Cocycle Invariants of Knots, Graphs and Surfaces

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Dedicated to Professor Yukio Matsumoto for his 60th birthday

1 Introduction

Quandle cohomology theory was developed [6] to define invariants, called quandle cocycle (knot) invariants, for classical knots and knotted surfaces in state-sum form. The quandle cohomology theory is a modification of rack cohomology theory which was defined in [11]. The cocycle knot invariants are analogous in their definitions to the Dijkgraaf-Witten invariants [9] of triangulated 3-manifolds with finite gauge groups. In this paper, we give a brief review with a few new progresses on quandle cocycle invariants, on two aspects: (1) constructions and computations, and (2) applications.

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2 Definitions

A quandle, $X$, is a set with a binary operation $(a, b) \mapsto a * b$ such that

(I) For any $a \in X$, $a * a = a$.

(II) For any $a, b \in X$, there is a unique $c \in X$ such that $a = c * b$.

(III) For any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$.

A rack is a set with a binary operation that satisfies (II) and (III). Racks and quandles have been studied in, for example, [2, 10, 15, 16].

The following are typical examples of quandles. A group $G$ with conjugation as the quandle operation: $a * b = bab^{-1}$, denoted by $X = \text{Conj}(G)$, is a quandle. Any subset of $G$ that is closed under such conjugation is also a quandle. Any $\Lambda(= \mathbb{Z}[t, t^{-1}])$-module $M$ is a quandle with $a * b = ta + (1 - t)b$, $a, b \in M$, that is called an Alexander quandle. Let $n$ be

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Figure 1: Quandle relation at a crossing

Let $X$ be a fixed quandle. Let $K$ be a given oriented classical knot or link diagram, and let $\mathcal{R}$ be the set of (over-)arcs. The normals (normal vectors) are given in such a way that the ordered pair (tangent, normal) agrees with the orientation of the plane, see Fig. 1. A (quandle) coloring $\mathcal{C}$ is a map $\mathcal{C} : \mathcal{R} \to X$ such that at every crossing, the relation depicted in Fig. 1 holds. The (ordered) colors $\mathcal{C}(\alpha), \mathcal{C}(\beta)$ are called source colors.

The cocycle invariant for classical knots [6] was defined as follows. Let $\phi \in C^2(X; A)$ be a 2-cocycle of a finite quandle $X$ with the coefficient group $A$. This $\phi$ is regarded as a function $X \times X \to A$ that satisfies the 2-cocycle condition

$$
\phi(x, y) - \phi(x, z) + \phi(x \ast y, z) - \phi(x \ast z, y \ast z) = 0,
$$

and $\phi(x, x) = 0, \forall x \in X$. Let $\mathcal{C}$ be a coloring of a given knot diagram $K$ by $X$. The Boltzmann weight $B(\mathcal{C}, \tau)$ at a crossing $\tau$ of $K$ is then defined by $B(\mathcal{C}, \tau) = \phi(x_\tau, y_\tau)^{\epsilon(\tau)}$, where $x_\tau, y_\tau$ are source colors at $\tau$ and $\epsilon(\tau)$ is the sign ($\pm 1$) of $\tau$. In Fig. 1, it is a positive crossing if the under-arc is oriented downward. Here $B(\mathcal{C}, \tau)$ is an element of $A$ written multiplicatively. The formal sum (called a state-sum) in the group ring $\mathbb{Z}[A]$

$$
\Phi_\kappa(K) = \sum_{\mathcal{C} \in \text{Col}_X(K)} \prod_\tau B(\mathcal{C}, \tau)
$$

is called the quandle cocycle invariant.

**Theorem 2.1** [6] The state-sum $\Phi_\kappa(K)$ does not depend on the choice of a diagram $D$ of a given knot $K$, so that it is a well-defined knot invariant.

Generalizations have been discovered [4, 5]. The quandle cocycle invariants have also been defined for knotted surfaces in 4-space, in a similar manner, using quandle 3-cocycles.

At first the computations relied on cocycles found by computer calculations, as the cocycle condition is a system of large number of over-determined linear equations. We have seen significant progress in computations after Mochizuki [17] discovered a family of cocycles for dihedral and other Alexander quandles written by polynomial expressions. Formulas for important families of knots and knotted surfaces and their applications followed [1, 14, 13]. We observe two aspects of Mochizuki’s construction; using polynomial expressions, and modifying coboundaries. Then we present generalizations.
3 Constructions and Computations: Polynomial Cocycles

A. Mochizuki’s 3-cocycle. For \( R_p = \{0, 1, \ldots, p-1\} = \mathbb{Z}_p \), \( x \ast y = 2y - x \mod p \) (prime), Mochizuki [17] showed that

\[
\theta(x, y, z) = (x - y)((2z^p - y^p) - (2z - y)^p)/p
\]
is a non-trivial quandle 3-cocycle. Here we observe that, for \( f(x, y) = (x - y)y^p \),

\[
\delta f(x, y, z) = f(x, z) - f(x, y) - f(x \ast y, z) - f(x \ast y, z) = (x - y)((2z^p - y^p) - (2z - y)^p).
\]

This is divisible by \( p \), and after dividing, a Mochizuki cocycle is obtained. We look at this situation more closely.

B. Algebra of Mochizuki’s 3-cocycle. Let \( X \) be a quandle, and we look at the short exact sequence \( 0 \to \mathbb{Z}_p(= N) \xrightarrow{\alpha=\times p} \mathbb{Z}_p(= E) \xrightarrow{\pi} \mathbb{Z}_p(= A) \to 0 \). Let \( s : \mathbb{Z}_p(= A) \to \mathbb{Z}_p(= E) \) be a section such that \( s(0) = 0 \). For a 2-cocycle \( f : X \times X \to A \), note that \( \delta(sf) \in \ker(\pi) = \text{im}(\alpha) \), so that there exists \( \vartheta : X^3 \to N = \mathbb{Z}_p \) such that \( \alpha(\vartheta) = \delta(sf) \). Hence for \( f(x, y) = (x - y)y^p \) and \( X = R_p \), we obtain a Mochizuki 3-cocycle as \( \theta \) by this construction. We remark here that this construction appears in [5] for the twisted theory of quandle cohomology. Since Mochizuki cocycles are non-trivial, Proposition 7. 15. in [5] is incorrect (that states that the invariant is trivial for cocycles obtained this way). This opens a possibility of constructing non-trivial invariants by this method.

C. Extension and Diagrams for Mochizuki’s 3-cocycle. Let \( f : X \times X \to A \) be a 2-cocycle of a quandle \( X \) with coefficients in \( A \). Then \( \tilde{X} = A \times X \), \( (a, x) \ast (b, y) = (a + f(x, y), x \ast y) \) defines a quandle, called an extension [3]. In the exact sequence \( 0 \to \mathbb{Z}_p(= N) \xrightarrow{\alpha=\times p} \mathbb{Z}_p(= E) \xrightarrow{\pi} \mathbb{Z}_p(= A) \to 0 \) and the section \( s \) in the preceding section B, \( \tilde{X} = E \times X \), \( (a, x) \ast (b, y) = (a + sf(x, y), x \ast y) \) may not be self-distributive. The discrepancy in the first factor is in \( \text{im}(\alpha) \), and is written as \( \alpha(\vartheta(x, y, z)) \). As we noted, this \( \vartheta \) gives a 3-cocycle, and Mochizuki cocycle is described this way. In Fig. 2, a diagram representing this situation is depicted by shadow (or, face) colors (see [8, 19]). For a given shadow coloring \( C \) of a knot diagram by \( X \), we try to lift the colors to \( \tilde{X} \), and further to \( \tilde{X}' \). If the knot has trivial cocycle invariant with \( f \), the color lifts to \( \tilde{X} \) [3]. Since \( s(0) = 0 \), the colors on knot strings lift to \( \tilde{X}' \). The shadow colors (colors of regions), however, may not lift in a well-defined manner because of the discrepancy, as depicted in the figure. Thus a Mochizuki cocycle is interpreted as a monodromy of non-self-distributivity at a shadow colored crossing.

D. Generalizations, polynomial cocycles. We have the following generalization.

Lemma 3.1 Let \( X = \mathbb{Z}_p[t, t^{-1}]/h(t) \), \( A = \mathbb{Z}_p[t, t^{-1}]/g(t) \), where \( h(t), g(t) \in \mathbb{Z}_p[t, t^{-1}] \) such that \( g(t) \) divides \( h(t) \). Let \( a_i = p^{m_i} \), for \( i = 1, \ldots, n - 1 \), where \( p \) is a prime and \( m_i \) are positive integers. Let \( f : X^n \to A \) be defined by

\[
f(x_1, x_2, \ldots, x_n) = (x_1 - x_2)^{a_1} (x_2 - x_3)^{a_2} \cdots (x_{n-1} - x_n)^{a_{n-1}} x_n^{a_n}
\]
Figure 2: Monodromy of a colored crossing

where the $x_i$s in the right-hand side are regarded as elements of $A$ via the quotient map. Then $f$ is an $n$-cocycle ($\in Z^n_Q(X; A)$), if either $a = 0$, or $a = p^m$ (a positive integer $m$) and $g(t)$ divides $1 - t^{a_1 + a_2 + \ldots + a_n - 1 + a}$.

These cocycles evaluate non-trivially for many Alexander quandles, and the cocycle invariants can be computed for some links.

Example 3.2 Let $p = 2$ and take the cocycle $f(x, y) = (x - y)^4y$ for $X = A = \mathbb{Z}_2[t, t^{-1}]/h(t)$, where $h(t) = t^4 + t^3 + t^2 + t + 1$. For the top color vector $(1, 0)$, the closure of $\sigma_1^n$ is colored non-trivially, and gives a contribution $f(1, 0) + f(0, t) + f(t, t^2 + t) + f(t^2 + t, t^3 + t^2 + t) + f(t^3 + t^2 + t, 1) = t^3 + t + 1$. The invariant written as a family of state-sum contributions is $\{\sqcup_{16}(0), \sqcup_{80}(t + 1), \sqcup_{80}(t^3), \sqcup_{80}(t^3 + t + 1)\}$.

4 Applications

A. Non-invertibility. Quandle cocycle invariants detect non-invertibility of some knotted surfaces. This was first discovered in [6] and extended to many new examples in [1, 4, 14].

B. Triple point numbers. The quandle cocycle invariants were used to obtain a lower bound for the triple point number (the minimal number of triple points in projections in [22], where a triple point number of the 2-twist spun trefoil was determined to be 4 for the first time. See [13, 23, 24] for other results on triple points and quandles.

C. Ribbon concordance. Quandle cocycle invariants were used in [7] to give obstructions to ribbon concordance for knotted surfaces.

D. Chirality of graphs. Suzuki’s $\theta_n$-curves are known for being knotted but their proper subgraphs being unknotted. Chirality is previously known for $n = 3$ [25] and $n = 4$ [12]. Fox coloring of graphs have been also studied by several authors. In [21] cocycle invariants (shadow colored version with Mochizuki 3-cocycles) were used to prove that Suzuki’s $\theta_n$-curves are chiral for $n \equiv \pm 2 \mod 6$. In Fig. 3, a possible 3-coloring of a Suzuki $\theta_n$-curve is depicted.
E. Chirality with colors. For a property of knots in question, we can consider the same property with a condition on the homomorphisms from the fundamental groups or quandles added. For example, it is well known that the figure-eight knot is amphicheiral and 5-colorable. Then we ask whether if there is a non-trivial 5-coloring such that it is isotopic to its mirror image with the colors also its mirror, as depicted in Fig. 4.

Figure 4:

We now observe that the answer to this question is negative for the specific coloring given in Fig. 5, since the sum of 3-cocycle values (a state-sum contribution) of this coloring is distinct from that of its mirror, as computed in Fig. 5, and this sum should be the same for isotopic knots with a given coloring. Since it is computed that any non-trivial 5-coloring has the same property as above, it is concluded that there is no non-trivial coloring that is equivalent to its mirror.

Figure 5: Cocycle weights for a colored figure-eight
F. Embedding Tangles. The number of Fox colorings was used obstructions to tangle embeddings, as well as branched coverings and quantum invariants (see, for example, [18]). Cocycle invariants can be used as obstructions as follows. Consider the state-sum of cocycles of a tangle \( T \) over colorings such that the boundary points of the tangle are required to have the same color: \( \Phi_\phi(T; x) = \{ \sum_\tau B_\phi(\tau, \mathcal{C}) : \mathcal{C}(\partial T) = x \in X \} \), where \( X \) is a quandle. For a shadow colorings, we pick and fix the color of \( \Phi_\theta(T; x, s) = \{ \sum_\tau B_\theta(\tau, \mathcal{C}) : \mathcal{C}(\partial T) = x \in X, \mathcal{C}(\text{the left most region}) = s \} \). If the tangle \( T \) is a subtangle of a link \( L \), such a coloring extends to a coloring of \( L \) by choosing the color of the boundary points of \( T \) for all arcs outside of \( T \). Hence we obtain

**Proposition 4.1** If \( T \) is embedded in a link diagram \( L \), then \( \Phi_\phi(T; x) \subset \Phi_\phi(L) \) for any \( x \in X \) and \( \Phi_\theta(T; x, s) \subset \Phi_\theta(L) \) for any \( x, s \in X \).

G. Minimal sheet numbers of knotted surfaces. In diagrams of knotted surfaces, under-sheets are broken and the resulting components of the surface in 3-space are called sheets. It was proved in [20] using an extension of a quandle that the minimal number of sheets in projections of the spun-trefoil is four. A standard diagram shown in Fig. 6 indeed realizes this number.

![Figure 6: Minimal sheet number of spun-trefoil is 4](image)

H. Number of type III moves. The number of Reidemeister type III moves needed to change a knot diagram to another is closely related to the minimal triple number in projections. For example, when one twists a standard diagram of a trefoil once, two type III moves are needed, as Satoh and Shima’s result on triple point number for the 2-twist spun trefoil shows: this minimal number is four. Here we show that cochains give lower bounds. Let \( f : R_3 = \{0, 1, 2\} \rightarrow \mathbb{Z}, \quad f(x, y) = (x - y)y^n \) for a positive integer \( n \). Then the sum of weight \( f \) is invariant by type I and II, but changes by \( 0, \pm (2^n + 1) \) or \( \pm (2^{n+1} - 1) \) by type III moves \( \delta f(a, b, c) = 0, \pm (2^n + 1), \) or \( \pm (2^{n+1} - 1) \) for any \( a, b, c \in R_3 \).

Figure 7 shows a coloring of a trefoil on the left-hand side, and the coloring after one twist on the right. The sum of weights by \( f \) for these colorings are also indicated. Note that for \( (2, 3k) \)-torus knot \( T(2, 3k) \) for any positive integer \( k \), this pattern is repeated \( k \) times, so that the difference of the sums between LHS and RHS for \( T(2, 3k) \) is

\[
k[(f(0, 1) + f(1, 2) + f(2, 0)) - (f(0, 2) + f(2, 1) + f(1, 0))] = k(2^n - 2).
\]
Hence for solutions \((x, y) \in \mathbb{Z} \times \mathbb{Z}\) to the integral equation \((2^n+1)x+(2^{n+1}-1)y = k(2^n-2)\), the minimum of \(|x|+|y|\) gives a lower bound of the number of type III moves for one twist of \(T(2,3k)\). For example, in the \(n = 4\) case, \(17x + 31y = 14k\) gives lower bounds \(2k\) for \(1 \leq k \leq 13\). For a given \(k\), by choosing an appropriately large even \(n\), we obtain:

**Proposition 4.2** One twist of \(T(2,3k)\) requires at least \(2k\) type III moves, for any \(k \in \mathbb{Z}\).

Though an upper bound \(2(3k - 2)\) is known [22] and conjectured to be the minimum, better lower bounds seem unknown.

**References**


