∞ stable maps. Then property (1) (or (2)) above implies that $S^\infty(M, N)$ is open (resp. dense) in $C^\infty(M, N)$.

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[1] For developments until around 1996, refer to the author's survey [30].

For a smooth map $f : M \to N$, a point $p \in M$ is a *singular point* of f if its differential $df_p : T_pM \to T_{f(p)}N$ has rank strictly smaller than $\min\{\dim M, \dim N\}$. Singularities of C^∞ stable maps have been extensively studied. For example, there are classification results by means of algebraic invariants (for example, see [4]). For low dimensions, it is a routine work to obtain a list of singularities that appear for C^∞ stable maps (see [9]).

Remark 1.2. We warn the reader that a branched covering map is NOT generic in most situations. Therefore, those nice maps which are studied in [11, Chap. 7] are not generic in our sense.

The essential point of Problem 1.1 is the following. If we allow all the singularities that can appear for C^∞ stable maps, then by virtue of property (2) above, there always exists a generic map $M \to N$ which has only the allowed singularities. However, if we restrict the set of allowed singularities, then there may not exist a generic map $M \to N$ having only the restricted set of singularities. In fact, we will see later that such phenomena do occur and that they sometimes reflect the differentiable structures of manifolds.

Let us start by considering Problem 1.1 in some specific situations.

Example 1.3. If $M = N$, then clearly we need no singularities. More generally, if M is a covering space of N, then the same holds. Even more generally, if M is the total space of a C^∞ fiber bundle over N, or if M immerses into N, then we need no singularities.

Example 1.4. Let M be a smooth closed manifold of dimension m and set $N = \mathbf{R}$, the real line. Then it is known that a smooth map (i.e. a smooth function) $f : M \to \mathbf{R}$ is generic if and only if the following two hold (see [9]).

(1) The critical points (= singular points) of f are all nondegenerate: i.e. f is a *Morse function*.
(2) Distinct critical points have distinct values.

Note that nondegenerate critical points are classified by their indices $0, 1, \dots, m$, according to the Morse Lemma (for example, see [15]). If we allow all the indices, then every manifold M admits such a smooth function as above. However, if we restrict the allowed indices, then not all manifolds can admit an allowed function. For example, if we prohibit critical points of index 1, then it is necessary that the manifold M is simply connected for it to admit a generic function with allowed singularities.[2] In this case, Problem 1.1 can be interpreted as follows. *What is the minimal set of indices that appear for Morse functions on a given manifold?* Note that this problem was completely solved for simply connected manifolds of dimensions greater than or equal to 6 by Smale [38].

In Example 1.4, we cannot exclude the index 0 nor m, since any Morse function on a closed manifold attains its minimum and maximum and the corresponding points are critical points of index 0 and m, respectively. In the extremal case where we allow indices 0 and n only, we have the following theorem due to Reeb [23].

Theorem 1.5. *If a smooth closed connected m-dimensional manifold M admits a Morse function $f : M \to \mathbf{R}$ having only critical points of indices 0 and m, then M is necessarily homeomorphic to the m-dimensional sphere S^m.*

In fact, the above theorem can be refined as follows.

[2]For dimensions $m \neq 3, 4$, this is also sufficient. For $m = 3$, it is sufficient if and only if the Poincaré conjecture is affirmative. For $m = 4$, it is still an open problem whether it is sufficient or not.

Theorem 1.6. *Let M be a smooth closed connected manifold of dimension m. Then there exists a Morse function $f : M \to \mathbf{R}$ having only critical points of indices 0 and m if and only if the following holds.*

(1) *For $m \leq 6$, M is diffeomorphic to the standard sphere S^m.*
(2) *For $m \geq 7$, M is a homotopy m-sphere.*

The above theorem follows from various deep results due to Smale [38], Cerf [3], etc. Note that for $m \geq 5$, homotopy m-spheres can be classified (see [12] or [39]). Note also that a homotopy m-sphere with $m \geq 7$ is always homeomorphic to S^m, but may not be diffeomorphic to S^m. So the existence of a Morse function with only critical points of extremal indices cannot detect the smooth structures for dimensions $m \geq 7$, in general.

Example 1.7. Let us consider an equi-dimensional case. Let M and N be smooth closed manifolds of the same dimension m. Let us assume that there exists a homotopy equivalence $h : M \to N$. Then we can approximate h by a smooth map and then by a C^∞ stable map. It is known that in most cases, such an approximation can be chosen so that it has only the fold singularities. A singular point $p \in M$ of a smooth map $f : M \to N$ is a *fold singular point* if there exist local coordinates $(x_1, x_2, \ldots, x_m)$ around p and $(y_1, y_2, \ldots, y_m)$ around $f(p)$ such that f has the form

$$y_i \circ f = \begin{cases} x_i, & 1 \leq i \leq m-1, \\ x_m^2, & i = m. \end{cases}$$

For example, every homotopy equivalence between homotopy m-spheres can be approximated by a smooth map with only fold singularities [6].[3] Of course, no singularity is necessary if and only if the homotopy m-spheres are diffeomorphic.

2. Detecting differentiable structures of spheres

As we have seen in Theorems 1.5 and 1.6, the existence of Morse functions with only critical points of extremal indices cannot detect the differentiable structures on spheres. Instead of using functions (i.e. maps into $\mathbf{R}$), let us consider maps into $\mathbf{R}^n$, $n \geq 1$, with the following singularities.

Definition 2.1. Let $f : M \to N$ be a smooth map between smooth manifolds, where $m = \dim M \geq \dim N = n$. A singular point $p \in M$ is a *fold singular point* if there exist local coordinates $(x_1, x_2, \ldots, x_m)$ around p and $(y_1, y_2, \ldots, y_n)$ around $f(p)$ such that f has the form

$$y_i \circ f = \begin{cases} x_i, & 1 \leq i \leq n-1, \\ \pm x_n^2 \pm x_{n+1}^2 \pm \cdots \pm x_m^2, & i = n. \end{cases}$$

If all the signs appearing on the right hand side of $y_n \circ f$ are the same, then we call p a *definite* fold singular point, otherwise an *indefinite* fold singular point.

It is known that any smooth map $f : M \to N$ exhibits definite fold singular points, provided that M is closed and N is open.

A smooth map $f : M \to N$ is a *fold map* if f has only fold singular points (see [8, 27]). Furthermore, a fold map f is a *special generic map* if it has only definite fold singular points (see [2, 22, 28]).

Example 2.2. Let us consider the case where $N = \mathbf{R}$. Then, a smooth map $f : M \to \mathbf{R}$ is a fold map if and only if it is a Morse function. Furthermore, it is a special generic map if and only if its critical points have extremal indices. When $m = n$, a fold map is automatically a special generic map.

[3] We warn the reader that such a map is NOT a branched covering in general, since the restriction to the set of the fold singular points may not be an embedding.

Example 2.3. (1) Let S^m be the unit sphere in $\mathbf{R}^{m+1}$. For $n = 1, 2, \ldots, m$, the standard projection $\pi : \mathbf{R}^{m+1} \to \mathbf{R}^n$ restricted to S^m, $f = \pi|_{S^m} : S^m \to \mathbf{R}^n$, is easily seen to be a special generic map.

(2) If $f : M \to \mathbf{R}^n$ is a special generic map, and Q is a submanifold of $\mathbf{R}^{n+k}$ of codimension n with trivial normal bundle, then the composition

$$Q \times M \xrightarrow{\mathrm{id}_Q \times f} Q \times \mathbf{R}^n \hookrightarrow \mathbf{R}^{n+k}$$

is a special generic map, where the last map is the composition of a trivialization of the open tubular neighborhood of Q in $\mathbf{R}^{n+k}$ and the inclusion. Thus, for example, $S^p \times S^q$ admits special generic maps into $\mathbf{R}^\ell$ with $\min\{p, q\} + 1 \le \ell \le p + q$. We can show that it cannot admit any special generic maps into $\mathbf{R}^\ell$ with $\ell \le \min\{p, q\}$ (see [28, 29]).

In view of these examples, the following definition seems reasonable.

Definition 2.4. Let M be a smooth closed manifold of dimension m. We denote by $\mathcal{S}(M)$ (or $\mathcal{F}(M)$) the set of all integers n such that $1 \le n \le m$ and M admits a special generic map (resp. fold map) into $\mathbf{R}^n$. Note that we always have

$$\mathcal{S}(M) \subset \mathcal{F}(M) \subset \{1, 2, \ldots, \dim M\}$$

and that $\mathcal{S}(M)$ and $\mathcal{F}(M)$ are diffeomorphism invariants of M.

By Example 2.3, we have

$$\mathcal{S}(S^m) = \{1, 2, \ldots, m\}$$

and

$$\mathcal{S}(S^p \times S^q) = \{\min\{p, q\} + 1, \min\{p, q\} + 2, \ldots, p + q\}.$$

In [28, 29], the author showed the following.

Theorem 2.5. *A smooth closed connected manifold M of dimension m is diffeomorphic to S^m if and only if $\mathcal{S}(M) = \{1, 2, \ldots, m\}$.*

In other words, the standard m-sphere can be characterized as a smooth manifold for which the definite fold singularity is the minimal set of singularities for generic maps into the Euclidean spaces of dimensions $\le m$. For example, if Σ^7 is a 7-dimensional exotic sphere [16], then we have $\{1, 2, 7\} \subset \mathcal{S}(\Sigma^7) \subset \{1, 2, 3, 7\}$ (see [28]).

As the above theorem suggests, special generic maps are closely related to differentiable structures on spheres. Let us introduce the following definition.

Definition 2.6. Two special generic maps $f : M \to \mathbf{R}^n$ and $g : N \to \mathbf{R}^n$ of m-dimensional closed manifolds are said to be *cobordant* if there exist a compact $(m+1)$-dimensional manifold V with $\partial V = M \amalg N$ and a smooth map $F : V \to \mathbf{R}^n \times [0, 1]$ with only definite fold singularities such that

$$\begin{aligned} F|_{M \times [0,\varepsilon)} &= f \times \mathrm{id}_{[0,\varepsilon)} : M \times [0, \varepsilon) \to \mathbf{R}^n \times [0, \varepsilon), \quad \text{and} \\ F|_{N \times (1-\varepsilon,1]} &= g \times \mathrm{id}_{(1-\varepsilon,1]} : N \times (1 - \varepsilon, 1] \to \mathbf{R}^n \times (1 - \varepsilon, 1] \end{aligned}$$

for some sufficiently small $\varepsilon > 0$, where we identify the collar neighborhoods of M and N in V with $M \times [0, \varepsilon)$ and $N \times (1-\varepsilon, 1]$ respectively. It is easy to show that this is an equivalence relation and that the set of cobordism classes of all special generic maps of closed (but not necessarily connected) m-dimensional manifolds into $\mathbf{R}^n$ forms an abelian group under disjoint union. This is called the *cobordism group of special generic maps* and is denoted by $\Gamma(m, n)$. We can also define the oriented version of this group: we just restrict ourselves to special generic maps of *oriented* closed manifolds and the manifold V should be oriented so that $\partial V = M \amalg (-N)$ in the above definition. Then the corresponding group is called the *oriented cobordism group of special generic maps* and is denoted by $\widetilde{\Gamma}(m, n)$.

In [31], the author showed the following.

Theorem 2.7. *Suppose $m \geq 6$. The group $\widetilde{\Gamma}(m,1)$ is isomorphic to Θ_m, the h-cobordism group of homotopy m-spheres. Furthermore, the group $\Gamma(m,1)$ is isomorphic to $\Theta_m \otimes \mathbf{Z}_2$.*

Recently Sadykov [26] showed that $\widetilde{\Gamma}(m,1)$ is in fact isomorphic to the mapping class group $\pi_0(\mathrm{Diff}^+ S^{m-1})$ of S^{m-1} for all $m > 0$.

3. 4-dimensional manifolds

Let us concentrate on 4-dimensional manifolds. In [28] the author showed the following (for other related results, see [37, 34, 35]).

Theorem 3.1. *Let M be a closed simply connected 4-manifold.*

1. *The 4-manifold M admits a special generic map into $\mathbf{R}$ (i.e. $1 \in \mathcal{S}(M)$) if and only if it is diffeomorphic to the standard 4-sphere S^4.*
2. *$2 \in \mathcal{S}(M)$ if and only if M is diffeomorphic to the standard 4-sphere S^4.*
3. *$3 \in \mathcal{S}(M)$ if and only if M is diffeomorphic to a connected sum of the form*
$$\Sigma^4 \sharp \left(\sharp^k S^2 \times S^2\right) \sharp \left(\sharp^\ell S^2 \widetilde{\times} S^2\right), \quad k, \ell \geq 0,$$
where $S^2 \widetilde{\times} S^2 \cong \mathbf{C}P^2 \sharp \overline{\mathbf{C}P^2}$ is the total space of the nontrivial S^2-bundle over S^2, and Σ^4 is a homotopy 4-sphere of the form $\partial(\Delta^3 \times D^2)$ for a compact contractible 3-manifold Δ^3.
4. *$4 \in \mathcal{S}(M)$ if and only if M is stably parallelizable: i.e. if and only if the second Stiefel-Whitney class $w_2(M) \in H^2(M; \mathbf{Z}_2)$ and the signature $\sigma(M) \in \mathbf{Z}$ of M both vanish.*

Using Theorem 3.1 (3), we obtain the following remarkable example.

Example 3.2. Let M be a simply connected spin[4] 4-manifold with vanishing signature. By Freedman [7] such a 4-manifold is homeomorphic to a connected sum M' of some copies of $S^2 \times S^2$.

Suppose that the second betti number $b_2(M)$ of M is big. If we can show that M is not diffeomorphic to a manifold of the form $(S^2 \times S^2) \sharp N$ for any smooth closed 4-manifold N, then M cannot admit a special generic map into $\mathbf{R}^3$ in view of Theorem 3.1 (3), while the connected sum M' of any copies of $S^2 \times S^2$ admits such a map. Such a 4-manifold M does exist. See [18, 19, 21].

In other words, we have $M \approx M'$, while
$$\mathcal{S}(M) = \{4\} \neq \{3, 4\} = \mathcal{S}(M'),$$
where the symbol "$\approx$" denotes a homeomorphism. This means that for generic smooth maps of M' into $\mathbf{R}^3$, the minimal set of singularities is the definite fold singularities, while for maps of M into $\mathbf{R}^3$, that is not enough, although M and M' are homeomorphic.

For fold maps, in contrast to special generic maps, we have the following (see [13, 32, 25]).

Theorem 3.3. *Let M be a closed connected oriented 4-manifold.*

1. *Every such 4-manifold M admits a fold map into $\mathbf{R}$, i.e. $1 \in \mathcal{F}(M)$.*
2. *$2 \in \mathcal{F}(M)$ if and only if M has even Euler characteristic.*
3. *$3 \in \mathcal{F}(M)$ if and only if the intersection form of M is not isomorphic to*
$$\pm(1) \quad \text{nor} \quad \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

[4] A manifold is *spin* if its second Stiefel-Whitney class vanishes.

(4) $4 \in \mathcal{F}(M)$ *if and only if* M *is stably parallelizable.*

Note that in the above theorem, all the conditions depend only on the homotopy type of M. In other words, if M admits a fold map into $\mathbf{R}^n$, then any 4-manifold homotopy equivalent to M also admits a fold map into $\mathbf{R}^n$.

4. A generalization of the Morse theory

In this section, we consider the following problem.

Problem 4.1. Given a generic map $f : M \to N$, how can we get information on the (differential) topology of M?

In the case where $N = \mathbf{R}$, i.e. in the case of Morse functions, we know that we can get a lot of information on the topology of M: betti numbers, Euler characteristic, handlebody decomposition, etc. (for example, see [15, 39]).

In general, if we use generic maps instead of Morse functions, we can get information on the characteristic classes (see, for example, [40, 10, 1]).[5] The following has been proved in [20] and [25] (see also [37]). Recall that three times the signature of a closed oriented 4-manifold coincides with the first Pontrjagin number (see [17]).

Theorem 4.2. *Let* M *be a closed oriented* 4*-manifold and* N *an orientable* 3*-manifold. For any generic map* $f : M \to N$*, we have*

$$\sigma(M) = S(f) \cdot S(f),$$

where $\sigma(M)$ *stands for the signature of* M*,* $S(f) \subset M$ *denotes the* (*possibly nonorientable*) *surface of singular points of* f*, and* $S(f) \cdot S(f)$ *is the self-intersection number of* $S(f)$ *in* M*.*

The above theorem is, in a sense, a *horizontal* signature formula. In order to get a *vertical* signature formula, we need to look at the singular fibers of generic maps.[6] The most natural equivalence among singular fibers would be the following [33].

Definition 4.3. Let $f_i : M_i \to N_i$ be smooth maps and take points $y_i \in N_i$, $i = 0, 1$. We say that the fibers over y_0 and y_1 are C^∞ *equivalent* if for some open neighborhoods U_i of y_i in N_i, there exist diffeomorphisms $\widetilde{\varphi} : (f_0)^{-1}(U_0) \to (f_1)^{-1}(U_1)$ and $\varphi : U_0 \to U_1$ with $\varphi(y_0) = y_1$ which make the following diagram commutative:

$$\begin{array}{ccc} ((f_0)^{-1}(U_0), (f_0)^{-1}(y_0)) & \xrightarrow{\widetilde{\varphi}} & ((f_1)^{-1}(U_1), (f_1)^{-1}(y_1)) \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ (U_0, y_0) & \xrightarrow{\varphi} & (U_1, y_1). \end{array}$$

When $y \in N$ is a regular value of a smooth map $f : M \to N$ between smooth manifolds, we call $f^{-1}(y)$ a *regular fiber*; otherwise, a *singular fiber.*

In [33], the author classified the singular fibers of proper generic maps of orientable 4-manifolds into 3-manifolds[7] as follows.

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[5] The theory of the so-called *Thom polynomials* offers a lot of information on this subject.

[6] The author was inspired by Yukio Matsumoto's works and lectures on singular fibers of certain singular fibrations of 4-manifolds over surfaces. Unfortunately, or fortunately, such singular fibrations studied by Matsumoto are not generic in our sense. Any way, the author would like to thank Prof. Matsumoto for his enthusiasm for singular objects and for his constant encouragement!

[7] For nonorientable 4-manifolds, a similar classification has been obtained by T. Yamamoto [41, 42].

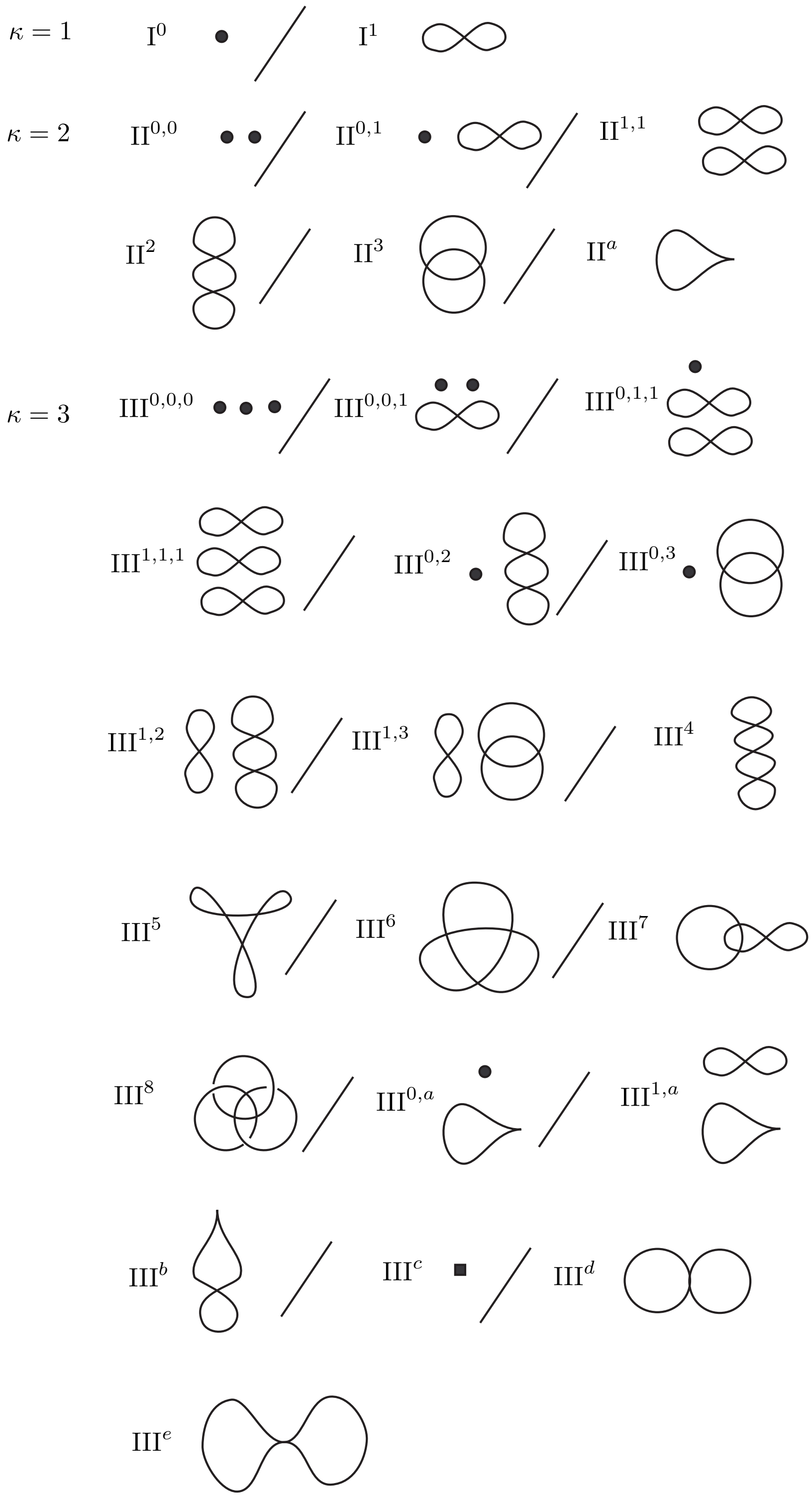

FIGURE 1. List of singular fibers of proper C^∞ stable maps of orientable 4-manifolds into 3-manifolds

Theorem 4.4. *Let $f : M \to N$ be a proper C^∞ stable map of an orientable 4-manifold M into a 3-manifold N. Then, every singular fiber of f is equivalent to one of the fibers as in Fig. 1 up to disjoint union with regular fibers.*

In [36], T. Yamamoto and the author observed that if the source 4-manifold M is oriented, then one can define a sign ($= \pm 1$) for each singular fiber of type III^8. Furthermore, we obtained the following vertical signature formula.

Theorem 4.5. *Let M be a closed oriented 4-manifold and N a 3-manifold. Then for any C^∞ stable map $f : M \to N$, the sum of the signs over all singular fibers of f of III^8-type coincides with the signature $\sigma(M)$ of M.*

It should be a very interesting problem to construct and to study diffeomorphism invariants derived from singular fibers of generic maps. One of the major advantages of using generic maps is that any manifold admits a generic map: i.e. we do not need to restrict the study to certain classes of 4-manifolds.

Furthermore, as has been observed in Theorem 2.7, study of cobordisms of singular maps may serve to find some diffeomorphism invariants. For this subject, see [24] and [33].

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