ON THE SPACE OF FOLD MAPS OF S^2 TO $\mathbf{R}^2$ WITH A CONNECTED SINGULAR SET

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Dedicated to Professor Yukio Matsumoto in celebration of his 60th birthday

ABSTRACT. Let S^2 be the oriented 2-sphere, $S^1_0 \subset S^2$ its equator and $\mathbf{R}^2$ the oriented plane. We denote by $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$ the space of all fold maps of S^2 to $\mathbf{R}^2$ such that the singular set coincides with S^1_0. Here, a fold map is a smooth map with only fold singularities. In 1970's, Eliashberg proved that $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$ is homotopic to disjoint union of four circles. In his paper, he only mentioned that this is an application of his elegant theory, the homotopy principle for fold maps. In this talk, we introduce another method to determine the number of connected components of $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$. Comparing to Eliashberg's method, our method is concrete, combinatorial.

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1. INTRODUCTION

In this talk, all manifolds and maps are differentiable of class C^∞. Let M and N be connected surfaces without boundary. For a map $f : M \to N$, we define $S(f) = \{q \in M | \text{rank } df_q < 2\}$. We call $S(f)$ the singular set of f. We say that a point $q \in S(f)$ is a *fold singularity* if there exist local coordinate systems (x_1, x_2) around q and (y_1, y_2) around $f(q)$ such that $(y_1 \circ f, y_2 \circ f) = (x_1, x_2^2)$. If $S(f)$ only consists of fold singularities, we call f a *fold map*. Note that for a fold map $f : M \to N$, $S(f)$ is a 1-dimensional submanifold (not necessary connected) of M and $f|S(f)$ is an immersion. If $f|S(f) : S(f) \looparrowright N$ is an immersion with normal crossings, then we call f a *stable fold map*.

Let V be a closed 1-dimensional submanifold of M and $\mathcal{F}(M, N; V)$ is the space of all fold maps $f : M \to N$ with $S(f) = V$. Note that $\mathcal{F}(M, N; V)$ is endowed with the Whitney C^∞-topology. Eliashberg [1] studied the existence problem of fold maps in $\mathcal{F}(M, N; V)$ and obtained the homotopy principle for fold maps. (His theorem works for any dimension $\dim M = \dim N \geq 1$.) In the supplements of [2], Eliashberg mentioned that we can determine the homotopy type of $\mathcal{F}(M, N; V)$ by applying his homotopy principle for fold maps. As an example, he stated the following. See Theorem 2.4, too.

Theorem 1.1 (Eliashberg [2]). *Let $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$ be the space of fold maps of the oriented 2-sphere S^2 to the oriented plane $\mathbf{R}^2$ such that the singular set equals to the equator S^1_0 of S^2. Then $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$ is homotopic to disjoint union of four circles.*

Eliashberg's homotopy principle for fold maps is very elegant. But the supplements of [2] are very short and difficult. So in this talk, we introduce another method to determine the number of connected components of $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$. Our method comes from the extension theory for generic immersions established by Blank [5] and Francis [3], etc. Comparing to Eliashberg's method, our method is concrete, combinatorial.

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2. Eliashberg's homotopy principle for fold maps

Let M and N be connected surfaces without boundary. Let V be a closed 1-dimensional submanifold of M. Suppose that $f : M \to N$ is a fold map with $S(f) = V$. We denote by $\mathcal{F}(M, N; V)$ the space of all such fold maps. Let $\Phi : TM \to TN$ be a homomorphism of vector bundles, where TM and TN are the tangent bundles of M and N respectively. We say that $q \in M$ is a singular point of Φ if the rank of $\Phi_q : T_qM \to T_{\bar{\Phi}(q)}N$ is strictly less than 2, where $\bar{\Phi} : M \to N$ is the continuous map induced by Φ. By $\mathrm{Mon}_V(M, N)$ we denote the space of all homomorphisms $\Phi : TM \to TN$ with $S(\Phi) = V$ such that for each singular point $q \in V$, there exists a neighborhood $U \subset M$ of q and a fold map $f_U : U \to N$ with $df_U = \Phi|TU$. Note that $\mathrm{Mon}_V(M, N)$ is endowed with the compact-open topology. We have the natural map $d : \mathcal{F}(M, N; V) \to \mathrm{Mon}_V(M, N)$ defined by $d(f) = df : TM \to TN$. Then, Eliashberg [1] proved the following theorem which is the homotopy principle for fold maps.

Theorem 2.1 (Eliashberg [1]). *Let M and N be connected surfaces. Then for any non-empty closed 1-dimensional submanifold V of M, the map $d_* : \pi_0(\mathcal{F}(M, N; V)) \to \pi_0(\mathrm{Mon}_V(M, N))$ is surjective.*

Remark 2.2. Eliashberg [2] proved the similar theorem when $\dim M \geq \dim N \geq 2$.

In general, the map d_* is not injective. As an example, Eliashberg [1] showed that the map $d_* : \pi_0(\mathcal{F}(S^2, \mathbf{R}^2; S^1_0)) \to \pi_0(\mathrm{Mon}_{S^1_0}(S^2, \mathbf{R}^2))$ is not injective. Here, both of S^2 and $\mathbf{R}^2$ are oriented. In addition, Eliashberg [2, Section 6] determined the homotopy type of $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$. To characterize this space, we need the following definition. We set $S^2 \setminus S^1_0 = D_N \cup D_S$. We denote by $\overline{D_N}$ (resp. $\overline{D_S}$) the closure of D_N (resp. D_S) in S^2. Note that $\partial\overline{D_N} = \partial\overline{D_S} = S^1_0$.

Definition 2.3. Let f be an element of $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$. We say that $f|\overline{D_N}$ and $f|\overline{D_S}$ are *equivalent extensions* of $f|S^1_0$ if there exists an orientation reversing diffeomorphism $h : \overline{D_N} \to \overline{D_S}$ such that $h|S^1_0 = \mathrm{id}_{S^1_0}$ and $f|\overline{D_S} \circ h = f|\overline{D_N}$. Otherwise, we say that $f|\overline{D_N}$ and $f|\overline{D_S}$ are *inequivalent extensions* of $f|S^1_0$. Here, $\mathrm{id}_{S^1_0}$ is the identity map.

Using this definition, we have the following four subspaces of $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$:

$$\mathcal{F}_{11} = \left\{ f \in \mathcal{F}(S^2, \mathbf{R}^2; S^1_0) \;\middle|\; \begin{array}{l} f|D_N \text{ is an orientation preserving immersion,} \\ f|D_S \text{ is an orientation reversing immersion,} \\ f|\overline{D_N} \text{ and } f|\overline{D_S} \text{ are equivalent extensions of } f|S^1_0 \end{array} \right\}, \tag{2.1}$$

$$\mathcal{F}_{12} = \left\{ f \in \mathcal{F}(S^2, \mathbf{R}^2; S^1_0) \;\middle|\; \begin{array}{l} f|D_N \text{ is an orientation preserving immersion,} \\ f|D_S \text{ is an orientation reversing immersion,} \\ f|\overline{D_N} \text{ and } f|\overline{D_S} \text{ are inequivalent extensions of } f|S^1_0 \end{array} \right\}, \tag{2.2}$$

$$\mathcal{F}_{21} = \left\{ f \in \mathcal{F}(S^2, \mathbf{R}^2; S^1_0) \;\middle|\; \begin{array}{l} f|D_N \text{ is an orientation reversing immersion,} \\ f|D_S \text{ is an orientation preserving immersion,} \\ f|\overline{D_N} \text{ and } f|\overline{D_S} \text{ are equivalent extensions of } f|S^1_0 \end{array} \right\}, \tag{2.3}$$

$$\mathcal{F}_{22} = \left\{ f \in \mathcal{F}(S^2, \mathbf{R}^2; S^1_0) \;\middle|\; \begin{array}{l} f|D_N \text{ is an orientation reversing immersion,} \\ f|D_S \text{ is an orientation preserving immersion,} \\ f|\overline{D_N} \text{ and } f|\overline{D_S} \text{ are inequivalent extensions of } f|S^1_0 \end{array} \right\}. \tag{2.4}$$

It is easy to see that $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0) = \mathcal{F}_{11} \cup \mathcal{F}_{12} \cup \mathcal{F}_{21} \cup \mathcal{F}_{22}$. Then, Eliashberg's theorem is stated as follows. See Theorem 1.1, too.

Theorem 2.4 (Eliashberg [2]). *Let $\mathcal{F}_{11}, \mathcal{F}_{12}, \mathcal{F}_{21}$ and $\mathcal{F}_{22}$ be the subspaces of $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$ which are defined by* (2.1)– (2.4). *Then*

(1) *$\mathcal{F}_{ij} \cap \mathcal{F}_{kl} = \emptyset$ ($i \neq k$ or $j \neq l$), each $\mathcal{F}_{ij}$ is connected and*
(2) *each $\mathcal{F}_{ij}$ is homotopic to S^1.*

Remark 2.5. Eliashberg [2] mentioned that $\mathrm{Mon}_{S^1_0}(S^2, \mathbf{R}^2)$ is homotopic to disjoint union of two circles and $d(\mathcal{F}_{i1}) = d(\mathcal{F}_{i2})$ holds.

3. SPIRAL FOLD MAPS

3.1. **Spiral fold maps.** Let I_i and be closed interval $[0, 1]$ and $\pi_i : I_1 \times I_2 \to I_i$ standard projections, $i = 1, 2$. Let Q_m be a set of m interior points of I_1 and fix the points, $m \geq 1$. We define the orientations of I_1 and I_2 such that the orientation of I_1 "plus" the orientation of I_2 equal to the orientation of $\mathbf{R}^2$. Fix an order to Q_m. That is, we name the m points with integers $1, 2, \ldots, m$. For $m > 1$, two immersions τ^m_i and σ^m, $1 \leq i \leq m$ are depicted as in FIGURE 1 and FIGURE 2, respectively. If $m = 1$, τ^1 is a trivial arc such that $\pi_2 \circ \tau^1 = \mathrm{id}$ and the end points of τ^1 are in Q_1.

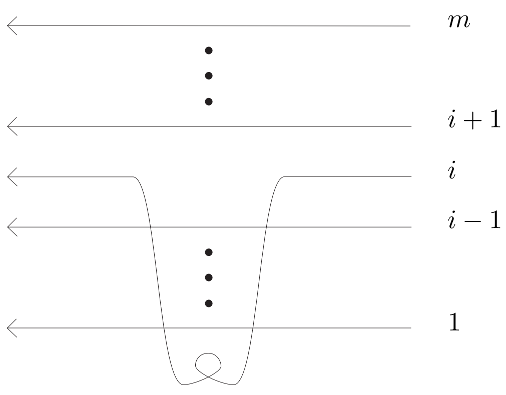

FIGURE 1. Definition of τ^m_i

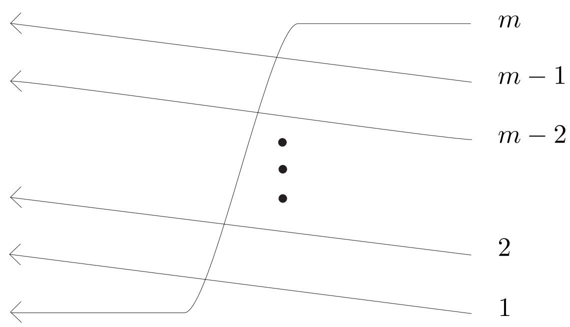

FIGURE 2. Definition of σ^m

For a fixed m, let $a \subset I_1 \times I^1_2$ and $b \subset I_1 \times I^2_2$ be τ^m_i, σ^m or τ^1. Identifying I^1_2 with $[0, 1/2]$ and I^2_2 with $[1/2, 1]$, we have $a \cup b$ in $I_1 \times (I^1_2 \cup I^2_2) = I_1 \times I_2$. We call it a product of a and b and denote it by ab. Let $b_j \in \{\tau^m_1, \ldots, \tau^m_m\}$. Then we have $b^m = b_1 \cdots b_{m-1}\sigma^m$ if $m > 1$ and $b^1 = \tau^1$ if $m = 1$. Put $I_1 \times I_2$ in $\mathbf{R}^2$ and attach m arcs to b^m trivially. We have an oriented immersed closed curve call it an *m-spiral curve*. We abuse an m-spiral curve and b^m for simplicity. Note that the rotation number of b^m equals to 1.

If $f(S_0^1)$ is an m-spiral curve, we call $f \in \mathcal{F}(S^2, \mathbf{R}^2; S_0^1)$ an *m-spiral fold map*. Let f and g be two stable fold maps in $\mathcal{F}(S^2, \mathbf{R}^2; S_0^1)$. If there is a homotopy $f_t \in \mathcal{F}(S^2, \mathbf{R}^2; S_0^1)$ such that $f_{-1} = f$ and $f_1 = g$, we call f_t a fold homotopy between f and g ($t \in [-1, 1]$).

Proposition 3.1. *For any stable fold map $f \in \mathcal{F}(S^2, \mathbf{R}^2; S_0^1)$, there exists a fold homotopy between f and an m-spiral fold map for some $m \in \mathbf{N}$.*

3.2. **Blank and Francis's theorem for spiral fold map.** For $m > 1$, let f^m be an m-spiral fold map such that $f^m(S_0^1)$ is described by $b_1 \cdots b_{m-1}\sigma^m$, where $b_i \in \{\tau_1^m, \ldots, \tau_m^m\}$. Let α be a collection of $m-1$ mutually disjoint arcs $\alpha_i : [0, c] \to \mathbf{R}^2$ defined as follows ($c > 0$). For each i, $\alpha_i(0)$ is in the inside of the negatively oriented loop of b_i and α_i intersects each arc of b_i exactly one time. Suppose that $b_i = \tau_{\delta(i)}^m$ and that $X(f^m, \alpha_i)$ is the set of crossings between $f^m(S_0^1)$ and α_i, $1 \le i \le m-1, 1 \le \delta(i) \le m$. We set $X(f^m, \alpha_i) = \{\delta(i), 1, \ldots, \delta(i-1), \delta(i+1), \ldots, m\}$ such that these numbers are induced from the order of Q_m. See Figure 3 as an example.

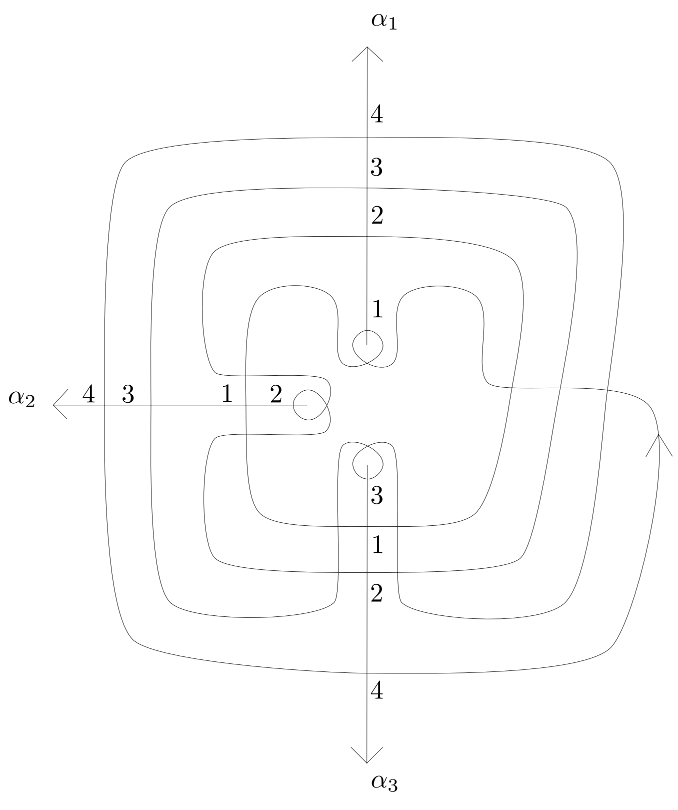

FIGURE 3. The case of $f^4 \in \mathcal{F}(S^2, \mathbf{R}^2; S_0^1)$ with $f^4(S_0^1) = \tau_1^4\tau_2^4\tau_3^4\sigma^4$

For $\delta(i) \in X(f^m, \alpha_i)$, the preimages $(f^m|\overline{D_N})^{-1}([\delta(i), \alpha_i(c)])$ and $(f^m|\overline{D_S})^{-1}([\delta(i), \alpha_i(c)])$ from the unique point $(f^m|S_0^1)^{-1}(\delta(i))$, must terminate at some boundary points $\hat{u}(i) \in S_0^1$ and $\hat{l}(i) \in S_0^1$ respectively. Put $u(i) = f^m(\hat{u}(i))$ and $l(i) = f^m(\hat{l}(i))$. Note that $u(i)$ and $l(i) \in X(f^m, \alpha_i) \setminus \{\delta(i)\}$, $1 \le i \le m-1$. Using these letters, we have two sequences of pairs $u(f^m)$ and $l(f^m)$ induced from $f^m|\overline{D_N}$ and $f^m|\overline{D_S}$. Both of them are described as follows:

$$u(f^m) = (\delta(1), u(1))(\delta(2), u(2)) \cdots (\delta(m-1), u(m-1)), \tag{3.1}$$

$$l(f^m) = (\delta(1), l(1))(\delta(2), l(2)) \cdots (\delta(m-1), l(m-1)). \tag{3.2}$$

By applying the theorem of Blank [5] and Francis [3], we have the following.

Proposition 3.2. *Let $f^m \in \mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$ be an m-spiral fold map. Suppose that $u(f^m)$ and $l(f^m)$ are two sequences written as* (3.1) *and* (3.2)*.*

(1) *If we consider both $u(f^m)$ and $l(f^m)$ as products of transpositions, then the results of these products equal to the cyclic permutation $(m, m-1, \dots, 1)$.*
(2) *If every pairs $(\delta(i), u(i))$ and $(\delta(i), l(i))$ are exactly the same for any $1 \le i \le m-1$, then $f^m \in \mathcal{F}_{11} \cup \mathcal{F}_{21}$. If there exists at least one i such that $(\delta(i), u(i))$ and $(\delta(i), l(i))$ are different, then $f^m \in \mathcal{F}_{12} \cup \mathcal{F}_{22}$.*

Note that any 1- and 2-spiral fold maps are in $\mathcal{F}_{11} \cup \mathcal{F}_{21}$ and there exists a 3-spiral fold map in $\mathcal{F}_{12} \cup \mathcal{F}_{22}$.

4. Operations for spiral fold maps

In this section, we define five types of operations for (3.1) and (3.2). Each operation corresponds to a fold homotopy between two spiral fold maps. Let $f^m \in \mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$ be an m-spiral fold map written as (3.1) and (3.2). We may assume that $\delta(1) = 1$.

4.1. **Operation I.** The operation I changes (3.1) and (3.2) as follows:

$$I(u(f^m)) = (\delta(2), u(2))(\delta(3), u(3)) \cdots (\delta(m-1), u(m-1))(1, u(1)), \tag{4.1}$$

$$I(l(f^m)) = (\delta(2), l(2))(\delta(3), l(3)) \cdots (\delta(m-1), l(m-1))(1, l(1)). \tag{4.2}$$

We have the inverse operation I^{-1}. There exists a fold homotopy between f^m and an m-spiral fold map g^m such that $X(g^m) = I(X(f^m))$ for $X = u$ or l. The inverse operation corresponds to the inverse path.

4.2. **Operation II.** Let t and i be elements of $\{1, 2, \dots, m\}$. The operations $II^A_{(t,i)}$ and $II^B_{(t,i)}$ change (3.1) and (3.2) as follows:

$$\begin{aligned} II^A_{(t,i)}(u(f^m)) = (\overline{1}, \overline{u(1)}) \cdots (\overline{\delta(i-1)}, \overline{u(i-1)})(t, t+1) \\ (\overline{\delta(i)}, \overline{u(i)}) \cdots (\overline{\delta(m-1)}, \overline{u(m-1)}), \end{aligned} \tag{4.3}$$

$$\begin{aligned} II^A_{(t,i)}(l(f^m)) = (\overline{1}, \overline{l(1)}) \cdots (\overline{\delta(i-1)}, \overline{l(i-1)})(t, t+1) \\ (\overline{\delta(i)}, \overline{l(i)}) \cdots (\overline{\delta(m-1)}, \overline{l(m-1)}). \end{aligned} \tag{4.4}$$

$$II^B_{(t,i)}(u(f^m)) = (\overline{1}, \overline{u(1)}) \cdots (\overline{\delta(i-1)}, \overline{u(i-1)})(t+1, t) \\ (\overline{\delta(i)}, \overline{u(i+1)}) \cdots (\overline{\delta(m-1)}, \overline{u(m-1)}), \tag{4.5}$$

$$II^B_{(t,i)}(l(f^m)) = (\overline{1}, \overline{l(1)}) \cdots (\overline{\delta(i-1)}, \overline{l(i-1)})(t+1, t) \\ (\overline{\delta(i)}, \overline{l(i+1)}) \cdots (\overline{\delta(m-1)}, \overline{l(m-1)}). \tag{4.6}$$

Here, $\overline{\kappa}$ is defined by

$$\begin{cases} \kappa & \text{if } \kappa \leq t, \\ \kappa + 1 & \text{if } \kappa > t. \end{cases} \tag{4.7}$$

We have the inverse operations $(II^A_{(t,i)})^{-1}$ and $(II^B_{(t,i)})^{-1}$. There exists a fold homotopy between f^m and an $(m+1)$-spiral fold map g^{m+1} such that $X(g^{m+1}) = II^Z_{(t,i)}(X(f^m))$. Here $X = u, l$ and $Z = A, B$. Each inverse operation corresponds to the inverse path.

4.3. **Operation III.** Let $1 \leq i \leq m-2$. We set $X = u, X' = l$ or vice versa.

(1) Suppose that $\delta(i) \neq u(i) \neq \delta(i+1) \neq u(i+1)$ and $\delta(i) \neq l(i) \neq \delta(i+1) \neq l(i+1)$ hold. Then, the operation III^A_i changes (3.1) and (3.2) as follows:

$$III^A_i(u(f^m)) = (1, u(1)) \cdots (\delta(i-1), u(i-1))(\delta(i+1), u(i+1))(\delta(i), u(i)) \\ (\delta(i+2), u(i+2)) \cdots (\delta(m-1), u(m-1)), \tag{4.8}$$

$$III^A_i(l(f^m)) = (1, l(1)) \cdots (\delta(i-1), l(i-1))(\delta(i+1), l(i+1))(\delta(i), l(i)) \\ (\delta(i+2), l(i+2)) \cdots (\delta(m-1), l(m-1)). \tag{4.9}$$

(2) Suppose that $\delta(i) = X(i+1)$ and $\delta(i+1) \neq X'(i)$. Then, the operation III^B_i changes (3.1) and (3.2) as follows:

$$III^B_i(X(f^m)) = (1, X(1)) \cdots (\delta(i-1), X(i-1))(\delta(i+1), X(i))(\delta(i), X(i)) \\ (\delta(i+2), X(i+2)) \cdots (\delta(m-1), X(m-1)), \tag{4.10}$$

$$III^B_i(X'(f^m)) = (1, X'(1)) \cdots (\delta(i-1), X'(i-1))(\delta(i+1), \overline{X'(i+1)})(\delta(i), \overline{X'(i)}) \\ (\delta(i+2), X'(i+2)) \cdots (\delta(m-1), X'(m-1)). \tag{4.11}$$

Here, $(\delta(i+1), \overline{X'(i+1)})(\delta(i), \overline{X'(i)})$ is defined by

$$
(4.12)\quad \begin{cases} (\delta(i+1), X'(i+1))(\delta(i), X'(i)) & \text{if } X'(i+1) \neq X'(i) \neq \delta(i), \\ (\delta(i+1), X'(i))(\delta(i), X'(i)) & \text{if } X'(i+1) = \delta(i), \\ (\delta(i+1), \delta(i))(\delta(i), X'(i)) & \text{if } X'(i+1) = X'(i). \end{cases}
$$

(3) Suppose that $\delta(i+1) = X(i)$ and $\delta(i) \neq X'(i+1)$. Then, the operation III_i^C changes (3.1) and (3.2) as follows:

$$
(4.13)\quad III_i^C(X(f^m)) = (1, X(1)) \cdots (\delta(i-1), X(i-1))(\delta(i+1), X(i+1))(\delta(i), X(i+1)) \\ (\delta(i+2), X(i+2)) \cdots (\delta(m-1), X(m-1)),
$$

$$
(4.14)\quad III_i^C(X'(f^m)) = (1, X'(1)) \cdots (\delta(i-1), X'(i-1))(\delta(i+1), \overline{X'(i+1)})(\delta(i), \overline{X'(i)}) \\ (\delta(i+2), X'(i+2)) \cdots (\delta(m-1), X'(m-1)).
$$

Here, $(\delta(i+1), \overline{X'(i+1)})(\delta(i), \overline{X'(i)})$ is defined by

$$
(4.15)\quad \begin{cases} (\delta(i+1), X'(i+1))(\delta(i), X'(i)) & \text{if } X'(i) \neq X'(i+1) \neq \delta(i+1), \\ (\delta(i+1), X'(i+1))(\delta(i), X'(i+1)) & \text{if } X'(i) = \delta(i+1), \\ (\delta(i+1), X'(i))(\delta(i), \delta(i+1)) & \text{if } X'(i) = X'(i+1). \end{cases}
$$

For each operation, we have the inverse operations $(III_i^A)^{-1}$, $(III_i^B)^{-1}$ and $(III_i^C)^{-1}$. There exists a fold homotopy between f^m and an m-spiral fold map g^m such that $X(g^m) = III_i^Z(X(f^m))$ and $X'(g^m) = III_i^Z(X'(f^m))$, where $Z = A, B$ or C. Each inverse operation corresponds to the inverse path.

4.4. **Operation IV.** Suppose that we have $1 \leq i \leq m-1$, $u(i) = l(i)$ and $u(i) \neq \delta(i) \pm 1 \pmod{m}$. Then the operation IV_i changes (3.1) and (3.2) as follows:

$$
(4.16)\quad IV_i(u(f^m)) = (1, u(1)) \cdots (\delta(i-1), u(i-1))\sharp \\ (\delta(i+1), u(i+1)) \cdots (\delta(m-1), u(m-1)),
$$

$$
(4.17)\quad IV_i(l(f^m)) = (1, l(1)) \cdots (\delta(i-1), l(i-1))\sharp \\ (\delta(i+1), l(i+1)) \cdots (\delta(m-1), l(m-1)).
$$

In (4.16) and (4.17), the symbol $\sharp$ means the connected sum of an i-spiral fold map g^i and an $(m-i)$-spiral fold map g^{m-i}. That is, there exists a fold homotopy between f^m and $g^i \sharp g^{m-i}$ such that $X(g^i) = (1, X(1)) \cdots (\delta(i-1), X(i-1))$ and $X(g^{m-i}) = (\delta(i+1), X(i+1)) \cdots (\delta(m-1), X(m-1))$. Here X is u or l. We have the inverse operation $(IV_i)^{-1}$ and it corresponds to the inverse path.

4.5. **Operation V.** Let $1 \leq i \leq m-1$. Suppose that the products of pairs (as transpositions) equal to following cyclic permutations; $(1, u(1)) \cdots (\delta(i), u(i)) = (1, l(1)) \cdots (\delta(i), l(i)) = (i+1, i, \ldots, 1)$ and $(\delta(i+1), u(i+1)) \cdots (\delta(m-1), u(m-1)) = (\delta(i+1), l(i+1)) \cdots (\delta(m-1), l(m-1)) = (m, m-1, \ldots, i+1)$. Then the operation V_i changes (3.1) and (3.2) as follows:

$$V_i(u(f^m)) = (1, u(1)) \cdots (\delta(i), u(i)) \sharp (\delta(i+1), u(i+1)) \cdots (\delta(m-1), u(m-1)), \tag{4.18}$$

$$V_i(l(f^m)) = (1, l(1)) \cdots (\delta(i), l(i)) \sharp (\delta(i+1), l(i+1)) \cdots (\delta(m-1), l(m-1)). \tag{4.19}$$

In (4.18) and (4.19), the symbol $\sharp$ means the connected sum of an $(i+1)$-spiral fold map g^{i+1} and an $(m-i)$-spiral fold map g^{m-i}. That is, there exists a fold homotopy between f^m and $g^{i+1} \sharp g^{m-i}$ such that $X(g^{i+1}) = (1, X(1)) \cdots (\delta(i), X(i))$ and $X(g^{m-i}) = (\delta(i+1), X(i+1)) \cdots (\delta(m-1), X(m-1))$. Here X is u or l. We have the inverse operation $(V_i)^{-1}$ and it corresponds to the inverse path.

4.6. **Idea of the proof of Theorem 2.4 (1).** By Definition 2.3, we have $\mathcal{F}_{ij} \cap \mathcal{F}_{kl} = \emptyset$ for $i \neq k$ or $j \neq l$. We will prove that both of $\mathcal{F}_{11}$ and $\mathcal{F}_{12}$ are connected. In the sense of Mather [4], the subspace of all stable fold maps in $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$ is open dense. Because of this fact and Proposition 3.1, it is sufficient to show that any two spiral fold maps in $\mathcal{F}_{1j}$ can be connected by a fold homotopy. Suppose that an m-spiral fold map f^m is in $\mathcal{F}_{11}$, $m \geq 1$. Using the operations II and IV finitely many times, we show that there exists a fold homotopy between f^m and a fixed 1-spiral fold map $f^1 \in \mathcal{F}_{11}$. Suppose that an m-spiral fold map g^m is in $\mathcal{F}_{12}$, $m \geq 3$. Using the operations I–V and a theorem in [6] finitely many times, we show that there exists a fold homotopy between g^m and a fixed 3-spiral fold map $g^3 \in \mathcal{F}_{12}$.

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