Cocycle Invariants of Knots, Graphs and Surfaces

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Dedicated to Professor Yukio Matsumoto for his 60th birthday

1 Introduction

Quandle cohomology theory was developed [6] to define invariants, called quandle cocycle (knot) invariants, for classical knots and knotted surfaces in state-sum form. The quandle cohomology theory is a modification of rack cohomology theory which was defined in [11]. The cocycle knot invariants are analogous in their definitions to the Dijkgraaf-Witten invariants [9] of triangulated 3-manifolds with finite gauge groups. In this paper, we give a brief review with a few new progresses on quandle cocycle invariants, on two aspects: (1) constructions and computations, and (2) applications.

Professor Yukio Matsumoto was the Master's advisor for Masahico Saito. This author especially would like to express his deepest gratitude for Professor Matsumoto's continuing personal and professional guidance and support. The remaining authors have been profoundly influenced by Professor Matsumoto and his work albeit less directly.

2 Definitions

A *quandle*, X, is a set with a binary operation $(a, b) \mapsto a * b$ such that

(I) For any $a \in X$, $a * a = a$.

(II) For any $a, b \in X$, there is a unique $c \in X$ such that $a = c * b$.

(III) For any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$.

A *rack* is a set with a binary operation that satisfies (II) and (III). Racks and quandles have been studied in, for example, [2, 10, 15, 16].

The following are typical examples of quandles. A group G with conjugation as the quandle operation: $a * b = bab^{-1}$, denoted by $X =$Conj(G), is a quandle. Any subset of G that is closed under such conjugation is also a quandle. Any $\Lambda(= \mathbb{Z}[t, t^{-1}])$-module M is a quandle with $a * b = ta + (1 - t)b$, $a, b \in M$, that is called an *Alexander quandle*. Let n be

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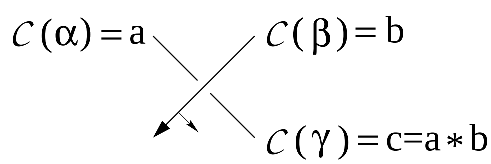

Figure 1: Quandle relation at a crossing

a positive integer, and for elements $i, j \in \{0, 1, \dots, n-1\}$, define $i * j \equiv 2j - i \pmod{n}$. Then $*$ defines a quandle structure called the *dihedral quandle*, R_n.

Let X be a fixed quandle. Let K be a given oriented classical knot or link diagram, and let $\mathcal{R}$ be the set of (over-)arcs. The normals (normal vectors) are given in such a way that the ordered pair (tangent, normal) agrees with the orientation of the plane, see Fig. 1. A (quandle) *coloring* $\mathcal{C}$ is a map $\mathcal{C} : \mathcal{R} \to X$ such that at every crossing, the relation depicted in Fig. 1 holds. The (ordered) colors $\mathcal{C}(\alpha)$, $\mathcal{C}(\beta)$ are called *source* colors. Let $\mathrm{Col}_X(K)$ denote the set of colorings of a knot diagram K by a quandle X.

The cocycle invariant for classical knots [6] was defined as follows. Let $\phi \in C^2_{\mathrm{Q}}(X; A)$ be a 2-cocycle of a finite quandle X with the coefficient group A. This ϕ is regarded as a function $X \times X \to A$ that satisfies the 2-cocycle condition

$$\phi(x, y) - \phi(x, z) + \phi(x * y, z) - \phi(x * z, y * z) = 0, \quad \forall x, y, z \in X$$

and $\phi(x, x) = 0, \forall x \in X$. Let $\mathcal{C}$ be a coloring of a given knot diagram K by X. The *Boltzmann weight* $B(\mathcal{C}, \tau)$ at a crossing τ of K is then defined by $B(\mathcal{C}, \tau) = \phi(x_\tau, y_\tau)^{\epsilon(\tau)}$, where x_τ, y_τ are source colors at τ and $\epsilon(\tau)$ is the sign (± 1) of τ. In Fig. 1, it is a positive crossing if the under-arc is oriented downward. Here $B(\mathcal{C}, \tau)$ is an element of A written multiplicatively. The formal sum (called a state-sum) in the group ring $\mathbb{Z}[A]$

$$\Phi_\kappa(K) = \sum_{\mathcal{C} \in \mathrm{Col_X(K)}} \prod_{\tau} B(\mathcal{C}, \tau)$$

is called the quandle cocycle invariant.

Theorem 2.1 [6] *The state-sum $\Phi_\kappa(K)$ does not depend on the choice of a diagram D of a given knot K, so that it is a well-defined knot invariant.*

Generalizations have been discovered [4, 5]. The quandle cocycle invariants have also been defined for knotted surfaces in 4-space, in a similar manner, using quandle 3-cocycles.

At first the computations relied on cocycles found by computer calculations, as the cocycle condition is a system of large number of over-determined linear equations. We have seen significant progress in computations after Mochizuki [17] discovered a family of cocycles for dihedral and other Alexander quandles written by polynomial expressions. Formulas for important families of knots and knotted surfaces and their applications followed [1, 14, 13]. We observe two aspects of Mochizuki's construction; using polynomial expressions, and modifying coboundaries. Then we present generalizations.

3 Constructions and Computations: Polynomial Cocycles

A. Mochizuki's 3-cocycle. For $R_p = \{0, 1, \dots, p-1\} = \mathbb{Z}_p$, $x * y = 2y - x \bmod p$ (prime), Mochizuki [17] showed that

$$\theta(x, y, z) = (x - y)[(2z^p - y^p) - (2z - y)^p]/p$$

is a non-trivial quandle 3-cocycle. Here we observe that, for $f(x, y) = (x - y)y^p$,

$$\begin{aligned} \delta f(x, y, z) &= f(x, z) - f(x, y) - f(x * y, z) + f(x * z, y * z) \\ &= (x - y)[(2z^p - y^p) - (2z - y)^p]. \end{aligned}$$

This is divisible by p, and after dividing, a Mochizuki cocycle is obtained. We look at this situation more closely.

B. Algebra of Mochizuki's 3-cocycle. Let X be a quandle, and we look at the short exact sequence $0 \to \mathbb{Z}_p (= N) \stackrel{\alpha = \times p}{\to} \mathbb{Z}_{p^2} (= E) \stackrel{\pi}{\to} \mathbb{Z}_p (= A) \to 0$. Let $s : \mathbb{Z}_p (= A) \to \mathbb{Z}_{p^2} (= E)$ be a section such that $s(0) = 0$. For a 2-cocycle $f : X \times X \to A$, note that $\delta(sf) \in \mathrm{Ker}(\pi) = \mathrm{Im}(\alpha)$, so that there exists $\theta : X^3 \to N = \mathbb{Z}_p$ such that $\alpha(\theta) = \delta(sf)$. Hence for $f(x, y) = (x - y)y^p$ and $X = R_p$, we obtain a Mochizuki 3-cocycle as θ by this construction. We remark here that this construction appears in [5] for the twisted theory of quandle cohomology. Since Mochizuki cocycles are non-trivial, Proposition 7. 15. in [5] is incorrect (that states that the invariant is trivial for cocycles obtained this way). This opens a possibility of constructing non-trivial invariants by this method.

C. Extension and Diagrams for Mochizuki's 3-cocycle. Let $f : X \times X \to A$ be a 2-cocycle of a quandle X with coefficients in A. Then $\tilde{X} = A \times X$, $(a, x) * (b, y) = (a + f(x, y), x * y)$ defines a quandle, called an extension [3]. In the exact sequence $0 \to \mathbb{Z}_p (= N) \stackrel{\alpha = \times p}{\to} \mathbb{Z}_{p^2} (= E) \stackrel{\pi}{\to} \mathbb{Z}_p (= A) \to 0$ and the section s in the preceding section **B**, $\tilde{X}' = E \times X$, $(a, x) * (b, y) = (a + sf(x, y), x * y)$ may not be self-distributive. The discrepancy in the first factor is in $\mathrm{Im}(\alpha)$, and is written as $\alpha(\theta(x, y, z))$. As we noted, this θ gives a 3-cocycle, and Mochizuki cocycle is described this way. In Fig. 2, a diagram representing this situation is depicted by shadow (or, face) colors (see [8, 19]). For a given shadow coloring $\mathcal{C}$ of a knot diagram by X, we try to lift the colors to $\tilde{X}$, and further to $\tilde{X}'$. If the knot has trivial cocycle invariant with f, the color lifts to $\tilde{X}$ [3]. Since $s(0) = 0$, the colors on knot strings lift to $\tilde{X}'$. The shadow colors (colors of regions), however, may not lift in a well-defined manner because of the discrepancy, as depicted in the figure. Thus a Mochizuki cocycle is interpreted as a monodromy of non-self-distributivity at a shadow colored crossing.

D. Generalizations, polynomial cocycles. We have the following generalization.

Lemma 3.1 *Let $X = \mathbb{Z}_p[t, t^{-1}]/h(t)$, $A = \mathbb{Z}_p[t, t^{-1}]/g(t)$, where $h(t), g(t) \in \mathbb{Z}_p[t, t^{-1}]$ such that $g(t)$ divides $h(t)$. Let $a_i = p^{m_i}$, for $i = 1, ..., n-1$, where p is a prime and m_i are positive integers. Let $f : X^n \to A$ be defined by*

$$f(x_1, x_2, ..., x_n) = (x_1 - x_2)^{a_1} (x_2 - x_3)^{a_2} ... (x_{n-1} - x_n)^{a_{n-1}} x_n^a$$

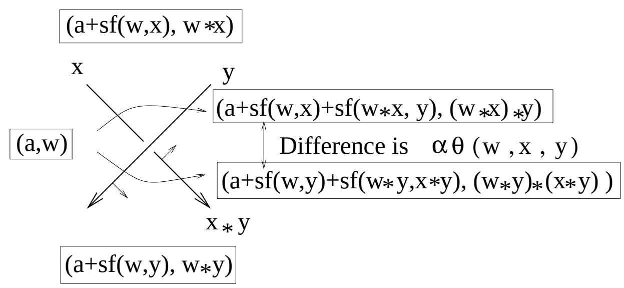

Figure 2: Monodromy of a colored crossing

where the x_is in the right-hand side are regarded as elements of A via the quotient map. Then f is an n-cocycle ($\in Z^n_Q(X;A)$), if either $a=0$, or $a=p^m$ (a positive integer m) and $g(t)$ divides $1-t^{a_1+a_2+\ldots+a_{n-1}+a}$.

These cocycles evaluate non-trivially for many Alexander quandles, and the cocycle invariants can be computed for some links.

Example 3.2 Let $p=2$ and take the cocycle $f(x,y)=(x-y)^4y$ for $X=A=\mathbb{Z}_2[t,t^{-1}]/h(t)$, where $h(t)=t^4+t^3+t^2+t+1$. For the top color vector $(1,0)$, the closure of σ_1^5 is colored non-trivially, and gives a contribution $f(1,0)+f(0,t)+f(t,t^2+t)+f(t^2+t,t^3+t^2+t)+f(t^3+t^2+t,1)=t^3+t+1$. The invariant written as a family of state-sum contributions is $\{\sqcup_{16}(0),\sqcup_{80}(t+1),\sqcup_{80}(t^3),\sqcup_{80}(t^3+t+1)\}$.

4 Applications

A. Non-invertibility. Quandle cocycle invariants detect non-invertibility of some knotted surfaces. This was first discovered in [6] and extended to many new examples in [1, 4, 14].

B. Triple point numbers. The quandle cocycle invariants were used to obtain a lower bound for the *triple point number* (the minimal number of triple points in projections in [22], where a triple point number of the 2-twist spun trefoil was determined to be 4 for the first time. See [13, 23, 24] for other results on triple points and quandles.

C. Ribbon concordance. Quandle cocycle invariants were used in [7] to give obstructions to ribbon concordance for knotted surfaces.

D. Chirality of graphs. Suzuki's θ_n-curves are known for being knotted but their proper subgraphs being unknotted. Chirality is previously known for $n=3$ [25] and $n=4$ [12]. Fox coloring of graphs have been also studied by several authors. In [21] cocycle invariants (shadow colored version with Mochizuki 3-cocycles) were used to prove that Suzuki's θ_n-curves are chiral for $n\equiv\pm2$ mod 6. In Fig. 3, a possible 3-coloring of a Suzuki θ_n-curve is depicted.

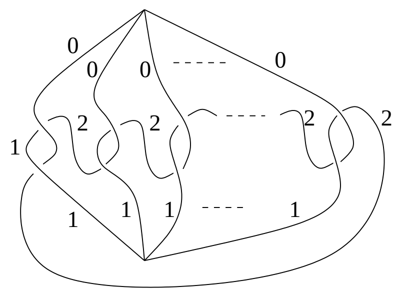

Figure 3: Some Suzuki θ-curves are chiral

E. Chirality with colors. For a property of knots in question, we can consider the same property with a condition on the homomorphisms from the fundamental groups or quandles added. For example, it is well known that the figure-eight knot is amphicheiral and 5-colorable. Then we ask whether if there is a non-trivial 5-coloring such that it is isotopic to its mirror image with the colors also its mirror, as depicted in Fig. 4.

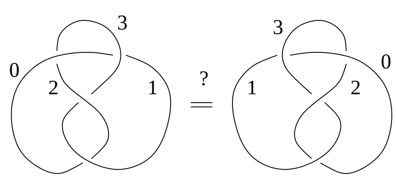

Figure 4:

We now observe that the answer to this question is negative for the specific coloring given in Fig. 5, since the sum of 3-cocycle values (a state-sum contribution) of this coloring is distinct from that of its mirror, as computed in Fig. 5, and this sum should be the same for isotopic knots with a given coloring. Since it is computed that any non-trivial 5-coloring has the same property as above, it is concluded that there is no non-trivial coloring that is equivalent to its mirror.

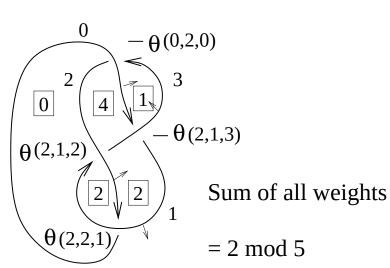

Figure 5: Cocycle weights for a colored figure-eight

F. Embedding Tangles. The number of Fox colorings was used obstructions to tangle embeddings, as well as branched coverings and quantum invariants (see, for example, [18]). Cocycle invariants can be used as obstructions as follows. Consider the state-sum of cocycles of a tangle T over colorings such that the boundary points of the tangle are required to have the same color: $\Phi_\phi(T;x) = \{\sum_\tau B_\phi(\tau, \mathcal{C}) : \mathcal{C}(\partial T) = x \in X\}$, where X is a quandle. For a shadow colorings, we pick and fix the color of $\Phi_\theta(T;x,s) = \{\sum_\tau B_\theta(\tau, \mathcal{C}) : \mathcal{C}(\partial T) = x \in X, \mathcal{C}(\text{the left most region}) = s\}$. If the tangle T is a subtangle of a link L, such a coloring extends to a coloring of L by choosing the color of the boundary points of T for all arcs outside of T. Hence we obtain

Proposition 4.1 *If T is embedded in a link diagram L, then $\Phi_\phi(T;x) \subset \Phi_\phi(L)$ for any $x \in X$ and $\Phi_\theta(T;x,s) \subset \Phi_\theta(L)$ for any $x, s \in X$.*

G. Minimal sheet numbers of knotted surfaces. In diagrams of knotted surfaces, under-sheets are broken and the resulting components of the surface in 3-space are called sheets. It was proved in [20] using an extension of a quandle that the minimal number of sheets in projections of the spun-trefoil is four. A standard diagram shown in Fig. 6 indeed realizes this number.

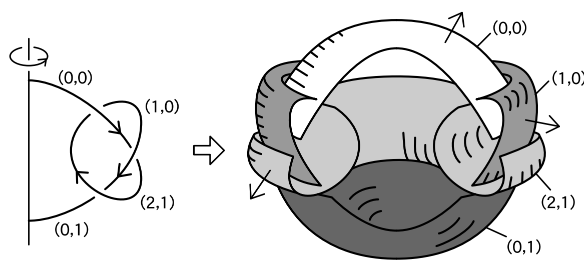

Figure 6: Minimal sheet number of spun-trefoil is 4

H. Number of type III moves. The number of Reidemeister type III moves needed to change a knot diagram to another is closely related to the minimal triple number in projections. For example, when one twists a standard diagram of a trefoil once, two type III moves are needed, as Satoh and Shima's result on triple point number for the 2-twist spun trefoil shows: this minimal number is four. Here we show that cochains give lower bounds. Let $f : R_3 = \{0,1,2\} \to \mathbb{Z}$, $\quad f(x,y) = (x-y)y^n$ for a positive integer n. Then the sum of weight f is invariant by type I and II, but changes by $0, \pm(2^n+1)$ or $\pm(2^{n+1}-1)$ by type III moves ($\delta f(a,b,c) = 0, \pm(2^n+1)$, or $\pm(2^{n+1}-1)$ for any $a,b,c \in R_3$).

Figure 7 shows a coloring of a trefoil on the left-hand side, and the coloring after one twist on the right. The sum of weights by f for these colorings are also indicated. Note that for $(2,3k)$-torus knot $T(2,3k)$ for any positive integer k, this pattern is repeated k times, so that the difference of the sums between LHS and RHS for $T(2,3k)$ is

$$k[(f(0,1) + f(1,2) + f(2,0)) - (f(0,2) + f(2,1) + f(1,0))] = k(2^n - 2).$$

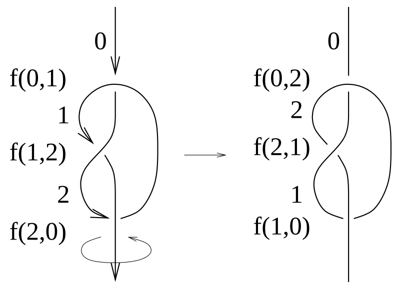

Figure 7: Trefoil twisted

Hence for solutions $(x, y) \in \mathbb{Z}\times\mathbb{Z}$ to the integral equation $(2^n+1)x+(2^{n+1}-1)y = k(2^n-2)$, the minimum of $|x|+|y|$ gives a lower bound of the number of type III moves for one twist of $T(2, 3k)$. For example, in the $n = 4$ case, $17x + 31y = 14k$ gives lower bounds $2k$ for $1 \leq k \leq 13$. For a given k, by choosing an appropriately large even n, we obtain:

Proposition 4.2 *One twist of $T(2, 3k)$ requires at least $2k$ type III moves, for any $k \in \mathbb{Z}$.*

Though an upper bound $2(3k-2)$ is known [22] and conjectured to be the minimum, better lower bounds seem unknown.

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