REPRESENTING HOMOLOGY CLASSES OF 4-MANIFOLDS WITH TOPOLOGICALLY EMBEDDED 2-SPHERES

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Tokyo, 9 November 2004, the day after Professor Yukio Matsumoto's 60th birthday.

Matsumoto (1976): There exist topological embeddings of surfaces in 4-manifolds that cannot be approximated by piecewise linear or smooth embeddings.

This result showed that the Bing program for understanding the topology of 4-manifolds would not work.

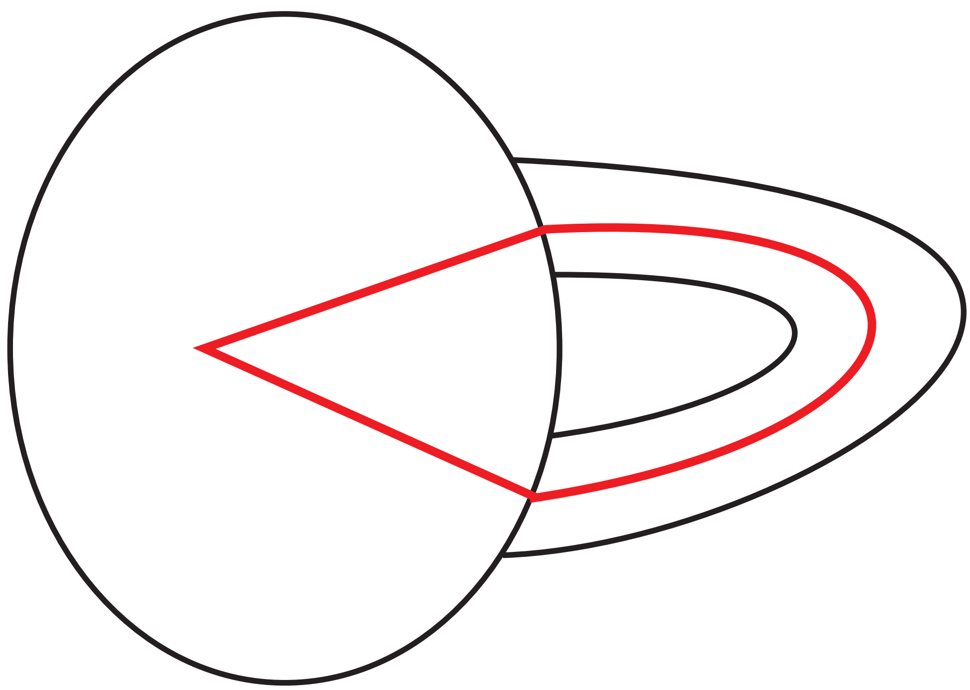
Let W be a simply connected, compact, piecewise linear (PL) 4-manifold (with boundary).

Each element of $H_2(W)$ can be represented by an immersed PL S^2. Many can be represented with PL embeddings and/or locally flat topological embeddings.

Question. *Can a given element of $H_2(W)$ be represented by a topological embedding $h : S^2 \to M$?*

The topological embedding may be a wild embedding.

If the homology class can be represented by a PL embedded 2-sphere, then it is represented by a 4-manifold consisting of a 4-ball with a single 2-handle attached.



The topological analogue of this should be a contractible 4-manifold with a single 2-handle attached.

The two theorems below are joint work with Vo Thanh Liem.

Theorem 1. *If W is a compact, simply connected, PL 4-manifold, then each element of $H_2(W)$ can be represented by a compact PL submanifold $M \subset W$ such that M consists of a Mazur-like contractible 4-manifold with a single 2-handle attached.*

Theorem 2. *If W is a compact, simply connected, PL submanifold of S^4, then each element of $H_2(W)$ can be represented by a locally flat topological embedding of S^2.*

Definitions. A compact contractible PL 4-manifold is *Mazur-like* if it has a handle decomposition in which there is one 0-handle, no handles of index greater than 2, and the attaching map for the ith 2-handle is homotopic to the loop represented by the ith 1-handle.

The manifold M represents a specified element of $H_2(W)$ in the sense that a generator of $H_2(M) \cong \mathbb{Z}$ is homologous in W to the given element of $H_2(W)$.

A diagram of the Mazur manifold

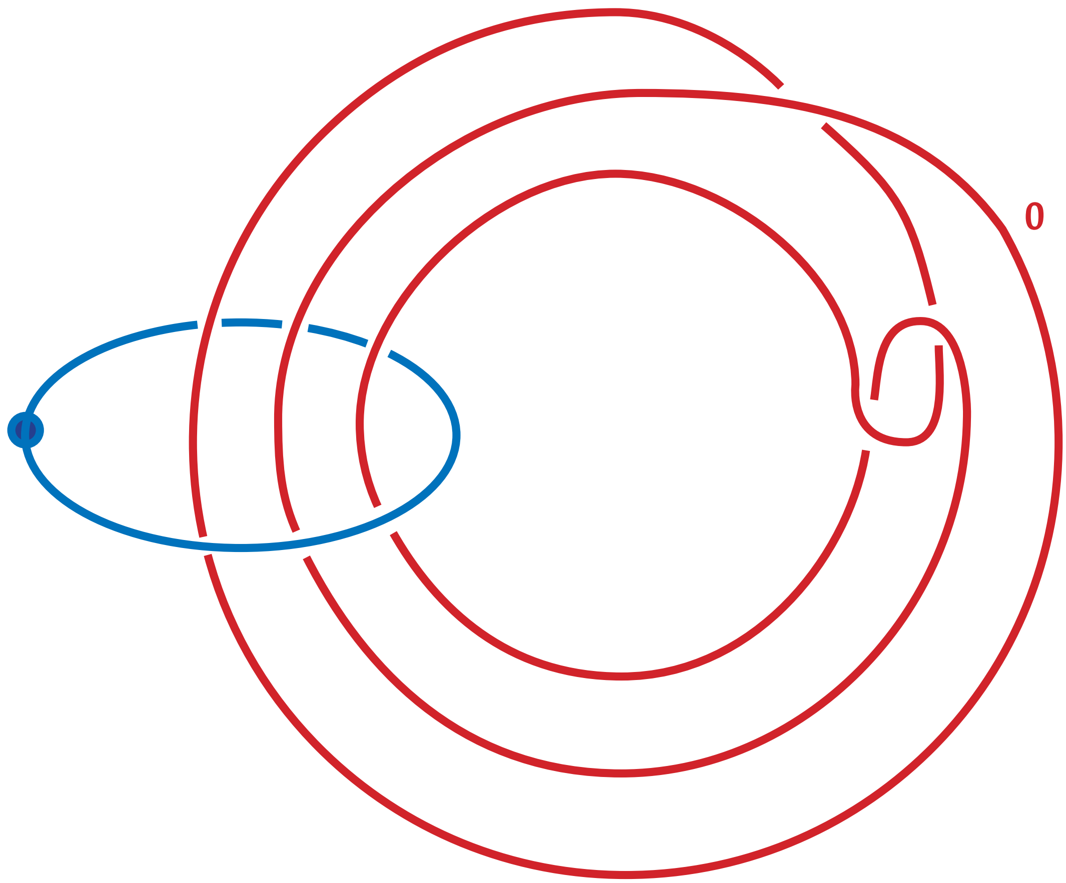
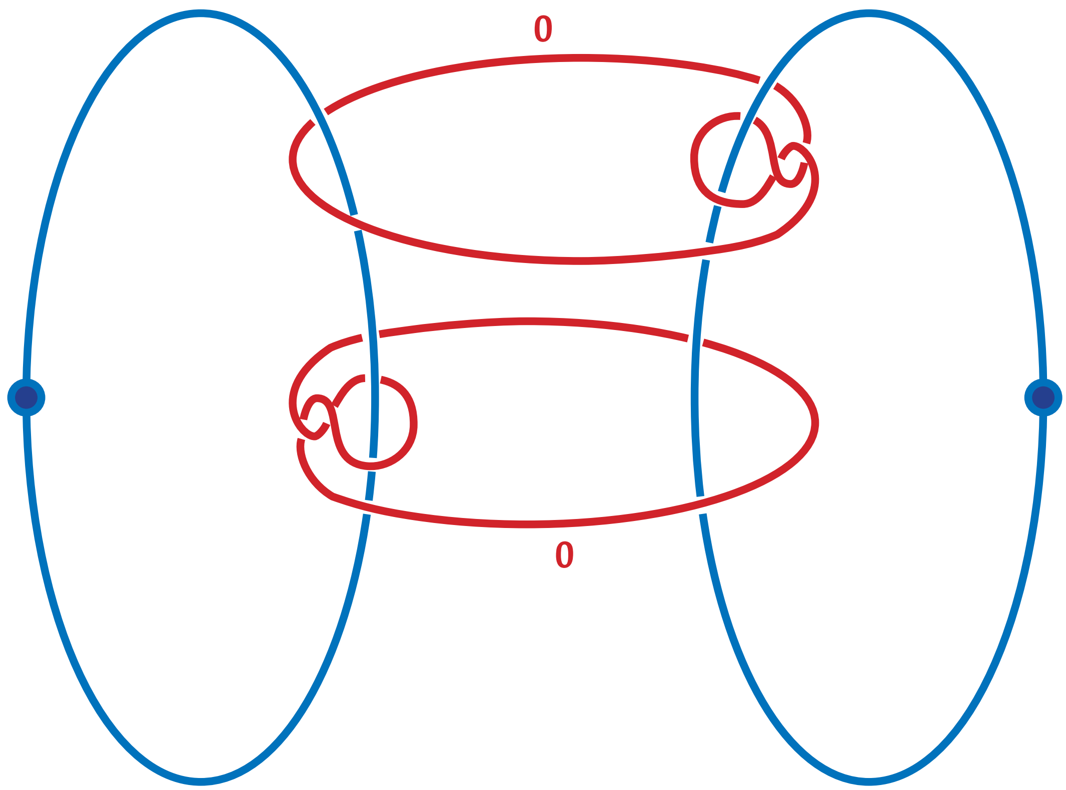
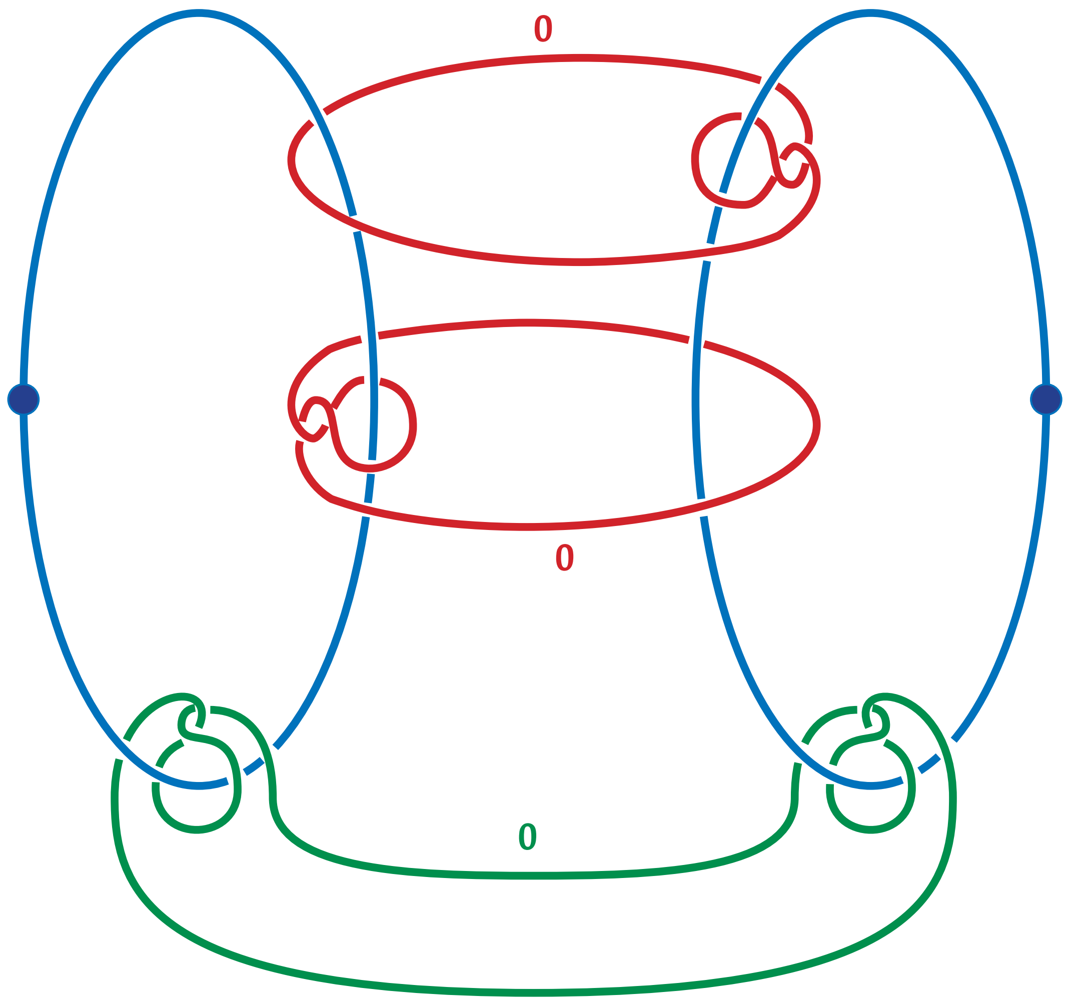


Diagram of a Mazur-like manifold



A diagram of M

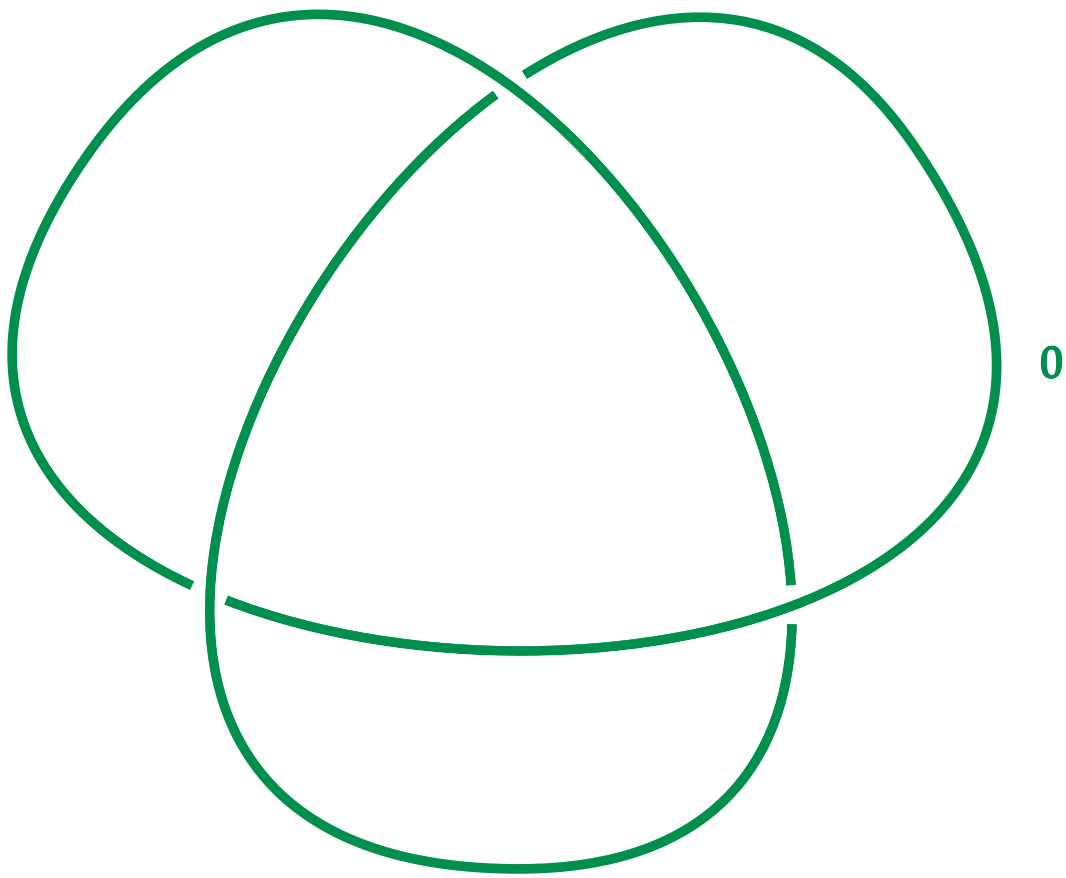
M is a Mazur-like manifold with a 2-handle attached



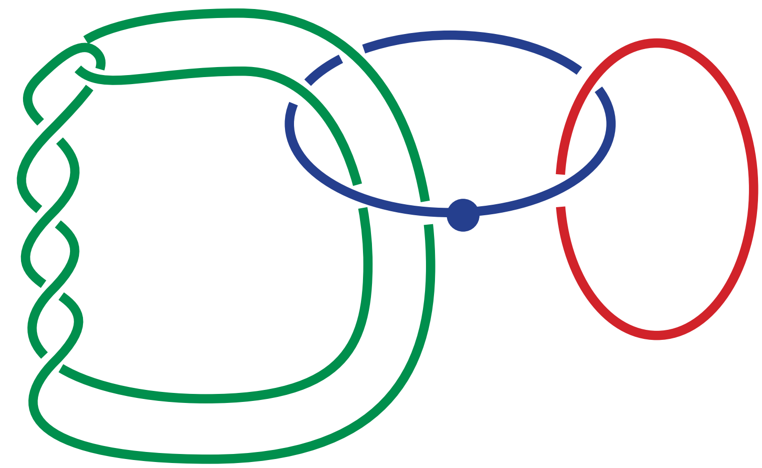
Both Theorems 1 and 2 are false without the hypothesis that W is compact.

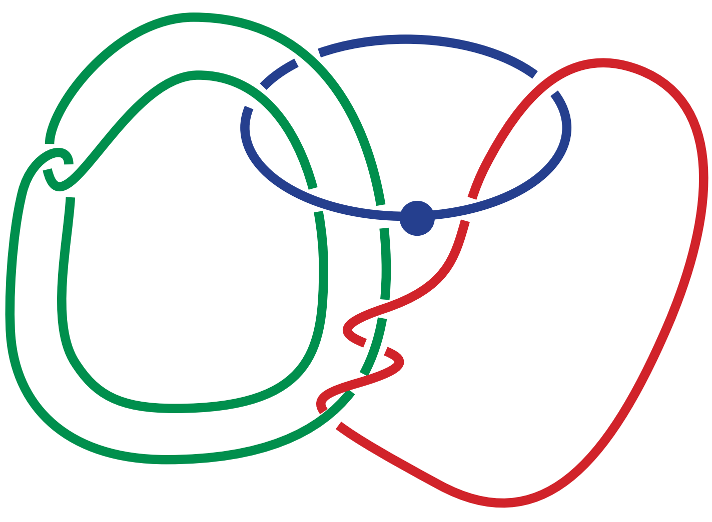
Theorem (Matsumoto-V)**.** *There exists an open subset W of S^4 such that W has the homotopy type of S^2, but there is no compact subset $X \subset W$ such that $X \hookrightarrow W$ is a homotopy equivalence.*

The conclusion of Theorem 2 is also false if W does not embed in S^4 (Kawauchi).



Another example.





Theorems 1 and 2 leave the following two problems unresolved.

Question. *If V is a compact PL (Mazur-like) contractible 4-manifold, then does every loop on the boundary of V bound a topologically embedded disk in V?*

Question. *If W is a compact PL simply connected 4-manifold, then can every element of $H_2(W)$ be represented by a topologically embedded 2-sphere in W?*

Two relevant facts:

Fact 1. *There exist loops on the boundary of the Mazur manifold that bound locally flat topological disks but do not bound PL disks (not even PL disks with nonlocally flat points). (Akbulut)*

Fact 2. *There exist classes in*

$$H_2(S^2 \times S^2)$$

that can be represented by locally flat topological 2-spheres but not by smooth (or locally flat PL) 2-spheres. (Freedman-Kuga)

Proof of Theorem 1.

Step 1. Choose an immersed PL 2-sphere representing the given homology class. Let N_0 denote a regular neighborhood of the immersed 2-sphere.

Step 2. Take a handle decomposition $\mathcal{H}$ of N_0 such that

- $\mathcal{H}$ has one 0-handle,
- $\mathcal{H}$ has n 1-handles,
- $\mathcal{H}$ has one 2-handle, and
- $\mathcal{H}$ has no handles of index ≥ 3.

Note that

$$N_0 \cong S^2 \vee (S^1 \vee \cdots \vee S^1)$$

Step 3. Extend $\mathcal{H}$ to a handle decomposition of W.

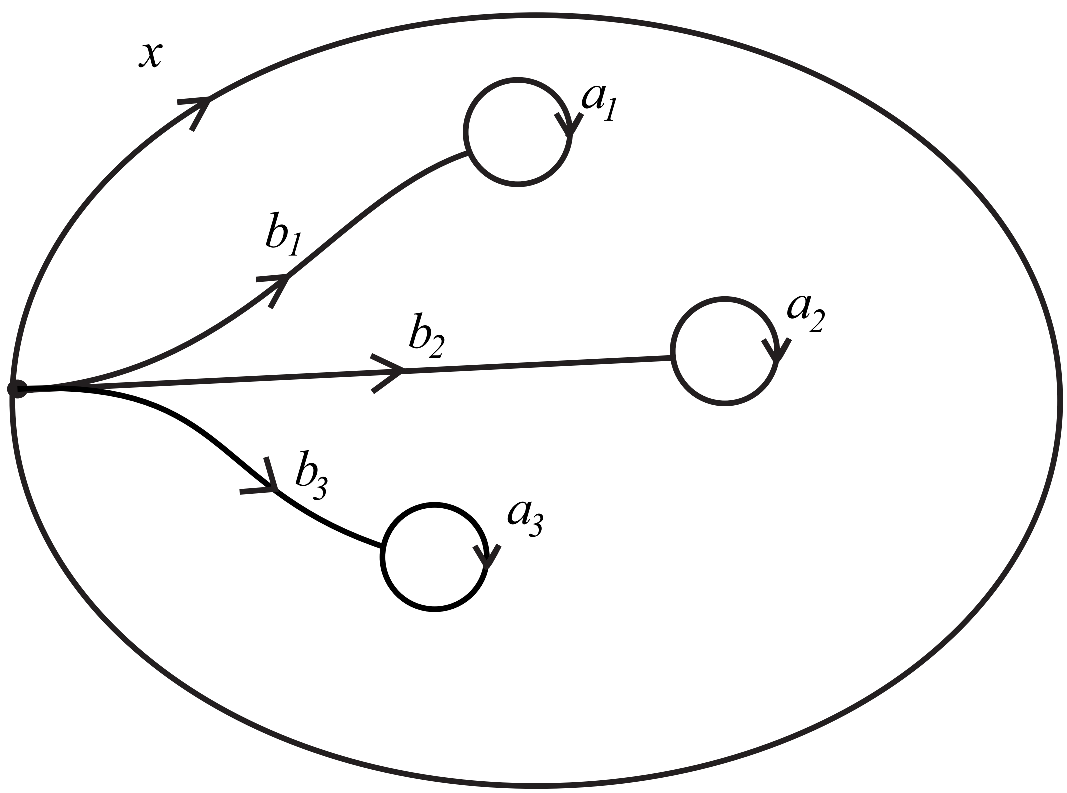
Cancel all the new 0-handles.

Let $N_1 = N_0 \cup$ (all 1-handles).

Then $\pi_1(N_1)$ is free and

$$\pi_1(N_1) \xrightarrow{0} \pi_1(W).$$

Hence each generator of $\pi_1(N_1)$ dies in some combination of the 2-handles of W.



Step 4. Introduce k canceling (2,3)-handle pairs. Slide the new 2-handles into position so that they homotopically cancel the 1-handles.

Define

$$V = N_1 \cup (\text{ new 2-handles }).$$

This completes the proof of Theorem 1.

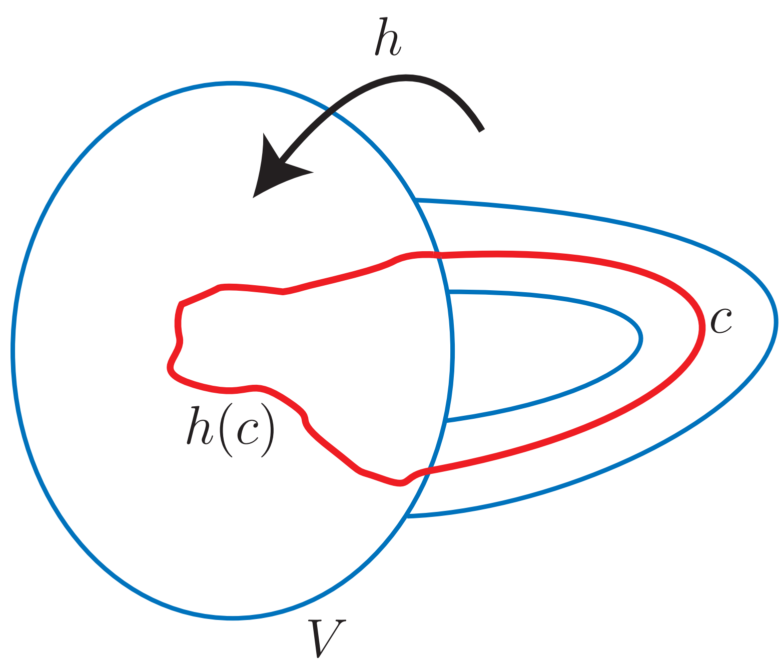
Proof of Theorem 2.

It suffices to show that the construction can be done in such a way that V is π_1-negligible in S^4.

In that case, there is a topological homeomorphism

$$h : \overline{S^4 - V} \to V$$

that is the identity on ∂V. Hence there is an interior 2-handle with the same boundary as the exterior 2-handle. The union of the cores of the two 2-handles is the locally flat 2-sphere.



Observation.
Let V and V' be two PL codimension-0 submanifolds of S^4 such that V has a handle decomposition with only handles of index ≤ 2 and V' is built from V by attaching 1- and 2-handles. If V' is π_1-negligible, then V is π_1-negligible.