THE BRAID INDEX OF SURFACE-KNOTS AND QUANDLE COLORINGS

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Dedicated to Professor Yukio Matsumoto on the occasion of his 60th birthday

Abstract. The braid index of a surface-knot F is the minimal number among the degrees of all simple surface braids whose closures are ambient isotopic to F. We prove that an S^2-knot which is the connected sum of k copies of the spun $(2, p)$-torus knot has the braid index $k+2$. To prove it, we use colorings of surface-knots by quandles and give lower bounds of the braid index of surface-knots.

1. Introduction

1.1. **Surface-knots, surface braids and closures.** A *surface-knot* is a closed, connected, oriented surface embedded locally flatly in $\mathbb{R}^4$. The notion of a *surface braid* was defined by Viro [15] and extensively studied by Kamada [9]. A similar notion was investigated also by Rudolph [13, 14]. A *surface braid of degree* m is a compact oriented surface S embedded properly and locally flatly in $B_1^2 \times B_2^2$, where B_i^2 is a 2-disk $(i = 1, 2)$, such that

(i) the restriction map $\pi|_S$ of the projection $\pi : B_1^2 \times B_2^2 \to B_2^2$ is a branched covering map of degree m, and
(ii) $\partial S = P_m \times \partial B_2^2 \left(\subset B_1^2 \times \partial B_2^2\right)$ for a fixed set P_m of m distinct interior points of B_1^2.

A surface braid is called *simple* if the covering $\pi|_S$ is simple (i.e., the preimage of each branch locus consists of $m - 1$ points).

Let S^2 be a 2-sphere obtained from B_2^2 by attaching a 2-disk $\overline{B_2^2}$ along the boundary of B_2^2. A surface braid S of degree m is extended to a closed surface $\widehat{S}$ in $B_1^2 \times S^2$ such that

$$\widehat{S} \cap (B_1^2 \times B_2^2) = S \text{ and } \widehat{S} \cap (B_1^2 \times \overline{B_2^2}) = X_m \times \overline{B_2^2} \ .$$

Identifying $B_1^2 \times S^2$ with the tubular neighborhood of a standard 2-sphere in $\mathbb{R}^4$, we assume that $\widehat{S}$ is a closed oriented surface embedded in $\mathbb{R}^4$ and call it the *closure* of S.

Surface braids are closely related to surface-knots; as an analogue to Alexander's theorem in classical knot theory, Viro [15] and Kamada [7] proved that any surface-knot is ambient isotopic to the closure of a simple surface braid. Refer to [9, 1] for more details.

1.2. **Braid index of surface-knots.** The *braid index* of a surface-knot F, denoted by $\mathrm{Braid}(F)$, is defined to be the minimal number among the degrees of all simple surface braids whose closures are ambient isotopic to F in $\mathbb{R}^4$.

There are several results on the braid index of a surface-knot; see [6, 8, 10], for example. Surface-knots with braid index less than three are unknotted, and those with braid index three are "ribbon" [6]. The 2-twist spun trefoil, for example, is not ribbon, and hence has the braid index four [6]. However, the braid index, for higher than four, has never been precisely determined for any specific examples of surface-knots. In this talk, we prove:

Theorem 1. *The connected sum of k copies of the spun $(2,p)$-torus knot has the braid index $k+2$, where p is an odd integer with $p \geq 3$.*

2. Quandles, diagrams and colorings

To prove Theorem 1, we use colorings of surface-knots by quandles.

2.1. Quandles.

A *quandle* [2, 5, 11] is a non-empty set X equipped with a binary operation $(a,b) \mapsto a*b$ such that (i) $a*a = a$ for any $a \in X$, (ii) the map $*a : X \to X$ $(x \mapsto x*a)$ is bijective for each $a \in X$, and (iii) $(a*b)*c = (a*c)*(b*c)$ for any $a, b, c \in X$.

The *dihedral quandle of order p*, denoted by R_p, is a quandle consisting of the set $\{0, 1, \ldots, p-1\}$ with the binary operation defined by $i * j \equiv 2j - i \pmod{p}$.

2.2. Diagrams of surface-knots.

For a fixed projection $\pi : \mathbb{R}^4 \to \mathbb{R}^3$, by perturbing a surface-knot F if necessary, we may assume that the projection $\pi|_F$ is generic, that is, $\pi|_F$ has double points, isolated triple points and isolated branch points in the image as its singularities. A *diagram* of a surface-knot is a generic projection image in $\mathbb{R}^3$ where one of the two sheets near the double point curve is broken depending on the relative height (See Figure 1). This convention is similar to classical knot diagrams. A diagram consists of *broken sheets*, that are mutually disjoint compact oriented surfaces in $\mathbb{R}^3$, and the orientations are specified by normal vectors. Refer to [1] for more details.

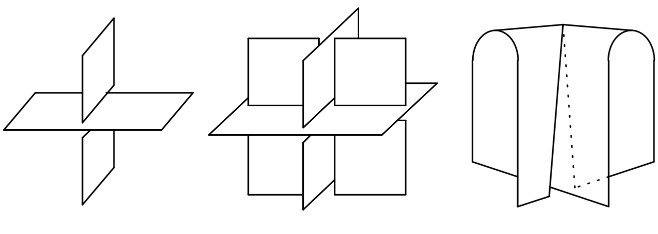

Figure 1

2.3. Quandle colorings.

A *coloring* of a surface-knot diagram by a quandle X is an assignment of an element of X to each broken sheet such that $a * b = c$ holds along each double point curve, where a (resp. c) is the color of under-sheet that is behind (resp. in front of) the over-sheet colored b with respect to the normal vector of the over-sheet (See Figure 2). We remark that the number of the colorings is an invariant of a surface-knot and that the coloring by R_p is coincident with the Fox *p-coloring* [3, 4].

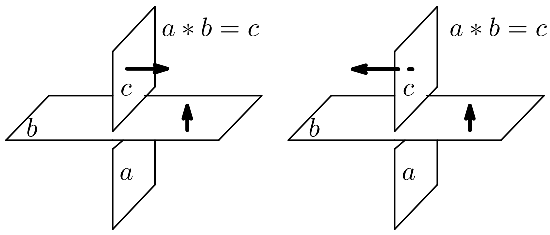

FIGURE 2

3. Key proposition and proof of Theorem 1

The following is the key proposition to prove Theorem 1.

Proposition 2. *Let F be a surface-knot which is not a trivial S^2-knot. If there is a finite quandle X with n elements such that F admits at least n^s colorings by X for some integer $s > 0$, then the braid index of F is at least $s+1$.*

Outline of the proof. Let m be the braid index of F. Consider a simple surface braid S of degree m whose closure $\hat{S}$ presents F. Since F is not a trivial S^2-knot, the diagram of S has branch points. Further, we may assume that the first and second boundary circles belong to the same broken sheet (cf. [10, Lemma 12]); Figure 3 shows this situation where a branch point connects the first and second broken sheets near the boundary circles. It follows that the surface-knot F admits at most n^{m-1} colorings by X. Thus we obtain $n^s \leq n^{m-1}$, that is, $m \geq s+1$. □

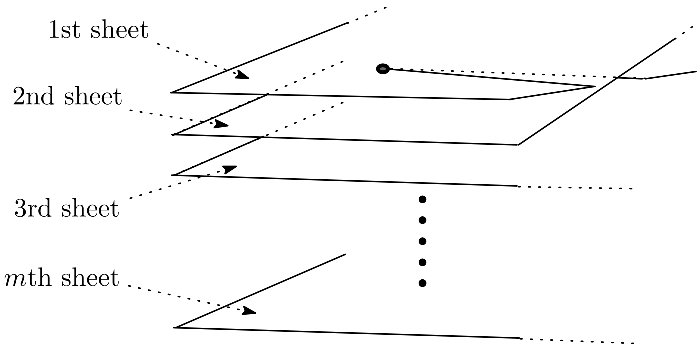

FIGURE 3

Proof of Theorem 1. Let $F_p(k)$ be the connected sum of k copies of the spun $(2,p)$-torus knot. Since the number of the colorings of $F_p(1)$ by the dihedral quandle R_p of order p is equal to p^2, that of $F_p(k)$ is equal to p^{k+1} (cf. [12]). Hence the braid index of $F_p(k)$ is at least $k+2$ by Proposition 2.

On the other hand, the following was proved by Kamada, Satoh and Takabayashi [10, Theorem 3]: if neither F_1 nor F_2 is a trivial S^2-knot, then the inequality

$$(*) \qquad \mathrm{Braid}(F_1 \# F_2) \leq \mathrm{Braid}(F_1) + \mathrm{Braid}(F_2) - 2$$

holds for the connected sum $F_1 \# F_2$ of two surface-knots F_1 and F_2. Thus the braid index of $F_p(k)$ is at most $k+2$, since that of $F_p(1)$ is three. □

We obtain the following by an argument similar to that in the proof of Proposition 1.

Corollary 3. *The connected sum of k copies of the spun $(2,p)$-torus knot and g copies of the the trivial T^2-knot has the braid index $k+2$, where p is an odd integer with $p \geq 3$.*

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