FUKUMOTO-FURUTA AND OTHER INVARIANTS AND THEIR APPLICATIONS TO 4-MANIFOLDS WITH BOUNDARY

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On the occation of Professor Yukio Matsumoto's sixtieth birthday

Abstract. We give a survey for the Fukumoto-Furuta invariant, which is defined for a pair of a closed 3-manifold and a spin 4-orbifold bounded by it. In certain cases, such as Seifert rational homology 3-spheres, we can consider it as a spin homology cobordism invariant for a 3-manifold by choosing a "canonical" spin 4-orbifold. The proof is based on the orbifold 10/8 theorem.

1. The Fukumoto-Furuta invariant for a $\mathbf{Z}$-homology 3-sphere

Let us first recall the original definition of the Fukumoto-Furuta invariant $w(\Sigma, X, c)$ for a $\mathbf{Z}$ homology 3-sphere Σ and a spin 4-orbifold X bounded by Σ with spin structure c. We choose a spin 4-manifold Y with $\partial Y = -\Sigma$ to obtain a closed spin 4-orbifold $X \cup_\Sigma Y$.

Definition 1. [3] $w(\Sigma, X, c)$ is defined by

$$w(\Sigma, X, c) = \operatorname{ind} \mathcal{D}(X \cup_\Sigma Y) + \sigma(Y)/8 \in \mathbf{Z},$$

where $\operatorname{ind} \mathcal{D}(X \cup_\Sigma Y)$ denotes the index (over $\mathbf{C}$) of the spin Dirac operator on $X \cup_\Sigma Y$ and $\sigma(Y)$ is the signature of Y.

This invariant does not depend on Y due to the excision property of the index (or more explicitly by the V-index theorem [10]). Moreover $w(\Sigma, X, c) \mod 2$ is the Rochlin invariant $\mu(\Sigma)$ of Σ. In general $w(\Sigma, X, c)$ depends on (X, c), but in some particular cases (including Seifert homology 3-spheres), we can derive a homology cobordism invariant for a 3-manifold from $w(\Sigma, X, c)$.

Theorem 1. *Let Σ be a Seifert* $\mathbf{Z}$ *homology 3-sphere. Then*

(1) *we have a canonical choice of (X, c) such that $w(\Sigma, X, c)$ is equal to the Neumann-Siebenmann invariant $\overline{\mu}(\Sigma)$ (see §2 for the definition) up to sign* [4].

(2) *$\overline{\mu}(\Sigma)$ is a homology cobordism invariant for any Seifert* $\mathbf{Z}$ *homology 3-sphere Σ* ([4] *under some restrictions and* [17] *for general cases).*

The $\overline{\mu}$ invariant is defined also for plumbed homology 3-spheres, and for the plumbed cases the first claim is true [17], but the second claim is still open. The above claim for Seifert cases (and its extention to Seifert rational homology 3-spheres discussed below) is deduced from the following "orbifold" 10/8-theorem, which is based on Seiberg-Witten theory.

Theorem 2. [3] *Let Z be a closed spin 4-orbifold with $b_1(Z) = 0$. Then either* $\operatorname{ind}\mathcal{D}(Z) = 0$ *or*

$$1 - b_2^-(Z) \leq \operatorname{ind}\mathcal{D}(Z) \leq b_2^+(Z) - 1.$$

In particular if $b_2^{\pm}(Z) \leq 2$, then $\operatorname{ind}\mathcal{D}(Z) = 0$ *since* $\operatorname{ind}\mathcal{D}(Z)$ *is even.*

2. Fukumoto-Furuta invariants for rational homology 3-spheres

We extend the above results to those for a certain rational homology 3-sphere S. (The results for the cases with $b_1 > 0$ are discussed in [2], based on the stable homotopy Seiberg-Witten theory [5].) We have to consider a pair (S, c) of S and its spin structure c. Although the set of spin structures on S is identified with $H^1(S, \mathbf{Z}_2)$, there is no canonical way to determine the zero element in general. So we fix a framed link description $\mathcal{L} = \coprod(L_i)$ of S. Let (μ_i, λ_i) be the the meridian-longitude pair of L_i and α_i/β_i be the framing of L_i. Then the spin structure c on S is described by a homomorphism $H_1(S^3 \setminus \mathcal{L}, \mathbf{Z}) \to \mathbf{Z}_2$ (which is determined by assigning $c(\mu_i) \in \mathbf{Z}_2$ to each μ_i) such that $\alpha_i c(\mu_i) + \beta_i c(\lambda_i) + \alpha_i\beta_i \equiv 0 \mod 2$ (where $c(\lambda_i)$ is a linear combination of $c(\mu_j)$). Here $c(\mu_i) \equiv 0$ if and only if c extends to the spin structure over the meridian disk bounded by μ_i in S^3.

First we discuss the case where S is spherical. Note that the following relations among the classes of 3-manifolds.

$$\{\text{spherical}\} \subset \{\text{Seifert}\} \subset \{\text{plumbed}\}.$$

2.1. **The case where S is spherical.** It is not difficult to see that the spin structure c on S can be uniquely extended to that on the cone cS over S. Then if we take a spin 4-manifold (Y, c) with $\partial(Y, c) = (S, c)$, $b_1(Y) = 0$ (which always exists), a closed 4-orbifold $Z = cS \cup_S (-Y)$ has a spin structure with $b_1(Z) = 0$. Then

$$\operatorname{ind} \mathcal{D}(Z) = -(\sigma(Z) + \delta(S, c))/8 = (\sigma(Y) - \delta(S, c))/8$$

where $\delta(S, c) \in \mathbf{Z}$ is determined as follows. Note that we have

$$\operatorname{ind} \mathcal{D}(Z) = -\frac{1}{24} \int p_1 + \delta_{\text{Dirac}}, \quad \sigma(Z) = \frac{1}{3} \int p_1 + \delta_{\text{sign}},$$

where δ_{Dirac} and δ_{sign} are the contributions from the isolated singularities of Z determined by the V-index theorem. Then

$$\delta(S, c) = -8\delta_{\text{Dirac}} - \delta_{\text{sign}}.$$

- $\delta(S, c)$ depends only on (S, c). In fact, this is the Fukumoto-Furuta invariant for (S, c) (although the sign convention is opposite to that for $w(\Sigma, X, c)$).
- $\delta(S, c) \mod 16 = \mu(S, c)$ (The Rochlin invariant of (S, c)).

We have a complete list of the value of $\delta(S, c)$ [18]. In particular when S is a lens space $L(p, q)$, which is represented by $-p/q$ surgery on the unknot, the spin structure c is determined by a value $c(\mu) \in \mathbf{Z}_2$ for a meridian μ of the unknot satisfying $pc(\mu) + pq \equiv 0 \mod 2$. Then $\delta(L(p, q), c)$ is described by a "σ-function" $\sigma(q, p, \pm 1)$ as follows.

Proposition 1. [4]

$$\delta(L(p, q), c) = \sigma(q, p, (-1)^{c(\mu)-1}),$$

where $\sigma(q,p,\epsilon)$ $(\gcd(p,q)=1, \epsilon=\pm 1)$ *is uniquely demterined by the following recursive formula.*

(1) $\sigma(q+cp,p,\epsilon)=\sigma(q,p,(-1)^c\epsilon)$

(2) $\sigma(-q,p,\epsilon)=\sigma(q,-p,\epsilon)=-\sigma(q,p,\epsilon)$

(3) $\sigma(q,1,\epsilon)=0$

(4) $\sigma(p,q,-1)+\sigma(q,p,-1)=-\operatorname{sgn} pq \quad \textit{if } p+q\equiv 1 \pmod 2$

The σ function is easily computed by the above formula recursively. If S is spherical other than a lens space, then $\delta(S,c)$ is either described by a σ function or is determined more explicitly [18].

2.2. **The case where S is a general Seifert rational homology 3-sphere.** Let (S,c) be a rational homology 3-sphere with spin structure that bounds a spin 4-orbifold (X,c) with only isolated singularities and with $b_1(X)=0$. Define $\delta(X,c)$ to be the sum of $\delta(S.c|_S)$, where S runs over all the links of the isolated singularities of X. Then $\sigma(X)+\delta(X,c)$ can be considered as a Fukumoto-Furuta invariant for (S,X,c). For, if we take a spin 4-manifold (Y,c) with $\partial(Y,c)=(S,c)$, we have

$$\sigma(X)+\delta(X,c)=-8\operatorname{ind}\mathcal{D}(X\cup(-Y))+\sigma(Y).$$

In general its value depends on the choice of (X,c), but if S is a Seifert rational homology 3-sphere, this invariant is related to the Neumann-Siebenmann invariant for some "canonical" choice of (X,c).

Definition 2. Let Γ be an integrally weighted tree and $P(\Gamma)$ be an associated plumbed 4-manifold. We assume that each vertex v_i of Γ (with weight n_i) corresponds to a D^2-bundle over S^2 with euler class n_i, whose zero-section is also denoted by v_i. The plumbed 3-manifold $\partial P(\Gamma)$ has a canonical framed link representative so that each v_i corresponds to an unknot, whose meridian-longitude pair is denoted by (μ_i,λ_i). Then a spin structure c on $\partial P(\Gamma)$ is determined by assigning a $\mathbf{Z}_2$ value $c(\mu_i)$ to each μ_i satisfying $n_ic(\mu_i)+c(\lambda_i)+n_i\equiv 0 \mod 2$. For a pair $(P(\Gamma),c)$, a spherical Wu class $w(P(\Gamma),c)$ is defined by

$$w(P(\Gamma),c)=\sum \epsilon(\mu_i)[v_i]\in H_2(P(\Gamma),\mathbf{Z})$$

where $\epsilon(\mu_i)$ is either 0 or 1 and satisfies $\epsilon(\mu_i) \mod 2 \equiv c(\mu_i)$. Then the Neumann-Siebenmann invariant of $(\partial P(\Gamma), c)$ is defined by

$$\overline{\mu}(\partial P(\Gamma), c) = \sigma(P(\Gamma)) - w(P(\Gamma), c) \cdot w(P(\Gamma), c).$$

For a plumbed 3-manifold $S = \partial P(\Gamma)$, the value of $\overline{\mu}$ depends only on (S, c), although the choice of Γ satisfying $S = \partial P(\Gamma)$ is not unique. Moreover $\overline{\mu}(S, c) \mod 16$ is equal to the Rochlin invariant of (S, c) by the Rochlin formula and due to the fact that any spherical Wu class is represented by an embedded 2-sphere. We note that if S is a $\mathbf{Z}$ homology 3-sphere, the choice of c is unique, and the above $\overline{\mu}$ is divisible by 8. So usually $\overline{\mu}(S, c)/8$ is called the Neumann-Siebennmann invariant (just as in the statement in §1).

In particular if S is a Seifert rational homology 3-sphere, $\overline{\mu}(S, c)$ is related to the Fukumoto-Furuta invariant as follows.

Lemma 1. [20] *Let (S, c) be a Seifert rational homology 3-sphere with spin structure c. Then there exist two compact spin 4-orbifold $(X_\pm, c_\pm)$ with only isolated singularities such that*

(1) $\partial(X_\pm, c_\pm) = (S, c)$.
(2) $\sigma(X_+) + \delta(X_+, c_+) = \sigma(X_-) + \delta(X_-, c_-) = \overline{\mu}(S, c)$.
(3) $b_1 X_+ = b_1 X_- = 0$, $b_2^+ X_+ \leq 1$ *and* $b_2^- X_- \leq 1$.

Remark 1. If S is a spherical 3-manifold, then we can take $X_\pm$ to be the cone over S and we have $\delta(S, c) = \overline{\mu}(S, c)$ ([17] for lens spaces and [18] for the other cases).

We need the property (3) in the above lemma to prove the following result by the orbifold 10/8 theorem.

Theorem 3. *If two Seifert rational homology 3-spheres with spin structure (S_1, c_1) and (S_2, c_2) are spin cobordant via a 4-manifold with spin structure (W, c) with $b_2(W) = 0$, then $\overline{\mu}(S_1, c_1) = \overline{\mu}(S_2, c_2)$.*

Corollary 1. *For a Seifert $\mathbf{Z}_2$ homology 3-sphere (in which case the spin structure is unique), the $\overline{\mu}$ invariant is a $\mathbf{Z}_2$ homology cobordism invariant.*

Corollary 2. *Suppose that k copies of a Seifert rational homology 3-sphere with spin structure (S,c) bounds a spin rational acyclic 4-manifold for some k. Then $|\overline{\mu}(S,c)| < 8$. (If S is spherical or a Seifert $\mathbf{Z}$ homology 3-sphere, then $\overline{\mu}(S,c) = 0$.)*

3. Some remarks on the intersection forms of 4-manifolds with boundary

We give some remarks and problems concerning the constraints on the intersection forms of 4-manifolds with given boundary and relations between Fukumoto-Furuta invariants and other invariants.

3.1. If(S,c) is a spherical 3-manifold with $|\overline{\mu}(S,c)| \leq 18$ (this is true for most spherical manifolds), and if $(S.c)$ bounds a definite spin 4-manifold Y, then $\overline{\mu}(S,c) = \sigma(Y)$ [18]. For example, it follows that if the Poincare homology 3-sphere bounds a definite spin 4-manifold, then its intersection form must be E_8. This result was first observed by Furuta and also proved by Frøyshov [7] by using another invariant coming from the Seifert-Witten theory. But if S is not a $\mathbf{Z}$ homology 3-sphere, the uniqueness of the intersection form of the definite 4-manifolds bounded by S does not hold (see [18] for such examples).

3.2. The relation between the arguments on the homology cobordism classes of homology 3-spheres based on the instanton gauge theory and those on Fukumoto-Furuta theory is not clear. For example, the existence of infinitely many Brieskorn homology 3-spheres that are independent in the homology cobordism group [9] cannot be detected by Fukumoto-Furuta method. On the other hand T.Lawson [11] gave some examples of Seifert homology 3-spheres with 4 singular fibers whose homology cobordism classes cannot be determined by Fintushel-Stern invariant [6]. But some of them have nonzero $\overline{\mu}$ and so have infinite order in the homology cobordism group.

3.3. For a $\mathbf{Z}_2$ homology 3-sphere Σ, we can extend the invariants defined by Bohr and Lee [1] as follows [18]. The extended definition of the invariants are given by

$$\underline{m}(\Sigma) = \max\{(5\sigma(X) + \delta(X))/4 - b_2(X)\},$$
$$\overline{m}(\Sigma) = \min\{(5\sigma(X) + \delta(X))/4 + b_2(X)\},$$

where X runs over all spin 4-orbifolds with isolated singularities bounded by Σ and $\delta(X)$ is the contribution from the singularities of X as in the previous section. These invariants enable us to prove several results in [1] more straightforwardly.

3.4. In [16] Saveliev defined a ν invariant for a **Z** homology 3-sphere Σ by

$$\nu(\Sigma) = \frac{1}{2}\sum(-1)^{(n+1)(n+2)/2}\operatorname{rank} I_n(\Sigma),$$

where I_n denotes the instanton Floer homology and the sum is taken over its degree n defined modulo 8. He proved that in case of a Seifert homology 3-sphere, $\nu(\Sigma) = \overline{\mu}(\Sigma)$, which is the Fukumoto-Furuta invariant up to sign by our results. The relation between these invariants for more general cases is still unknown.

3.5. Frøyshov defined another integer valued homology cobordism invariant $h(\Sigma)$ for a **Z** homology 3-sphere Σ [8] based on the instanton Floer homology. But this is not an integral lift of the Rochlin invariant (in fact, $h(\Sigma(2,3,7)) = 0$, while $\mu(\Sigma(2,3,7)) \neq 0$).

3.6. Ozsváth-Szabó 's Floer homology theory has derived several results concerning the intersection forms of 4-manifolds (in particular definite 4-manifolds) [12]-[15]. The relation between their theory and Fukumoto-Furuta theory is also still unknown.

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