THE BRAID INDEX OF SURFACE-KNOTS
AND QUANDLE COLORINGS

KOKORO TANAKA

Dedicated to Professor Yukio Matsumoto on the occasion of his 60th birthday

Abstract. The braid index of a surface-knot $F$ is the minimal number among the degrees of all simple surface braids whose closures are ambient isotopic to $F$. We prove that an $S^2$-knot which is the connected sum of $k$ copies of the spun $(2, p)$-torus knot has the braid index $k + 2$. To prove it, we use colorings of surface-knots by quandles and give lower bounds of the braid index of surface-knots.

1. Introduction

1.1. Surface-knots, surface braids and closures. A surface-knot is a closed, connected, oriented surface embedded locally flatly in $\mathbb{R}^4$. The notion of a surface braid was defined by Viro [15] and extensively studied by Kamada [9]. A similar notion was investigated also by Rudolph [13, 14]. A surface braid of degree $m$ is a compact oriented surface $S$ embedded properly and locally flatly in $B^2_i \times B^2_2$, where $B^2_i$ is a 2-disk ($i = 1, 2$), such that

(i) the restriction map $\pi|_S$ of the projection $\pi : B^2_i \times B^2_2 \to B^2_2$ is a branched covering map of degree $m$, and

(ii) $\partial S = P_m \times \partial B^2_2 \subset B^2_i \times \partial B^2_2$ for a fixed set $P_m$ of $m$ distinct interior points of $B^2_i$.

A surface braid is called simple if the covering $\pi|_S$ is simple (i.e., the preimage of each branch locus consists of $m - 1$ points).

Let $S^2$ be a 2-sphere obtained from $B^2_2$ by attaching a 2-disk $\overline{B^2_2}$ along the boundary of $B^2_2$. A surface braid $S$ of degree $m$ is extended to a closed surface $\widehat{S}$ in $B^2_i \times S^2$ such that

$\widehat{S} \cap (B^2_i \times \overline{B^2_2}) = S$ and $\widehat{S} \cap (B^2_i \times B^2_2) = X_m \times \overline{B^2_2}$.

Identifying $B^2_i \times S^2$ with the tubular neighborhood of a standard 2-sphere in $\mathbb{R}^4$, we assume that $\widehat{S}$ is a closed oriented surface embedded in $\mathbb{R}^4$ and call it the closure of $S$.

Surface braids are closely related to surface-knots; as an analogue to Alexander’s theorem in classical knot theory, Viro [15] and Kamada [7] proved that any surface-knot is ambient isotopic to the closure of a simple surface braid. Refer to [9, 1] for more details.

1.2. Braid index of surface-knots. The braid index of a surface-knot $F$, denoted by $\text{Braid}(F)$, is defined to be the minimal number among the degrees of all simple surface braids whose closures are ambient isotopic to $F$ in $\mathbb{R}^4$.
There are several results on the braid index of a surface-knot; see [6, 8, 10], for example. Surface-knots with braid index less than three are unknotted, and those with braid index three are “ribbon” [6]. The 2-twist spun trefoil, for example, is not ribbon, and hence has the braid index four [6]. However, the braid index, for higher than four, has never been precisely determined for any specific examples of surface-knots. In this talk, we prove:

**Theorem 1.** The connected sum of \( k \) copies of the spun \((2, p)\)-torus knot has the braid index \( k + 2 \), where \( p \) is an odd integer with \( p \geq 3 \).

2. **Quandles, diagrams and colorings**

To prove Theorem 1, we use colorings of surface-knots by quandles.

2.1. **Quandles.** A *quandle* [2, 5, 11] is a non-empty set \( X \) equipped with a binary operation \((a, b) \mapsto a \ast b\) such that (i) \( a \ast a = a \) for any \( a \in X \), (ii) the map \( \ast a : X \to X \) \((x \mapsto x \ast a)\) is bijective for each \( a \in X \), and (iii) \((a \ast b) \ast c = (a \ast c) \ast (b \ast c)\) for any \( a, b, c \in X \).

The *dihedral quandle* of order \( p \), denoted by \( R_p \), is a quandle consisting of the set \( \{0, 1, \ldots, p - 1\} \) with the binary operation defined by \( i \ast j \equiv 2j - i \mod p \).

2.2. **Diagrams of surface-knots.** For a fixed projection \( \pi : \mathbb{R}^4 \to \mathbb{R}^3 \), by perturbing a surface-knot \( F \) if necessary, we may assume that the projection \( \pi|_F \) is generic, that is, \( \pi|_F \) has double points, isolated triple points and isolated branch points in the image as its singularities. A *diagram* of a surface-knot is a generic projection image in \( \mathbb{R}^3 \) where one of the two sheets near the double point curve is broken depending on the relative height (See Figure 1). This convention is similar to classical knot diagrams. A diagram consists of *broken sheets*, that are mutually disjoint compact oriented surfaces in \( \mathbb{R}^3 \), and the orientations are specified by normal vectors. Refer to [1] for more details.

![Figure 1](image)

2.3. **Quandle colorings.** A *coloring* of a surface-knot diagram by a quandle \( X \) is an assignment of an element of \( X \) to each broken sheet such that \( a \ast b = c \) holds along each double point curve, where \( a \) (resp. \( c \)) is the color of under-sheet that is behind (resp. in front of) the over-sheet colored \( b \) with respect to the normal vector of the over-sheet (See Figure 2). We remark that the number of the colorings is an invariant of a surface-knot and that the coloring by \( R_p \) is coincident with the Fox \( p \)-coloring [3, 4].
3. KEY PROPOSITION AND PROOF OF THEOREM 1

The following is the key proposition to prove Theorem 1.

**Proposition 2.** Let $F$ be a surface-knot which is not a trivial $S^2$-knot. If there is a finite quandle $X$ with $n$ elements such that $F$ admits at least $n^s$ colorings by $X$ for some integer $s > 0$, then the braid index of $F$ is at least $s + 1$.

**Outline of the proof.** Let $m$ be the braid index of $F$. Consider a simple surface braid $S$ of degree $m$ whose closure $\hat{S}$ presents $F$. Since $F$ is not a trivial $S^2$-knot, the diagram of $S$ has branch points. Further, we may assume that the first and second boundary circles belong to the same broken sheet (cf. [10, Lemma 12]); Figure 3 shows this situation where a branch point connects the first and second broken sheets near the boundary circles. It follows that the surface-knot $F$ admits at most $n^{m-1}$ colorings by $X$. Thus we obtain $n^s \leq n^{m-1}$, that is, $m \geq s + 1$. □

**Proof of Theorem 1.** Let $F_p(k)$ be the connected sum of $k$ copies of the spun $(2, p)$-torus knot. Since the number of the colorings of $F_p(1)$ by the dihedral quandle $R_p$ of order $p$ is equal to $p^2$, that of $F_p(k)$ is equal to $p^{k+1}$ (cf. [12]). Hence the braid index of $F_p(k)$ is at least $k + 2$ by Proposition 2.

On the other hand, the following was proved by Kamada, Satoh and Takabayashi [10, Theorem 3]: if neither $F_1$ nor $F_2$ is a trivial $S^2$-knot, then the inequality

(*) \[ \text{Braid}(F_1 \# F_2) \leq \text{Braid}(F_1) + \text{Braid}(F_2) - 2 \]

holds for the connected sum $F_1 \# F_2$ of two surface-knots $F_1$ and $F_2$. Thus the braid index of $F_p(k)$ is at most $k + 2$, since that of $F_p(1)$ is three. □
We obtain the following by an argument similar to that in the proof of Proposition 1.

**Corollary 3.** The connected sum of $k$ copies of the spun $(2,p)$-torus knot and $g$ copies of the trivial $T^2$-knot has the braid index $k + 2$, where $p$ is an odd integer with $p \geq 3$.

**References**


Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba Meguro, Tokyo 153-8914, Japan

E-mail address: k-tanaka@ms.u-tokyo.ac.jp