Degenerations of Riemann surfaces

Dedicated to Professor Yukio Matsumoto on the occation of his 60th birthday

Shigeru Takamura
Department of Mathematics, Kyoto University
e-mail: takamura@math.kyoto-u.ac.jp

概 要

We explain the shape of singular fibers of degenerations of Riemann surfaces. Then we discuss some topics on deformations of degenerations.

敬愛する松本幸夫先生が、このたび還暦をむかえられ、多くの方々といっしょにお祝いできることをうれしく思います。この講演のタイトルは「リーマン面の退化」なのですが、ご承知のようにこの分野にも先生の大きな貢献があります。単位円盤 $\Delta = \{s \in \mathbb{C} : |s| < 1\}$ でパラメータ付けされたコンパクトなリーマン面の族で、原点上のファイバーが特異点をもち、それ以外の点の上のファイバーはなめらかなリーマン面である族を**リーマン面の退化**といいます。すなわち、複素曲面 M から 単位円盤 Δ への正則写像 π で $X = \pi^{-1}(0)$ は特異ファイバー、$X = \pi^{-1}(s)$ $(s \neq 0)$ はなめらかなファイバーであるような組 $\pi : M \to \Delta$ をリーマン面の退化と呼びます。さて、なめらかなファイバーが種数 g のリーマン面であるような退化はどれくらいあるか、と言う問題がまず考えられます。松本幸夫先生がこの問題に興味をもたれたきっかけは、談話会で上野健爾氏が種数２の特異ファイバーの分類を説明し、そのとき配られた表に約１２０個もの特異ファイバーが載っているのに対しなんでこんなにたくさんあるのだろう、と思ったことがきっかけだそうです。私も含め、たいていの人は、リーマン面の種数が増えるとともに、猛烈な増え方をする特異ファイバーの数にびっくりすると思います。

— セミナーにうち出てみれば ファイバーのあまりの多さに目をまわしつつ —

この問題を考える上で鍵となるのは特異ファイバーのまわりのモノドロミー（これは写像類群の元になっています）と結びつけることです。松本-モンテシノス理論の核心は**退化の位相同値類とモノドロミーは一対一に対応する**という画期的な定理です。つまり退化全体の集合をモノドロミーの立場から理解できるわけです。モノドロミーは擬周期的な同相写像ということがわかるので、退化を理解することはリーマン面の擬周期的な同相写像を理解することに帰着されるわけです。その後、足利正氏と石坂

瑞穂氏は安定曲線の周期的同相写像の観点から「種数 g のリーマン面の退化の分類の具体的なアルゴリズム」（**足利-石坂アルゴリズム**）を与えました。（同時に両氏は種数３の分類を完成しています。それによると種数３の場合、約１６００個の退化が存在します。ちなみに種数１、２の場合はそれぞれ小平邦彦氏および浪川、上野両氏により約１０、１２０個の退化があることが示されています。）

もう少し補足しておくと、退化には２つの種類があります。すなわち、特異ファイバーの形状が**星形** (stellar) または**星座形** (constellar) によって２種類に大別されます。星座形というのは、文字通りいくつかの星形ファイバーを基本的な操作（**松本-モンテシノス結合**）で結合して得られたファイバーです。モノドロミーの立場から言うと、星形ファイバーはモノドロミーが周期的同相写像、一方、星座形ファイバーのモノドロミーは（周期的でない）擬周期的同相写像です。具体的には、星形ファイバーは**芯** (core) と呼ばれる既約成分から何本かの**枝** (branch)、すなわちリーマン球面の chain が出た形をしています。また、星座形ファイバーはいくつかの星形ファイバーの枝と枝を結び合わせてできた形をしています。このとき、枝と枝が結ばれてできた部分（これもリーマン球面の chain）は星座型ファイバーの**幹** (trunk) と言います。

さて、「多様体のトポロジーの未来へ」という趣旨のシンポジウムでありますので、松本-モンテシノス理論と関連した、最近になってわかってきた興味深い現象を少し述べたいと思います。星形ファイバーの枝の付け根の位置というのは、今まであまり注意が払われていなかったのですが、実は枝の付け根の位置は、退化の分裂族を構成する際に微妙な役割を果たします。まったく同じ分裂族の構成法（剥がし変形）を適用しても、付け根の位置によって、特異ファイバーへの分裂が異なった族ができてしまうことがある、ということが起こりうるのです（したがって分裂族の位相型が異なったものができる）。この現象を把握するためには、core がリーマン球面のときは粟田育子氏により導入された多項式 $J(z)$ が大切です。説明のため、星形ファイバーを

$$X = m_0\Theta_0 + \sum_{j=1}^{h} \mathrm{Br}^{(j)},$$

と表します。ここで Θ_0 は core で、リーマン球面と仮定し、$\mathrm{Br}^{(j)} = m_1^{(j)}\Theta_1^{(j)} + m_2^{(j)}\Theta_2^{(j)} + \cdots + m_{\lambda_j}^{(j)}\Theta_{\lambda_j}^{(j)}$ は枝とします。また、枝 $\mathrm{Br}^{(j)}$ の付け根の点を p_j $(j = 1, 2, \ldots, h)$ と書きます。さて、X の subdivisor Y で、特殊な性質を持つものを crust と言います。細かい説明は省きますが (Towards 3 参照)、ここでは単に「退化の変形の際、剥がされるような subdivisor」とだけ思っていて下さい。Y を X と同じように次のように表します。$Y = n_0\Theta_0 + \sum_{j=1}^{h} \mathrm{br}^{(j)}$、ここで、$0 < n_0 < m_0$ で、$\mathrm{br}^{(j)} = n_1^{(j)}\Theta_1^{(j)} + \cdots + n_{e_j}^{(j)}\Theta_{e_j}^{(j)}$ は $\mathrm{Br}^{(j)}$ の部分枝。このとき、

$$J(z) = \sum_{j=1}^{h} (n_0 m_1^{(j)} - m_0 n_1^{(j)}) P_j(z)$$

が、粟田氏によって導入された多項式（**粟田多項式**）です（正確に言うと、その最も

簡単な場合です)。ただし,

$$P_j(z) = (z-p_1)(z-p_2)\cdots(z-p_{j-1})(z-p_{j+1})\cdots(z-p_{h-1}).$$

ここで h 番目の枝の付け根 p_h は無限遠にとっています。粟田多項式と剥がし変形 $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ の関係は、$(z_0, \zeta_0) \in X_{s,t} := \Psi^{-1}(s,t)$ が core の近くの特異点ならば、z_0 は粟田多項式の根ということです。このことからわかるように、分裂族のファイバーの特異点をしらべるのに、粟田多項式は大事な役割を果たします。以下はテクニカルな話しで、まず一般の種数の core の場合に剥がし変形の特異ファイバーの特異点の満たす方程式系を導き、次に core がリーマン球面の場合は、core 上の関数の対数微分の立場から、粟田多項式の導出を説明したいと思います。(用語などは Towards 3 を参照, また以下は、core のまわりに限った話しで、直線束 N は core の normal bundle で, $C = \Theta_0$, $m_0 = m, n = n_0, m_j = m_1^{(j)}, n_j = n_1^{(j)}$ と略記する。)

1 Singularities

We investigate the singularities on curves $X_{s,t}$ appeared in multiple barking families:

$$X_{s,t}: \quad \sigma(z)\zeta^m - s + \sum_{k=1}^{l} {}_l\mathrm{C}_k t^k \sigma(z)\tau(z)^k \zeta^{m-kn} = 0,$$

where σ is a holomorphic section of $N^{\otimes(-m)}$ with a zero of order m_j at p_j $(j = 1, 2, \ldots, h)$, and τ is a meromorphic section of $N^{\otimes n}$ with a pole of order n_j $(0 \le n_j \le m_j)$ at p_j $(j = 1, 2, \ldots, h)$. For brevity, we often use the notation τ_z for the derivative $d\tau/dz$.

Now we shall study when $(z_0, \zeta_0) \in X_{s,t}$ is a singularity; we separate into two cases according to whether or not z_0 is a pole of τ.

Case 1. We first consider the case where z_0 is not a pole of τ.

Lemma 1.1 *Suppose that z_0 is not a pole of τ. Then $(z_0, \zeta_0) \in X_{s,t}$, $(s, t \neq 0)$ is a singularity if and only if*

(1.1)

$$n\sigma_z(z_0)\tau(z_0) + m\sigma(z_0)\tau_z(z_0) = 0$$

(1.2)

$$\zeta_0^n = \frac{ln-m}{m} t\tau(z_0).$$

Proof. For simplicity, setting $F = \sigma\zeta^{m-ln}(\zeta^n + t\tau)^l$, then $X_{s,t}: F(z, \zeta, t) - s = 0$, and we shall investigate the singularities on $X_{s,t}$ $(s, t \neq 0)$. Suppose that $(z_0, \zeta_0) \in X_{s,t}$ is a singularity; this is exactly the case

$$\frac{\partial(F-s)}{\partial\zeta}(z_0, \zeta_0) = \frac{\partial(F-s)}{\partial z}(z_0, \zeta_0) = 0,$$

which, under the assumption $s \neq 0$, is equivalent to

$$(1.3) \quad \frac{\partial \log F}{\partial \zeta}(z_0, \zeta_0) = \frac{\partial \log F}{\partial z}(z_0, \zeta_0) = 0.$$

(Note: As $s \neq 0$, from $X_{s,t} : F(z, \zeta, t) - s = 0$, we have $F \neq 0$, and $\log F$ is well-defined.) Since $\log F = \log \sigma + (m - ln) \log \zeta + l \log(\zeta^n + t\tau)$, we have

$$\frac{\partial \log F}{\partial \zeta} = \frac{m - ln}{\zeta} + \frac{ln\zeta^{n-1}}{\zeta^n + t\tau}, \qquad \frac{\partial \log F}{\partial z} = \frac{\sigma_z}{\sigma} + \frac{lt\tau_z}{\zeta^n + t\tau},$$

and so from (1.3), the necessary and sufficient condition for $(z_0, \zeta_0) \in X_{s,t}$ $(s, t \neq 0)$ to be a singularity is given as follows:

(1.4)
$$\zeta_0^n = \frac{ln - m}{m} t\tau(z_0),$$

(1.5)
$$\sigma_z(z_0)\zeta_0^n = -t\Big[\sigma_z(z_0)\tau(z_0) + l\sigma(z_0)\tau_z(z_0)\Big].$$

Thus it is enough to verify that (1.5) is equivalent to (1.1).

(1.5) $\Longrightarrow$ (1.1): Multiplying (1.4) by $\sigma_z(z_0)$, we have

$$\sigma_z(z_0)\zeta_0^n = \frac{ln - m}{m} t\sigma_z(z_0)\tau(z_0).$$

This equation with (1.5) yields

$$\frac{ln - m}{m} t\sigma_z(z_0)\tau(z_0) = -t\Big[\sigma_z(z_0)\tau(z_0) + l\sigma(z_0)\tau_z(z_0)\Big].$$

As $t \neq 0$ by assumption, we deduce $m\sigma(z_0)\tau_z(z_0) + n\sigma_z(z_0)\tau(z_0) = 0$.

(1.1) $\Longrightarrow$ (1.5): Just reverse the above argument. □

We then give a finer result according to $m > ln$ or $m = ln$.

Proposition 1.2 *Suppose that z_0 is not a pole of τ.*

(1) *In the case $ln < m$, a point $(z_0, \zeta_0) \in X_{s,t}$, $(s, t \neq 0)$ is a singularity if and only if*

(1.6)
$$n\sigma_z(z_0)\tau(z_0) + m\sigma(z_0)\tau_z(z_0) = 0$$

(1.7)
$$\zeta_0^n = \frac{ln - m}{m} t\tau(z_0).$$

In this case, s and t satisfy

$$(1.8)\quad \left(\frac{ln-m}{ln}\right)^{al} s^a = \left(\frac{ln-m}{m}\right)^b t^b \sigma(z_0)^a \tau(z_0)^b,$$

where a and b are the positive integers such that $\mathrm{lcm}(m,n) = am = bn$.

(2) *In the case $ln = m$, a point $(z_0, \zeta_0) \in X_{s,t}$, $(s, t \neq 0)$ is a singularity if and only if $\zeta_0 = 0$, and z_0 satisfies $n\sigma_z(z_0)\tau(z_0) + m\sigma(z_0)\tau_z(z_0) = 0$; in this case, s and t satisfy $(\sigma\tau^l)(z_0)\, t^l = s$*

Proof. (1): The first half of the assertion is nothing but Lemma 1.1, and so it suffices to show that s and t satisfy (1.8). The defining equation of $X_{s,t}$ evaluated at (z_0, ζ_0) is

$$\sigma(z_0)\zeta_0^{m-ln}\left(\zeta_0^n + t\tau(z_0)\right)^l - s = 0.$$

Substituting $t\tau(z_0) = \dfrac{m}{ln-m}\zeta_0^n$ (1.7) into this equation, we obtain

$$\sigma(z_0)\zeta_0^{m-ln}\left(\zeta_0^n + \frac{m}{ln-m}\zeta_0^n\right)^l - s = 0,$$

and so

$$(1.9)\quad \sigma(z_0)\zeta_0^m = \left(\frac{ln-m}{ln}\right)^l s.$$

Thus if $(z_0, \zeta_0) \in X_{s,t}$ is a singularity, then

$$\zeta_0^n = \frac{ln-m}{m} t\tau(z_0) \quad (1.7) \qquad \text{and} \qquad \sigma(z_0)\zeta_0^m = \left(\frac{ln-m}{ln}\right)^l s \quad (1.9).$$

Now write $\mathrm{lcm}(m,n) = am = bn$, and taking powers of the above equations, we obtain

$$\zeta_0^{bn} = \left(\frac{ln-m}{m}\right)^b t^b \tau(z_0)^b, \qquad \sigma(z_0)^a \zeta_0^{am} = \left(\frac{ln-m}{ln}\right)^{al} s^a.$$

Since $am = bn$, the comparison of these two equations gives (1.8):

$$\left(\frac{ln-m}{ln}\right)^{al} s^a = \left(\frac{ln-m}{m}\right)^b t^b \sigma(z_0)^a \tau(z_0)^b.$$

(2): In this case ($m = ln$), the equation (1.7) is simply $\zeta_0^n = 0$, and so $\zeta_0 = 0$. Moreover since $m = ln$, the defining equation of $X_{s,t}$ is $\sigma(\zeta^n + t\tau)^l - s = 0$. Substituting z_0 and $\zeta_0 = 0$ into this equation, we obtain $\sigma(z_0)\tau(z_0)^l t^l = s$. □

Case 2. Next in order to investigate the case where z_0 is a pole of τ, we need the following lemma.

Lemma 1.3 *If $m = ln$, then $m_j = ln_j$ for all $j = 1, 2, \ldots, h$* (so $X = lY$), *and τ has no zeros.*

Proof. From $\frac{n_1 + n_2 + \cdots + n_h}{n} \geq \frac{m_1 + m_2 + \cdots + m_h}{m}$, we have

$$\frac{ln_1 + ln_2 + \cdots + ln_h}{ln} \geq \frac{m_1 + m_2 + \cdots + m_h}{m}. \tag{1.10}$$

Since $m = ln$, we have $ln_1 + ln_2 + \cdots + ln_h \geq m_1 + m_2 + \cdots + m_h$. However as $m_j \geq ln_j$ ($j = 1, 2, \ldots, h$), the above inequality is actually an equality, and also $m_j = ln_j$ ($j = 1, 2, \ldots, h$) hold. Then the equality of (1.10) implies that τ has no zero. □

Now we demonstrate that the condition "z_0 is a pole of τ" poses a numerical constraint.

Lemma 1.4 *Suppose that z_0 is a pole of τ. Then $(z_0, \zeta_0) \in X_{s,t}$ $(s \neq 0)$ is a singularity if and only if $m = ln$; in this case, s and t satisfy $(\sigma\tau^l)(z_0)\zeta_0^{m-ln}t^l = s$, and (z_0, ζ_0) is a non-isolated singularity (hence $X_{s,t}$ is non-reduced).*

(Note: $m = ln$ insures $m_j = ln_j$ ($j = 1, 2, \ldots, h$) by Lemma 1.3.)

Proof. Taking $F = \sigma\tau^l\zeta^{m-ln}\left(\frac{\zeta^n}{\tau} + t\right)^l$, then $X_{s,t} : F(z, \zeta, t) - s = 0$, and we apply the same argument as in the proof of Lemma 1.1 to see that $(z_0, \zeta_0) \in X_{s,t}$ is a singularity if and only if

(1.11)
$$\frac{1}{\tau}(z_0)\zeta_0^n = \frac{ln - m}{m}t,$$

(1.12)
$$\frac{\sigma_z}{\tau\tau_z}(z_0)\zeta_0^n = -t\left[\frac{\sigma_z}{\tau_z}(z_0) + l\frac{\sigma}{\tau}(z_0)\right].$$

On the other hand, since z_0 is a pole of τ, we have $\frac{1}{\tau}(z_0) = 0$, and thus (1.11) is just $0 = \frac{ln - m}{m}t$, and by $t \neq 0$, we derive $ln = m$. (Note that from $\frac{1}{\tau}(z_0) = 0$ and $\frac{1}{\tau_z}(z_0) = 0$, the equation (1.12) trivially holds; hence ζ_0 is arbitrary.) Therefore $(z_0, \zeta_0) \in X_{s,t}$ is a singularity if and only if $ln = m$. The equation of $X_{s,t}$ evaluated at (z_0, ζ_0) is

$$(\sigma\tau^l)(z_0)\zeta_0^{m-ln}\left(\zeta_0^n\frac{1}{\tau}(z_0) + t\right)^l - s = 0.$$

Since $\frac{1}{\tau}(z_0) = 0$, we have $(\sigma\tau^l)(z_0)\zeta_0^{m-ln}t^l - s = 0$; so s and t satisfy $(\sigma\tau^l)(z_0)\zeta_0^{m-ln}t^l = s$. Note that though ζ_0 is arbitrary, for each ζ_0, there are at most a finite number of z_0 satisfying this equation. □

It is immediate to deduce

Corollary 1.5 *If $m > ln$, then there is no singularity $(z_0, \zeta_0) \in X_{s,t}$ $(s \neq 0)$ such that z_0 is a pole or a zero of τ, that is, $\sigma(z_0) = 0$ or $\tau(z_0) = 0$.*

Proof. If $ln < m$, then z_0 is not a pole of τ by Lemma 1.4. It remains to show that z_0 is not a zero of τ. In fact, if $\tau(z_0) = 0$, we substitute this into (1.8) of Proposition 1.2 (note that as z_0 is not a pole of τ, we may apply that proposition):

$$\left(\frac{ln-m}{ln}\right)^{al} s^a = \left(\frac{ln-m}{m}\right)^{b} t^b \sigma(z_0)^a \tau(z_0)^b,$$

and we derive $\left(\frac{ln-m}{ln}\right)^{al} s^a = 0$, and so $s = 0$ by $ln - m \neq 0$. This contradicts the assumption $s \neq 0$. □

Next we study singularities of $X_{s,t}$ from the viewpoint of a function $n\sigma_z\tau + m\sigma\tau_z$. We note that this function is rather a global object on C. Indeed, $\sigma^n\tau^m$ is a section of $N^{\otimes(-nm)} \otimes N^{\otimes(mn)} \cong \mathcal{O}_C$; so $\sigma^n\tau^m$ is a (meromorphic) function on C. Then $n\sigma_z\tau + m\sigma\tau_z$ appears in the logarithmic derivative of a function $\sigma^n\tau^m$:

$$\frac{d\log(\sigma^n\tau^m)}{dz} = \frac{n\sigma_z\tau + m\sigma\tau_z}{\sigma\tau},$$

where $\frac{n\sigma_z\tau + m\sigma\tau_z}{\sigma\tau}dz$ is a (meromorphic) 1-form on C.

Lemma 1.6 *Suppose that $m\sigma\tau_z + n\sigma_z\tau$ is not identically zero. Let $(z_0, \zeta_0) \in X_{s,t}$ $(s, t \neq 0)$ be a singularity.*

(1) *If $ln < m$, then z_0 is not a pole of τ, and (z_0, ζ_0) is an isolated singularity.*

(2) *If $ln = m$, then there are two cases: (2.1) z_0 is not a pole of τ and (z_0, ζ_0) is an isolated singularity, and (2.2) z_0 is a pole of τ and (z_0, ζ_0) is a non-isolated singularity (hence $X_{s,t}$ is non-reduced).*

Proof. (1): For the case $ln = m$, z_0 is not a pole of τ by Corollary 1.5, which also guarantees us to be able to apply Lemma 1.1: $(z_0, \zeta_0) \in X_{s,t}$ $(s, t \neq 0)$ is a singularity if and only if

$$(1.13)\quad n\sigma_z(z_0)\tau(z_0) + m\sigma(z_0)\tau_z(z_0) = 0, \qquad \zeta_0^n = \frac{ln-m}{m} t\tau(z_0).$$

Since $m\sigma\tau_z + n\sigma_z\tau$ is not identically zero, there are at most a finite number of z_0 which satisfy $m\sigma(z_0)\tau_z(z_0) + n\sigma_z(z_0)\tau(z_0) = 0$, and for each z_0, there are at most finite number (just n including multiplicity) of ζ_0 satisfying $\zeta_0^n = \frac{ln-m}{m}t\tau(z_0)$. Hence there are at most finite points (z_0, ζ_0) which are singularities on $X_{s,t}$, in other words, $X_{s,t}$ has only isolated singularities.

(2): Suppose that $m = ln$. (2.1) If z_0 is not a pole of τ, then $(z_0, \zeta_0) \in X_{s,t}$ is a singularity if and only if $\zeta_0 = 0$, and z_0 satisfies $n\sigma_z(z_0)\tau(z_0) + m\sigma(z_0)\tau_z(z_0) = 0$ by Lemma 1.2 (2). Since $m\sigma\tau_z + n\sigma_z\tau$ is not identically zero by assumption, there are at most finite number of z_0 satisfying this condition, and so (z_0, ζ_0), where $\zeta_0 = 0$, is an isolated singularity. (2.2) If z_0 is a pole of τ, then (z_0, ζ_0) is a singularity if and only if $ln = m$, and in this case ζ_0 is arbitrary as we saw in the proof of Lemma 1.4, and thus (z_0, ζ_0) is a non-isolated singularity. □

We summarize the results obtained above. Below, (I) is a consequence of Proposition 1.2 and Lemma 1.6 (1), while (II) is a consequence of Proposition 1.2 and Lemma 1.6 (2).

Proposition 1.7 *Suppose that the function $m\sigma\tau_z + n\sigma_z\tau$ of z is not identically zero.*

(1) *If $m > ln$, then $(z_0, \zeta_0) \in X_{s,t}$ $(s \neq 0)$ is a singularity if and only if z_0 is not a pole of τ, and z_0 and ζ_0 satisfy*

$$n\sigma_z(z_0)\tau(z_0) + m\sigma(z_0)\tau_z(z_0) = 0, \qquad \zeta_0^n = \frac{ln - m}{m} t\tau(z_0).$$

In this case, s and t satisfy $\left(\frac{ln-m}{ln}\right)^{al} s^a = \left(\frac{ln-m}{m}\right)^b t^b \sigma(z_0)^a \tau(z_0)^b$, where a and b are the positive integers such that $\mathrm{lcm}(m, n) = am = bn$.

(2) *If $m = ln$, then $(z_0, \zeta_0) \in X_{s,t}$ $(s \neq 0)$ is a singularity if and only if (2.1) z_0 is not a pole of τ, $n\sigma_z(z_0)\tau(z_0) + m\sigma(z_0)\tau_z(z_0) = 0$, and $\zeta_0 = 0$, or (2.2) z_0 is a pole of τ and ζ_0 is arbitrary. In both (2.1) and (2.2), s and t satisfy $(\sigma\tau)^l(z_0)t^l = s$.*

In (1) and (2), a singularity $(z_0, \zeta_0) \in X_{s,t}$ is isolated except (2.2) (that is, except the case where z_0 is a pole of τ).

We next consider the case where $m\sigma\tau_z + n\sigma_z\tau = 0$ (identically zero as a function of z). We note that

$$\begin{aligned} m\sigma\tau_z + n\sigma_z\tau = 0 &\Longleftrightarrow m\frac{\tau_z}{\tau} + n\frac{\sigma_z}{\sigma} = 0 \Longleftrightarrow \frac{d\log(\sigma^n\tau^m)}{dz} = 0 \\ &\Longleftrightarrow \log(\sigma^n\tau^m) \text{ is constant.} \end{aligned}$$

Thus $\sigma^n\tau^m$ is constant and clearly nonzero. Furthermore we note that $\sigma^n\tau^m$ is of the form $(z^{m_j})^n\left(\frac{1}{z^{n_j}}\right)^m h_j$, that is, $z^{m_jn - mn_j}h_j$ around p_j where $h_j = h_j(z, \zeta)$ is some non-vanishing holomorphic function. Since $\sigma^n\tau^m$ is constant, we have $m_jn - mn_j = 0$. Thus $\frac{m_j}{n_j} = \frac{m}{n}$ $(j = 1, 2, \ldots, h)$. Letting u and v be the relatively prime positive integers satisfying $\frac{u}{v} = \frac{m}{n}\left(= \frac{m_j}{n_j}\right)$, then $vm = un$ and $vm = un_j$ $(j = 1, 2, \ldots, h)$, so $vX = uY$. This confirms the following.

Lemma 1.8 *If $m\sigma\tau_z + n\sigma_z\tau$ is identically zero, then $vX = uY$ where u and v are the relatively prime positive integers satisfying $\frac{u}{v} = \frac{m}{n}$.*

Next under the assumption that $m\sigma\tau_z + n\sigma_z\tau$ is identically zero, we shall investigate the singularities on $X_{s,t}$ $(s, t \neq 0)$. Let us divide into two cases $ln < m$ and $ln = m$.

Case $ln < m$: By Corollary 1.5, z_0 is not a pole of τ. Then by Proposition 1.2 (1), $(z_0, \zeta_0) \in X_{s,t}$ $(s, t \neq 0)$ is a singularity if and only if $\zeta_0^n = \dfrac{ln - m}{m} t\tau(z_0)$, in which case s and t satisfy

$$\left(\frac{ln - m}{ln}\right)^{al} s^a = \left(\frac{ln - m}{m}\right)^b t^b \sigma(z_0)^a \tau(z_0)^b,$$

where a and b are the positive integers such that $\operatorname{lcm}(m, n) = am = bn$. In particular, z_0 is a zero of a holomorphic function of one variable z:

$$\left(\frac{ln - m}{m}\right)^b t^b \sigma(z)^a \tau(z)^b - \left(\frac{ln - m}{ln}\right)^{al} s^a,$$

where s and t are fixed. Such z_0 is isolated, and for each z_0 we have a finite number of ζ_0 satisfying $\zeta_0^n = \dfrac{ln - m}{m} t\tau(z_0)$. Thus $(z_0, \zeta_0) \in X_{s,t}$ $(s, t \neq 0)$ is an isolated singularity.

Case $ln = m$: For the case where z_0 is a pole, as we saw in Lemma 1.4, any singularity $(z_0, \zeta_0) \in X_{s,t}$ $(s, t \neq 0)$ is non-isolated. Next we consider the case where z_0 is not a pole. By Proposition 1.2 (2), (z_0, ζ_0) is a singularity if and only if $\zeta_0 = 0$ (as $m\sigma\tau_z + n\sigma_z\tau$ is identically zero, the condition $m\sigma(z_0)\tau_z(z_0) + n\sigma_z(z_0)\tau(z_0)$ is trivially fulfilled); in which case s and t satisfy $(\sigma\tau^l)(z_0)\, t^l = s$, and so z_0 is a zero of a holomorphic function $(\sigma\tau^l)(z)\, t^l - s$ of one variable z, where s and t are fixed. Such z_0 is isolated and hence the singularity (z_0, ζ_0), where $\zeta_0 = 0$, is isolated. Thus we obtain the following lemma.

Lemma 1.9 *Suppose that $m\sigma\tau_z + n\sigma_z\tau$ is identically zero. Let $(z_0, \zeta_0) \in X_{s,t}$ $(s, t \neq 0)$ be a singularity.*

(1) *If $ln < m$, then z_0 is not a pole of τ, and (z_0, ζ_0) is an isolated singularity.*

(2) *if $lm = m$, then there are two cases: (2.1) z_0 is not a pole of τ and (z_0, ζ_0) is an isolated singularity, and (2.2) z_0 is a pole of τ and (z_0, ζ_0) is a non-isolated singularity (hence $X_{s,t}$ is non-reduced).*

Lemma 1.9 with Proposition 1.2 gives us:

Proposition 1.10 *Suppose that the function $m\sigma\tau_z + n\sigma_z\tau$ of z is identically zero.*

(1) *If* $m > ln$*, then* $(z_0, \zeta_0) \in X_{s,t}$ $(s \neq 0)$ *is a singularity if and only if* z_0 *is not a pole of* τ*, and* z_0 *and* ζ_0 *satisfy* $\zeta_0^n = \frac{ln - m}{m} t\tau(z_0)$*. In this case,* s *and* t *satisfy*

$$\left(\frac{ln - m}{ln}\right)^{al} s^a = \left(\frac{ln - m}{m}\right)^b t^b \sigma(z_0)^a \tau(z_0)^b,$$

where a *and* b *are the positive integers such that* $\mathrm{lcm}(m, n) = am = bn$*.*

(2) *If* $m = ln$*, then* $(z_0, \zeta_0) \in X_{s,t}$ $(s, t \neq 0)$ *is a singularity if and only if* (2.1) z_0 *is not a pole of* τ *and* $\zeta_0 = 0$*, or* (2,2) z_0 *is a pole of* τ *and* ζ_0 *is arbitrary.*

In both (2.1) *and* (2.2)*,* s *and* t *satisfy* $(\sigma\tau^l)(z_0)\, t^l = s$*.*

In (1) *and* (2)*, a singularity* $(z_0, \zeta_0) \in X_{s,t}$ *is isolated except* (2.2) (*that is, except the case where* z_0 *is a pole of* τ)*.*

2 Awata polynomial

As we showed in Proposition 1.2 (1), if $m > ln$ and z_0 is not a pole of τ, then $(z_0, \zeta_0) \in X_{s,t}$, $(s, t \neq 0)$ is a singularity if and only if

$$n\sigma_z(z_0)\tau(z_0) + m\sigma(z_0)\tau_z(z_0) = 0, \qquad \zeta_0^n = \frac{ln - m}{m} t\tau(z_0).$$

Actually when C is the projective line, we may rewrite the equation on the left into a simpler form. To explain this, we prepare some notation. Taking the standard open covering $\mathbb{P}^1 = U \cup V$ where $z \in U$ is identified with $w \in V$ via $z = 1/w$, let N be a line bundle on $\mathbb{P}^1$ obtained by patching $(z, \zeta) \in U \times \mathbb{C}$ with $(w, \eta) \in V \times \mathbb{C}$ via $z = 1/w$ and $\zeta = w^r \eta$, where

$$r = \frac{m_1 + m_2 + \cdots + m_h}{m}.$$

Taking $p_h = \infty$, at first we consider the case:

$$\sigma = \begin{cases} (z - p_1)^{m_1}(z - p_2)^{m_2} \dots (z - p_{h-1})^{m_{h-1}} & \text{on } U \\ (1 - p_1 w)^{m_1}(1 - p_2 w)^{m_2} \dots (1 - p_{h-1} w)^{m_{h-1}} w^{m_h} & \text{on } V, \end{cases}$$

and

$$\tau = \begin{cases} \dfrac{1}{(z - p_1)^{n_1}(z - p_2)^{n_2} \dots (z - p_{h-1})^{n_{h-1}}} & \text{on } U \\ \dfrac{1}{(1 - p_1 w)^{n_1}(1 - p_2 w)^{n_2} \dots (1 - p_{h-1} w)^{n_{h-1}} w^{n_h}} & \text{on } V. \end{cases}$$

So σ is a holomorphic section of $N^{\otimes(-m)}$ and τ is a meromorphic section of $N^{\otimes n}$.

Remark 2.1 We may choose $p_1, p_2, \dots, p_h$ so that none of them is ∞. Unfortunately, such a choice may result in the appearance of a singularity $(z_0, w_0) \in C_{s,t}$ with $z_0 = \infty$. However, by taking $p_h = \infty$, we may insure this does not occur.

In this case, we have

$$\log(\sigma^n \tau^m) = \log \left[\prod_{j=1}^{h-1} (z - p_j)^{nm_j - mn_j} \right] = \sum_{j=1}^{h-1} (nm_j - mn_j) \log(z - p_j).$$

(We carry out the computation only on U; everything is the same as on V.) Thus

$$(2.1)\quad \frac{d \log(\sigma^n \tau^m)}{dz} = \sum_{j=1}^{h-1} \frac{nm_j - mn_j}{z - p_j}.$$

We set $P = (z - p_1)(z - p_2) \cdots (z - p_{h-1})$ and

$$P_j := \frac{P}{z - p_j} = (z - p_1)(z - p_2) \cdots (z - p_{j-1})(z - p_{j+1}) \cdots (z - p_{h-1}).$$

Multiplying the equation (2.1) by P, we deduce

$$(2.2)\quad P \frac{d \log(\sigma^n \tau^m)}{dz} = \sum_{j=1}^{h-1} (nm_j - mn_j) P_j.$$

Notice that "multiplying $\frac{d \log(\sigma^n \tau^m)}{dz}$ by P" corresponds to " multiplying $m\sigma\tau_z + n\sigma_z\tau$ by $\frac{P}{\sigma\tau}$", where we note that

$$\frac{P}{\sigma\tau} = \frac{1}{\prod_{j=1}^{h-1} (z - p_j)^{1 + m_j - n_j}}$$

is non-vanishing holomorphic provided that $(z, \zeta) \in X_{s,t}$ $(s, t \neq 0)$ is a singularity, because in that case, z is not a pole of τ (i.e. $z \neq p_j$) by Corollary 1.5.

We denote by $I(z)$ the polynomial on the right hand side of (2.2):

$$I(z) = \sum_{j=1}^{h-1} (nm_j - mn_j) P_j.$$

Then $m\sigma\tau_z(z_0) + n\sigma_z\tau(z_0) = 0$ is equivalent to $I(z_0) = 0$. We remark that if $nm_j - mn_j = 0$ for some j, we may rewrite $I(z_0) = 0$ into a simpler equation. Namely we may further simplify the polynomial $I(z)$ by dividing by some non-vanishing function. For instance, if $nm_{h-1} - mn_{h-1} = 0$, then

$$I(z) = \sum_{j=1}^{h-2} (nm_j - mn_j) P_j,$$

where note that P_j $(j = 1, 2, \ldots, h-2)$ is divisible by $z - p_{h-1}$, and so $\dfrac{I(z)}{z - p_{h-1}}$ is a polynomial. Hereafter we assume that not all of $nm_j - mn_j$ $(j = 1, 2, \ldots, h)$ are zero; for later application to the deformations of degenerations of compact curves with stellar singular fibers, we need to consider only this case. Without loss of generalty, we assume that $nm_h - mn_h \neq 0$. Now if there are v subscripts j satisfying $nm_j - mn_j = 0$, say, $j = h-v, h-v+1, \ldots, h-1$, then

$$I(z) = \sum_{j=1}^{h-1-v} (nm_j - mn_j) P_j,$$

which is divisible by $(z - p_{h-v})(z - p_{h-v+1}) \cdots (z - p_{h-1})$, and denote the polynomial

$$\frac{I(z)}{(z - p_{h-v})(z - p_{h-v+1}) \cdots (z - p_{h-1})}$$

by $J(z)$, called an *Awata polynomial* for the case $nm_j - mn_j = 0$ for $j = h - v + 1, h - v + 2, \ldots, h - 1$. (Of course if $nm_j - mn_j \neq 0$ for all j, then $J(z) = I(z) = \sum_{j=1}^{h-1} (nm_j - mn_j) P_j$.) Alternatively an Awata polynomial is defined as follows:

$$J(z) := \sum_j (nm_j - mn_j) R_j$$

where j runs over all subscripts $j \in \{1, 2, \ldots, h-1\}$ satisfying $nm_j - mn_j \neq 0$, and $R_j = \prod_i (z - p_i)$ where i runs over all subscripts $i \in \{1, 2, \ldots, h-1\}$ satisfying $i \neq j$ and $nm_i - mn_i \neq 0$. Letting v be the number of subscripts $j \in \{1, 2, \ldots, h-1\}$ satisfying $nm_j - mn_j = 0$, then the degree of $J(z)$ is of $h - 2 - v$, as that of R_j is $h - 2 - v$.

¿From the above discussion, $m\sigma(z_0)\tau_z(z_0) + n\sigma_z(z_0)\tau(z_0) = 0$ is equivalent to $J(z_0) = 0$, and so $(z_0, \zeta_0) \in X_{s,t}$, $(s, t \neq 0)$ is a singularity if and only if

$$J(z_0) = 0, \qquad \zeta_0^n = \frac{ln - m}{m} t\tau(z_0).$$

(Note: In the present case, i.e. C is the projective line and not all of $nm_j - mn_j$ $(j = 1, 2, \ldots, h)$ are zero, since $m\sigma\tau_z + n\sigma_z\tau$ is not identically zero, we do not need the assumption that z_0 is not a pole of τ. See Proposition 1.7 (I), p8.) So z_0 is a root of $J(z)$ and for each z_0, there are m solutions of $\zeta_0^n = \dfrac{ln - m}{m} t\tau(z_0)$, which also confirms again that a singular fiber $X_{s,t}$ $(s, t \neq 0)$ has only isolated singulalrties. Note that the number of roots of $J(z)$ varies as $p_1, p_2, \ldots, p_{h-3}$ vary (we ignore the multiplicity of a root, and we fixed $p_{h-2} = 0, p_{h-1} = 1$, and $p_h = \infty$; note that without loss of generality we may fix arbitrary three points on $\mathbb{P}^1$). Accordingly the number of singularities varies. In the generic case, the number of roots is $h - 2 - v$ where v is the number of subscripts $j \in \{1, 2, \ldots, h-1\}$ satisfying $nm_j - mn_j = 0$; letting

$D = D(p_1, p_2, \ldots, p_{h-3})$ be the discriminant of $J(z)$, we say that $p_1, p_2, \ldots, p_h$ are *in generic position* if $D(p_1, p_2, \ldots, p_{h-3}) \neq 0$. Note that "generic" is the notion only applied for $h \geq 4$.

More generally, we take τ with zeros, that is,

$$\tau = \begin{cases} \dfrac{(z-q_1)^{a_1}(z-q_2)^{a_2}\ldots(z-q_k)^{a_k}}{(z-p_1)^{n_1}(z-p_2)^{n_2}\ldots(z-p_{h-1})^{n_{h-1}}} & \text{on } U \\ \dfrac{(1-q_1w)^{a_1}(1-q_2w)^{a_2}\ldots(1-q_kw)^{a_k}}{(1-p_1w)^{n_1}(1-p_2w)^{n_2}\ldots(1-p_{h-1}w)^{n_{h-1}}w^{n_h}} & \text{on } V, \end{cases}$$

while σ is the same as before:

$$\sigma = \begin{cases} (z-p_1)^{m_1}(z-p_2)^{m_2}\ldots(z-p_{h-1})^{m_{h-1}} & \text{on } U \\ (1-p_1w)^{m_1}(1-p_2w)^{m_2}\ldots(1-p_{h-1}w)^{m_{h-1}}w^{m_h} & \text{on } V. \end{cases}$$

Then

$$\begin{aligned} \log(\sigma^n\tau^m) &= \log\left[\prod_{j=1}^{h-1}(z-p_j)^{nm_j-mn_j} \cdot \prod_{i=1}^{k}(z-q_i)^{ma_i}\right] \\ &= \sum_{j=1}^{h-1}(nm_j-mn_j)\log(z-p_j) + \sum_{i=1}^{k} ma_i \log(z-p_j). \end{aligned}$$

Thus we have

$$\text{(2.3)} \quad \frac{d\log(\sigma^n\tau^m)}{dz} = \sum_{j=1}^{h-1}\frac{nm_j-mn_j}{z-p_j} + \sum_{i=1}^{k} ma_i\frac{1}{z-q_i}.$$

We set $P = (z-p_1)(z-p_2)\cdots(z-p_{h-1})$ and

$$P_j := \frac{P}{z-p_j} = (z-p_1)(z-p_2)\cdots(z-p_{j-1})(z-p_{j+1})\cdots(z-p_{h-1}).$$

Likewise, we set $Q = (z-q_1)(z-q_2)\cdots(z-q_k)$ and

$$Q_i := \frac{Q}{z-q_i} = (z-q_1)(z-q_2)\cdots(z-q_{i-1})(z-q_{i+1})\cdots(z-q_k).$$

Multiplying (2.3) by PQ, we deduce

$$PQ\log(\sigma^n\tau^m) = Q\sum_{j=1}^{h-1}(nm_j-mn_j)P_j + P\sum_{i=1}^{k} ma_iQ_i.$$

Here "multiplying $\dfrac{d\log(\sigma^n\tau^m)}{dz}$ by PQ" corresponds to "multiplying $m\sigma\tau_z + n\sigma_z\tau$ by $\dfrac{PQ}{\sigma\tau}$", where we note

$$\frac{PQ}{\sigma\tau} = \frac{1}{\prod_{j=1}^{h-1}(z-p_j)^{1+m_j-n_j}\prod_{i=1}^{k}(z-q_i)^{a_i-1}}$$

is non-vanishing holomorphic if $(z,\zeta) \in X_{s,t}$ $(s,t \neq 0)$ is a singularity; in fact, by Corollary 1.5, z is neither p_j (a pole of τ) nor q_j (a zero of τ). We set

$$I(z) = Q\sum_{j=1}^{h-1}(nm_j - mn_j)P_j + P\sum_{i=1}^{k} ma_i Q_i,$$

and then $m\sigma\tau_z(z_0) + n\sigma_z\tau(z_0) = 0$ is equivalent to $I(z_0) = 0$. We note that if $nm_j - mn_j = 0$ for some j, we may further simplify the polynomial $I(z)$ by dividing by $\prod_j(z-p_j)$ where j runs over such subscripts $j \in \{1,2,\ldots,h-1\}$ that $nm_j - mn_j = 0$. If there are v subscripts j satisfying $nm_j - mn_j = 0$ (after reordering, we assume $j = h-v, h-v+1, \ldots, h$), then

$$I(z) = Q\sum_{j=1}^{h-1-v}(nm_j - mn_j)P_j + P\sum_{i=1}^{k} ma_i Q_i,$$

where note that both P and P_j are divisible by

$$(z-p_{h-v})(z-p_{h-v+1})\cdots(z-p_{h-1}).$$

We then consider polynomials

$$\begin{aligned}\widehat{P}_j &:= \frac{P_j}{(z-p_{h-v})(z-p_{h-v+1})\cdots(z-p_{h-1})},\\ \widehat{P} &:= \frac{P}{(z-p_{h-v})(z-p_{h-v+1})\cdots(z-p_{h-1})},\end{aligned}$$

and then a polynomial

$$J(z) := Q\sum_{j=1}^{h-1-v}(nm_j - mn_j)\widehat{P}_j + \widehat{P}\sum_{i=1}^{k} ma_i Q_i,$$

is called an *Awata polynomial* for the case $nm_j - mn_j = 0$ for $j = h-v, h-v+1, \ldots, h-1$.

Remark 2.2 In particular, if C is the projective line, as $m\sigma\tau_z + n\sigma_z\tau$ is a multiplication of $J(z)$ with some non-vanishing function, it is not idetically zero.

Theorem 2.3 (Awata) *Suppose that l is a positive integer and N is a line bundle on a curve C such that $N^{\otimes(-m)}$ has a holomorphic section σ with a zero of order m_j at p_j $(j=1,2,\ldots,h)$ and $N^{\otimes n}$ has a meromorphic section τ with a pole of order n_j $(0 \leq n_j \leq lm_j)$ at p_j $(j=1,2,\ldots,h)$. Let $X_{s,t}:\ \sigma(\zeta^n + t\tau)^l - s = 0$ be a barking family of (open) curves, parametrized by s and t. If $m > ln$ and C is the projective line, then a point $(z_0,\zeta_0) \in X_{s,t}$, $(s,t \neq 0)$ is a singularity if and only if $J(z_0) = 0$ and $\zeta_0^n = \dfrac{ln-m}{m}t\tau(z_0)$.*

Note that the number of roots of $J(z)$ varies as $p_1, p_2, \ldots, p_{h-3}$ and $q_1, q_2, \ldots, q_k$ vary (we ignore the multiplicity of a root and we fix $p_{h-2} = 0, p_{h-1} = 1$, and $p_h = \infty$); accordingly the number of singularities varies. Letting $D(p_1, p_2, \ldots, p_{h-3}, q_1, q_2, \ldots, q_k)$ be the discriminant of the Awata polynomial $J(z)$, we say that $p_1, p_2, \ldots, p_h$ and $q_1, q_2, \ldots, q_k$ are *in generic position* if

$$D(p_1, p_2, \ldots, p_{h-3}, q_1, q_2, \ldots, q_k) \neq 0.$$

In the generic case, the number of roots of $J(z)$ is of $h + k - 2 - v$, where v is the number of subscripts $j \in \{1, 2, \ldots, h-1\}$ satisfying $nm_j - mn_j = 0$.