重複度１の表現と複素多様体上の可視的な作用

Multiplicity-free representations and visible actions on complex manifolds

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アブストラクト：

正則ベクトル束に群作用が与えられているときに、その正則な大域切断の空間に実現された表現を考える。 この表現は、1) 底空間が非コンパクトならば一般に無限次元表現であり、2) 底空間における群軌道が無限個ならば一般に既約とは程遠い表現になる。

しかし、ある種の幾何的な条件（『可視的な作用』）が満たされていれば、重複度が１という性質がファイバーへの表現⇒大域切断における表現に伝播することが証明できる。

この概説講演では、有限次元および無限次元の表現における重複度１の表現のたくさんの例を紹介し、それが『可視的な作用』という幾何的な条件からどのように理解できるかを説明する予定です。

[1] Multiplicity-free theorem in branching problems of unitary highest weight modules, Proc. Symposium on Representation Theory held at Saga, Kyushu 1997 (ed. K. Mimachi), (1997), 9–17.

[2] Multiplicity one theorem in the orbit method (with Nasrin), in memory of Karpelevic "Lie Groups and Symmetric Spaces", (eds. S. Gindikin), 161–169 Amer. Math. Soc. 2003

[3] Geometry of Multiplicity-free representations of GL(n), visible actions on flag varieties, and triunity, Acta Appl. Math. 81 (2004), 129-146.

[4] Multiplicity-free representations and visible actions on complex manifolds, Publ. RIMS 41 (2005), 497-549.

Eg. 1 (Eigenspace decomposition)

$\mathcal{H}$: **Vector sp.** $/\mathbb{C}$, $\dim < \infty$

$A \in \mathrm{End}_{\mathbb{C}}(\mathcal{H})$

s.t. ① $\begin{cases} A \text{ is diagonalizable,} \\ \text{all eigenvalues are distinct.} \end{cases}$

$$\Rightarrow \mathcal{H} = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n \simeq \mathbb{C}^n \quad \textbf{(canonical)}$$

① & $\det A \neq 0$

$\Downarrow$

$$\pi_A : \begin{array}{ccc} \mathbb{Z} & \longrightarrow & GL_{\mathbb{C}}(\mathcal{H}) \\ \uplus & & \uplus \\ n & \longmapsto & A^n \end{array} \text{ is } \underline{\underline{\textbf{MF}}}$$

Eg. 2 (Fourier series expansion)

$$L^2(S^1) \simeq \sum_{n\in\mathbb{Z}}^{\oplus} \mathbb{C}\, e^{inx}$$

$$f(x) = \sum_{n\in\mathbb{Z}} a_n\, e^{inx}$$

Translation ($\Rightarrow$rep. of the group S^1)

$$f(\cdot) \mapsto f(\cdot - c) \qquad (c \in S^1 \simeq \mathbb{R}/2\pi\mathbb{Z})$$

$S^1 \curvearrowright L^2(S^1)$ is $\underline{\underline{\textbf{MF}}}$

$\pi : G \longrightarrow GL_{\mathbb{C}}(\mathcal{H})$

group

<u>Def. (naive)</u>

$(\pi, \mathcal{H})$ **is MF**

multiplicity-free

if $\dim \mathrm{Hom}_G(\tau, \pi) \leqq 1$

($\forall \tau$: **irred. rep. of** G**).**

$$L^2(S^1) \overset{\exists 1}{\hookleftarrow} e^{inx}$$

$\Leftrightarrow \dim \mathrm{Hom}_{S^1}(\tau, L^2(S^1)) = 1$

($\forall \tau$: **irred. rep. of** S^1**)**

$\Rightarrow S^1 \curvearrowright L^2(S^1)$ **is MF**

Eg. 3 (Taylor expansion, Laurent expansion)

$$f(z_1, \ldots, z_n) = \sum_{\alpha=(\alpha_1,\ldots,\alpha_n)} a_\alpha \, z_1^{\alpha_1} \cdots z_n^{\alpha_n}$$

Point (too obvious)

$\exists 1 \;\; a_\alpha \in \mathbb{C}$ **for each** α

$\Uparrow$

$\dim \mathrm{Hom}_{(S^1)^n}(\tau, \mathcal{O}(\{0\}) \leqq 1$

($\forall \tau$ **: irred. rep. of** $(S^1)^n$**)**

i.e. MF

Eg. 4 (Peter-Weyl)

G : **compact (Lie) group**

$$L^2(G) \simeq \sum_{\tau \in \widehat{G}}^{\oplus} \tau \boxtimes \tau^*$$

irred. rep. of $G \times G$

Translation ($\Rightarrow$ **rep. of** $G \times G$**)**

$f(\cdot) \mapsto f(g_1^{-1} \cdot g_2)$

$\Rightarrow \quad G \times G \curvearrowright L^2(G)$ **is MF**

Eg. 5 (Spherical harmonics)

$$\mathcal{H}_l := \left\{ f \in C^\infty(S^{n-1}) : \Delta_{S^{n-1}} f = -l(l+n-2)f \right\}$$

$$L^2(S^{n-1}) \simeq \sum_{l=0}^{\infty}{}^{\oplus} \mathcal{H}_l$$

$O(n)$ irred.

$$\Downarrow$$

$O(n) \curvearrowright L^2(S^{n-1})$ is **MF**

$\otimes$**-product rep.**

$$SL_2(\mathbb{C}) \overset{\pi_k}{\curvearrowright} S^k(\mathbb{C}^2)$$ irred.

$$(k = 0, 1, 2, \ldots)$$

Eg. 6 (Clebsch-Gordan)

$$\pi_k \otimes \pi_l \simeq \pi_{k+l} \oplus \pi_{k+l-2} \oplus \cdots \oplus \pi_{|k-l|}$$

↑ **MF**

Notation

Hightest weight

$$\lambda = (\lambda_1 \dots, \lambda_n) \in \mathbb{Z}^n, \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$$

$$\Downarrow$$

$\pi_\lambda^{GL_n} \equiv \pi_\lambda$ **: irred. rep. of** $GL_n(\mathbb{C})$

Eg.

$\lambda = (k, 0, \dots, 0) \quad \leftrightarrow \quad GL_n(\mathbb{C}) \curvearrowright S^k(\mathbb{C}^n)$

$\lambda = (\underbrace{1, \dots, 1}_{k}, 0, \dots, 0) \quad \leftrightarrow \quad GL_n(\mathbb{C}) \curvearrowright \Lambda^k(\mathbb{C}^n)$

$\otimes$-product rep. (GL_n-case)

Eg.7 (Pieri's law)

$$\pi_{(\lambda_1,\dots,\lambda_n)} \otimes \pi_{(k,0,\dots,0)}$$

$$\simeq \bigoplus_{\substack{\mu_1 \geq \lambda_1 \geq \dots \geq \mu_n \geq \lambda_n \\ \sum(\mu_i - \lambda_i) = k}} \pi_{(\mu_1,\dots,\mu_n)}$$

$\uparrow$

MF as a GL_n**-module.**

$\otimes$-product rep. (continued)

Eg. (counterexample)
$\pi_{(2,1,0)} \otimes \pi_{(2,1,0)}$ **is NOT MF as a $GL_3(\mathbb{C})$-module.**

$\otimes$-product rep. (continued)

$$\lambda = (\underbrace{a, \cdots, a}_{p}, \underbrace{b, \cdots, b}_{q}) \in \mathbb{Z}^n, \quad a \geq b$$

Eg. 8. (Stembridge 2001, K–)
$\pi_\lambda \otimes \pi_\nu$ is MF as a $GL_n(\mathbb{C})$ -module if

1) $\min(a-b, p, q) = 1$ (and ν is any),

or

2) $\min(a-b, p, q) = 2$ and ν is of the form ⋆
$$\nu = (\underbrace{x \cdots x}_{n_1} \underbrace{y \cdots y}_{n_2} \underbrace{z \cdots z}_{n_3}) \quad (x \geq y \geq z),$$

or

3) $\min(a-b, p, q) \geq 3$, ⋆ &
$\min(x-y, y-z, n_1, n_2, n_3) = 1$.

Eg.9 $(GL_n \downarrow GL_{n-1})$

$$\pi^{GL_n}_{(\lambda_1,\dots,\lambda_n)}\big|_{GL_{n-1}} \simeq \underbrace{\bigoplus_{\lambda_1 \ge \mu_1 \ge \cdots \ge \mu_{n-1} \ge \lambda_n} \pi^{GL_{n-1}}_{(\mu_1,\dots,\mu_{n-1})}}$$

$\uparrow$ **MF as a $GL_{n-1}(\mathbb{C})$-module**

$\Downarrow$ **Application**

Gelfand-Tsetlin basis

$GL_n \overset{\pi_\lambda}{\curvearrowright} V$

Find a 'good' basis of V

Remark. V **is not necessarily MF as a $(\mathbb{C}^\times)^n$-module.**

Idea of Gelfand-Tsetlin basis

$$\begin{array}{rlll}
 & GL_n(\mathbb{C}) & & \\
 & \cup & \circlearrowright & \text{MF} \\
\mathbb{C}^\times & \times GL_{n-1}(\mathbb{C}) & & \\
 & \cup & \circlearrowright & \text{MF} \\
(\mathbb{C}^\times)^2 & \times GL_{n-2}(\mathbb{C}) & & \\
 & \cup & \circlearrowright & \text{MF} \\
 & \vdots & \vdots & \\
 & \cup & \circlearrowright & \text{MF} \\
 & (\mathbb{C}^\times)^n & &
\end{array}$$

Infinite dimensional case (cf. Eg.9)

Eg.10. $(U(p,q) \downarrow U(p-1,q))$

$\forall\pi$: **irred. unitary rep. of** $U(p,q)$
with higest weight

$\Rightarrow$**restriction** $\pi|_{U(p-1,q)}$ **is MF**
as a $U(p-1,q)$**-module**

Eg.11. $(GL - GL$ **duality)**

$$N = mn$$

$$\Rightarrow GL_m \times GL_n \curvearrowright S(\mathbb{C}^N) \cong S(M(m,n;\mathbb{C}))$$

This rep. is MF

- **Explicit formula** $\cdots$ $GL_m - GL_n$ **duality**

- **Generalization**

  (Branching law of holo. disc.
  w.r.t. symmetric pair)

Hua-Kostant-Schmid- K-

each component $\cdots$ finite dim $\quad \infty -$ dim

Another generalization

Eg.12 (Kac's MF space)
$S(\mathbb{C}^N)$ **is still MF**
as a $GL_{m-1} \times GL_n$ **module**

We recall:

$$\pi : \underset{\text{group}}{G} \to GL_{\mathbb{C}}(\mathcal{H})$$

Def. (naive)
$(\pi, \mathcal{H})$ **is MF** (multiplicity-free)
if $\dim \mathrm{Hom}_G(\tau, \pi) \leq 1$
($\forall \tau$**: irred. rep. of** G**).**

Observation

$n \leq 1 \Leftrightarrow \mathrm{End}(\mathbb{C}^n)$ is commutative.

$(\pi, \mathcal{H})$: unitary rep. of G

Def. $(\pi, \mathcal{H})$ is MF if

$\mathrm{End}_G(\mathcal{H})$ is commutative.

More generally, (ϖ, W): top. rep.

Def. (ϖ, W) is MF if any

unitary subrep. $(\pi, \mathcal{H})$ is MF

($^\exists G$-inj. cont. hom. $\mathcal{H} \hookrightarrow W$)

Eg. 2 (Fourier series expansion)

$$L^2(S^1) \simeq \sum_{n\in\mathbb{Z}}^{\oplus} \mathbb{C}\, e^{inx}$$

$$f(x) = \sum_{n\in\mathbb{Z}} a_n\, e^{inx}$$

e^{inx} (cf. centrifugal separator)

Translation ($\Rightarrow$rep. of the group S^1)

$f(\cdot) \mapsto f(\cdot - c) \quad (c \in S^1 \simeq \mathbb{R}/2\pi\mathbb{Z})$

$S^1 \curvearrowright L^2(S^1)$ is MF

Eg.13 (Fourier transform)

$$L^2(\mathbb{R}) \simeq \int_{\mathbb{R}}^{\oplus} \mathbb{C} e^{i\zeta x} d\zeta$$

(direct integral of Hilbert sp.)

$$f(x) = \int_{\mathbb{R}} \tilde{f}(\zeta) e^{i\zeta x} d\zeta$$

Translation (⇒ regular representation on $L^2(\mathbb{R})$)

$$f(\cdot) \mapsto f(\cdot - c)$$

$\mathbb{R} \curvearrowright L^2(\mathbb{R})$ is "MF"

⋮

continuous spectrum

Symmetric Space

Eg.14.

G/K: Riemannian Symm. Space

$\Rightarrow G \curvearrowright L^2(G/K)$ is MF.

Eg.15. (counter example)

G/H: Semisimple Symm. Space

$\Rightarrow G \curvearrowright L^2(G/H)$

is NOT always MF.

Eg.14-A. $G/K = \mathbf{SL}(n,\mathbb{R})/\mathbf{SO}(n)$

$$L^2(G/K) \simeq \underbrace{\int^{\oplus}_{\substack{\lambda_1 \geq \cdots \geq \lambda_n \\ \Sigma\lambda_i = 0}} \mathcal{H}_\lambda \, \underbrace{d\lambda}_{\text{cont. spec.}}}_{\text{MF}}$$

$\mathcal{H}_\lambda$: ∞-dim, irred. rep. of G

Eg.15-A. $G/H = \mathbf{SL}(n,\mathbb{R})/\mathbf{SO}(p,q)$

Multiplicity of most cont. spec.

in $L^2(G/H)$

$$= \frac{n!}{p!q!} > 1 \text{ if } p, q > 0.$$

$\Rightarrow$ NOT MF

Vector bundle case

Eg.15-B. $G/K = \mathbf{GL}(n,\mathbb{R})/\mathbf{O}(n)$

$\mathcal{V}_\tau := G \times_K \tau \to G/K$

τ: unitary rep. of K

$\Rightarrow G \curvearrowright L^2(\mathcal{V}_\tau)$ unitary

$G \curvearrowright L^2(\mathcal{V}_\tau)$ is NOT always MF.

but it is MF if $\tau \simeq \Lambda^k(\mathbb{C}^n)$

$(0 \leq k \leq n)$

Examples of Multiplicity-Free reps

- Peter-Weyl theorem
- Cartan-Helgason theorem
- Branching laws: $GL_n \downarrow GL_{n-1}$, $O_n \downarrow O_{n-1}$
- Clebsch-Gordan formula
- Pieri's law
- GL_m-GL_n duality
- Plancherel formula for $L^2(G/K)$
  (G/K: Riemannian symmetric spaces)
- (Gelfand-Graev-Vershik) canonical representations
- Hua-Kostant-Schmid K-type formula
- (Kac) linear multiplicity-free spaces
- (Panyushev) spherical nilpotent orbits
- (Stembridge) multiplicity-free tensor product representations of GL_n

etc.

Accordingly, various techniques can be applied in each MF case.

For example, one can

1) look for an open orbit of a Borel subgroup.

2) apply Littlewood-Richardson rules and variants.

3) use computat'l combinatorics.

4) employ the commutativity of the Hecke algebra.

5) apply Schur-Weyl duality and Howe duality.

Aim $\cdots$

To give a **simple principle**
that explains the property MF
of all these examples,
and more.

§2 MF theorem

multiplicity free

$$\begin{cases} H, K\text{: Lie groups} \\ D\text{: complex mfd.} \\ P \to D\text{: } H\text{-equiv. principal K-b'dle} \\ \mu\text{: } K \to GL_{\mathbb{C}}(V) \end{cases}$$

$\Downarrow$

Setting 1 H-equiv. holo.vector b'dle:
$$\mathcal{V} := P \times_K V \to D$$

$\Downarrow$

$$H \curvearrowright \mathcal{O}(D, \mathcal{V}) = \{\text{holo. sections}\}$$

Setting 2

$$\sigma \quad \begin{array}{llll} \sigma_1 & \curvearrowright & P & \textbf{diffeo.} \\ \sigma_2 & \curvearrowright & K & \textbf{auto.} \\ \sigma_3 & \curvearrowright & H & \textbf{auto.} \end{array}$$

s.t.
$$\begin{cases} \sigma(h\,p\,k) = \sigma(h)\sigma(p)\sigma(k) \quad (h \in H,\ p \in P,\ k \in K) \\ \sigma \curvearrowright D \simeq P/K \ \text{anti-holomorphic} \end{cases}$$

Recall

$$\begin{array}{ccccc} H & \curvearrowright & P & \curvearrowleft & K \\ & & \downarrow & & \\ & & D & & \end{array}$$

$$\begin{array}{ccccccc} & & B & & M & & \\ & & \cap & & \cap & & \\ H & \curvearrowright & P & \curvearrowleft & K & \xrightarrow{\mu} & GL_{\mathbb{C}}(V) \\ & & \downarrow & & & & \\ & & D & & & & \end{array}$$

$B \subset P^\sigma := \{p \in P : \sigma(p) = p\}$

subset

$\Downarrow$

$M \equiv M_B := \{k \in K : bk \in Hb \ ({}^\forall b \in B)\}$

Assumption

(a) $HBK = P$

(b) $\mu|_M$ **is MF**

say, $\mu|_M \simeq \bigoplus_{i=1}^{l} \nu_i$

(c) $\mu \circ \sigma \simeq \mu^*$ **as** K**-modules**

$\nu_i \circ \sigma \simeq \nu_i^*$ **as** M**-modules** $({}^\forall i)$

Assumption

(a) HBK contains an interior point of P

(b) $\mu|_M$ is MF

say, $\mu|_M \simeq \bigoplus_{i=1}^{l} \nu_i$

(c) $\mu \circ \sigma \simeq \mu^*$ as K-modules

$\nu_i \circ \sigma \simeq \nu_i^*$ as M-modules $({}^\forall i)$

Point of Assumptions

(a) $\cdots$ base sp.
(b) $\cdots$ fiber
(c) $\cdots$ often automatic

Theorem (MF theorem)

Assume ${}^\exists\sigma$ and ${}^\exists B \subset P^\sigma$ satisfying (a)∼(c).

$\Rightarrow H \curvearrowright \mathcal{O}(D, \mathcal{V})$ is **MF**.

Point

- propagation of MF property

  fiber MF $\Rightarrow$ sections MF

- geometry of base space

  $\cdots$ '**visible action**'

$$\begin{array}{ccc} & \curvearrowright & \mathcal{V} \\ H & \curvearrowright & \downarrow \\ & & D \end{array} \quad \Rightarrow H \curvearrowright \mathcal{O}(D, \mathcal{V})$$

Assumption (c)

$\mu \circ \sigma \simeq \mu^*$ as K-modules

holds if σ is a Weyl involution

i.e.

$\sigma \in \mathrm{Aut}(K)$, $\sigma^2 = \mathrm{id}$ s.t.

$\sigma(g) = g^{-1}$ on some max torus of K

e.g. $K = U(n)$, $\sigma(g) = \bar{g}$

§3 Visible action

$H \overset{\text{holomorphic}}{\curvearrowright} D$ **complex mfd, connected**

Def. **The action is (strongly) visible**

if $\quad {}^\exists D' \subset D$ **open subset**
$\quad {}^\exists \sigma \curvearrowright D$ **anti-holomorphic**
$\quad {}^\exists N \subset D$ **totally real**

s.t.

$$\begin{cases} \sigma|_N = \mathrm{id} \\ N \text{ meets every } H\text{-orbit in } D' \\ \sigma \text{ stabilizes every } H\text{-orbit.} \end{cases}$$

Assumption(a) $\doteqdot$ **(strongly) visible action**

Assumption (a): $HBK = P$ **for some** $B \subset P^\sigma$

$\Rightarrow N := P^\sigma K/K$ **meets every** H**-orbit on** $D := P/K$

$\Rightarrow$ $H \curvearrowright D$ **visible**

(σ: **involusion** $\Rightarrow N$: **totally real)**

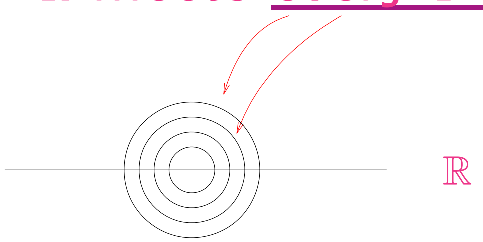
Example of Visible actions

$\mathbb{T} = \{a \in \mathbb{C} : |a| = 1\} \quad (\simeq S^1)$

Eg. $\mathbb{T} \curvearrowright \mathbb{C} \supset \mathbb{R}$

$a \in \mathbb{T}$: $z \mapsto az$

$\mathbb{R}$ **meets every $\mathbb{T}$-orbit**



$\Rightarrow$ **$\mathbb{T}$-action on $\mathbb{C}$ is visible.**

$H \overset{\text{holomorphic}}{\curvearrowright} (D, J)$ **complex mfd, connected**

Def. **Action is visible if**

$^{\exists}D' \underset{\text{open}}{\subset} D$,

$^{\exists}N \underset{\text{totally real}}{\subset} D$ **s.t.**

$$\begin{cases} N \text{ meets every } H\text{-orbit} \\ J_x(T_xN) \subset T_x(H \cdot x) \quad (x \in N) \end{cases}$$

$H \overset{\text{symplectic}}{\curvearrowright} (D, \omega)$ **symplectic mfd**

Def. **(Guillemin-Sternberg, Huckleberry-Wurzbacher)**

Action is coisotropic (or multiplicity-free)

if

$$T_x(H \cdot x)^{\perp \omega} \subset T_x(H \cdot x) \quad (x \in D)$$

$H \overset{\text{isometric}}{\curvearrowright} (D, g)$ **Riemannian mfd**

Def. **(Podestà-Thorbergsson)**

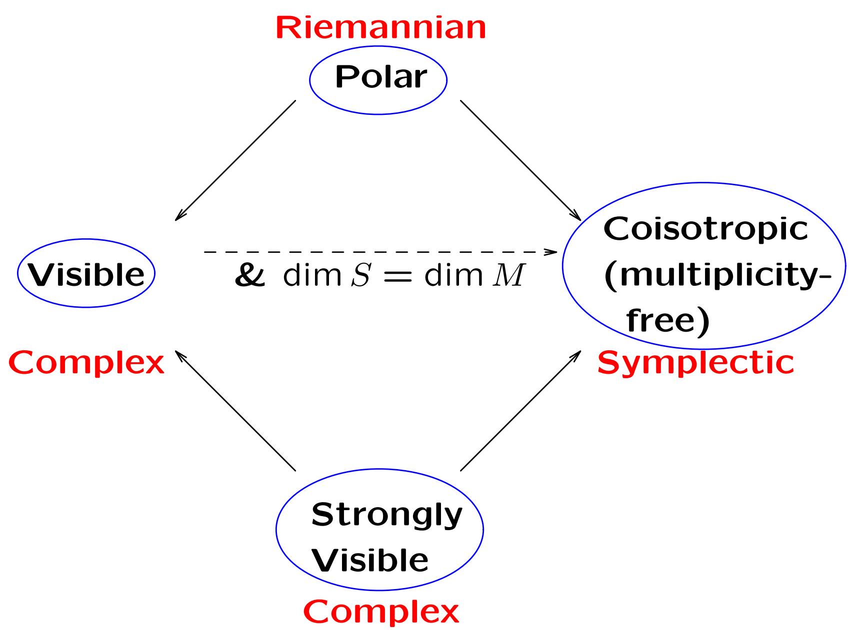
Action is polar if

$^{\exists} N \subset D$ **s.t.**

$$\begin{cases} N \text{ meets every } H\text{-orbit.} \\ T_x N \perp T_x(H \cdot x) \quad (x \in N) \end{cases}$$

$H \curvearrowright D$

compact compact, Kähler



Examples of Visible actions

Eg. $\mathbb{T} \curvearrowright \mathbb{C}$ is visible.

$\cup$

$\mathbb{R}$

$\Downarrow$

Eg. $\mathbb{T}^n \curvearrowright \mathbb{C}^n$ is visible.

$\cup$

$\mathbb{R}^n$

$\Downarrow$

Eg. $\mathbb{T}^n \curvearrowright \mathbb{P}^{n-1}\mathbb{C}$ is visible.

$\cup$

$\mathbb{P}^{n-1}\mathbb{R}$

$\Downarrow$

Eg. $U(1) \times U(n-1) \curvearrowright \mathcal{B}_n$ (full flag variety) is visible.

Eg. $U(n) \curvearrowright \mathbb{P}^{n-1}\mathbb{C} \times \mathcal{B}_n$ is visible.

Understanding of visible actions

$$\begin{pmatrix} H & & L \\ & \cap \quad \supset & \\ & G & \\ & \cup & \\ & G^\sigma & \end{pmatrix} := \begin{pmatrix} \mathbb{T}^n & & U(1) \times U(n-1) \\ & \cap \quad \supset & \\ & U(n) & \\ & \cup & \\ & O(n) & \end{pmatrix}$$

Geometry (visible action): $\mathbb{P}^{n-1}\mathbb{R}$ meets every $\mathbb{T}^n$-orbit on $\mathbb{P}^{n-1}\mathbb{C}$

$\updownarrow$ $\quad \mathbb{P}^{n-1}\mathbb{R} = G^\sigma / G^\sigma \cap L$, $\quad \mathbb{T}^n = H$, $\quad \mathbb{P}^{n-1}\mathbb{C} = G/L$

Group: $G = HG^\sigma L$ $\quad \Rightarrow$ Eg.3

$\updownarrow$

Group: $G = LG^\sigma H$ $\quad \Rightarrow$ Eg.9

$\updownarrow$

Group: $(G \times G) = \mathrm{diag}(G)(G^\sigma \times G^\sigma)(H \times L)$ $\quad \Rightarrow$ Eg.7

$\Downarrow$ **Theorem**

Various kinds of MF result including

- **(Fourier series)** $\mathbb{T} \curvearrowright L^2(\mathbb{T})$ — **Eg.2**
- **(Taylor series)** $\mathbb{T}^n \curvearrowright \mathcal{O}(\mathbb{C}^n)$ — **Eg.3**
- **($GL_n \downarrow GL_{n-1}$) Restriction** $\pi|_{GL_{n-1}}$ — **Eg.9**
- **(Pieri)** $\pi \otimes S^k(\mathbb{C}^n)$ — **Eg.7**
- **(Kac)** $GL_{m-1} \times GL_n \curvearrowright S(\mathbb{C}^{mn})$ — **Eg.12**

Eg.

G/K **compact symm. sp.**

$\Rightarrow G \curvearrowright G_{\mathbb{C}}/K_{\mathbb{C}}$ **is visible.**

$\Downarrow$ **Theorem**

Eg.16. $G \curvearrowright L^2(G/K)$ **is MF.**

Eg.17. (vector bundle case)

$\mathcal{V}_k = U(n) \underset{O(n)}{\times} \Lambda^k(\mathbb{C}^n) \to U(n)/O(n)$

$U(n) \curvearrowright L^2(\mathcal{V}_k)$ **is MF.**

Eg.

G/K **non-compact symm. sp.**

$\Rightarrow G \curvearrowright \Omega \underset{\text{crown}}{\subset} G_{\mathbb{C}}/K_{\mathbb{C}}$ **is visible.**

$\Downarrow$ **Theorem**

Eg.16′. $G \curvearrowright L^2(G/K)$ **is MF.**

Eg.17′. (vector bundle case)

$\mathcal{V}_k = GL(n,\mathbb{R}) \underset{O(n)}{\times} \Lambda^k(\mathbb{C}^n) \to U(n)/O(n)$

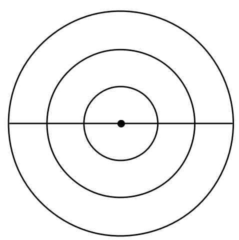
$GL(n,\mathbb{R}) \curvearrowright L^2(\mathcal{V}_k)$ **is MF.**

G/K **Hermitian symm. space**

Eg.

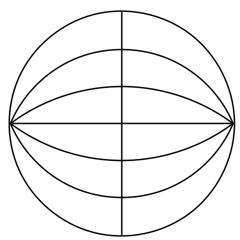
$$G = SL(2,\mathbb{R})$$
$$K = SO(2)$$
$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\}$$
$$G/K \simeq \{z \in \mathbb{C} : |z| < 1\}$$

$K \curvearrowright G/K$ **visible**



K-**orbits**

$H \curvearrowright G/K$ **visible**



H-**orbits**

Theorem $H \subset G \supset K$

Assume $\begin{cases} G/K & \text{Hermitian symm. sp.} \\ (G,H) & \text{symmetric pair} \end{cases}$

$\Rightarrow H \curvearrowright G/K$ **is visible**

$\Downarrow$ **Theorem**

Eg. 18 π_λ, π_μ **: highest wt. modules of scalar type**

$\Rightarrow \pi_\lambda \otimes \pi_\mu$ **is MF**

Eg. 19 $\begin{cases} \pi_\lambda : & \text{highest wt. module of scalar type} \\ (G,H) : & \text{symmetric pair} \end{cases}$

$\Rightarrow \pi_\lambda|_H$ **is MF**

Also, for finite dimensional case

$\Downarrow$ **Theorem**

Eg.20 (Okada, 1998)

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(n,\mathbb{C})$$

$$\lambda = (\underbrace{a,\cdots,a}_{p},\underbrace{b,\cdots,b}_{n-p}) \in \mathbb{Z}^n, a \geq b$$

$\pi_\lambda|_{\mathfrak{h}_{\mathbb{C}}}$ **is MF if**

$$\mathfrak{h}_{\mathbb{C}} = \begin{cases} \mathfrak{gl}(k,\mathbb{C}) + \mathfrak{gl}(n-k,\mathbb{C}) & (1 \leq k \leq n) \\ \mathfrak{o}(n,\mathbb{C}) & \\ \mathfrak{sp}(\frac{n}{2},\mathbb{C}) & (n : \text{even}) \end{cases}$$

G: **non-compact, simple Lie gp.,**

G/K **Hermitian**

Eg. $SU(p,q), SO(n,2), Sp(n,\mathbb{R}), SO^*(2n), E_{6(-14)}, E_{7(-25)}$

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}^+ + \mathfrak{p}^-$$

Def. $(\pi,V) \in \widehat{G}$ **unitary highest wt rep**

$\Leftrightarrow \{v \in V^\infty : d\pi(X)v = 0 \ ({}^\forall X \in \mathfrak{p}^+)\} \neq 0$

μ

K

Write

$$\pi = \pi^G(\mu) \quad (\mu \in \widehat{K})$$

Def. π : **holomorphic discrete series**

$\Leftrightarrow \mathrm{Hom}_G(\pi, L^2(G)) \neq 0$

π : **scalar type**

$\Leftrightarrow \dim \mu = 1$

Def. (G, H) **symmetric pair, holomorphic type**

$\Leftrightarrow {}^{\exists}\tau \in \mathrm{Aut}(G),\ \tau^2 = \mathrm{id}$ **s.t.**

$$\begin{cases} (G^\tau)_0 \subset H \subset G^\tau \\ \tau \curvearrowright G/K \text{ holomorphic} \end{cases}$$

$$\begin{array}{ccccccccc} & & \mathfrak{g}_{\mathbb{C}} & = & \mathfrak{k}_{\mathbb{C}} & + & \mathfrak{p}_+ & + & \mathfrak{p}_- \\ & & \cup & & \cup & & \cup & & \cup \\ \mathfrak{h}_{\mathbb{C}} & = & \mathfrak{g}_{\mathbb{C}}^\tau & = & \mathfrak{k}_{\mathbb{C}}^\tau & + & \mathfrak{p}_+^\tau & + & \mathfrak{p}_-^\tau \end{array}$$

$$\begin{array}{ccccc} & \mathfrak{t}^\tau & \subset & \mathfrak{k}^\tau & \\ \text{Cartan} & \cap & & \cap & \text{Cartan} \\ & \mathfrak{t} & \subset & \mathfrak{k} & \end{array}$$

$\{\nu_1, \ldots, \nu_k\}$ **maximal set of strongly orth. roots in** $\Delta(\mathfrak{p}_+^{-\tau}, \mathfrak{t}^\tau)$

Note: $k = \mathbb{R}$**-rank** G/H

Def. (G, H) **symmetric pair, holomorphic type**

$\Leftrightarrow {}^{\exists}\tau \in \mathrm{Aut}(G),\ \tau^2 = \mathrm{id}$ **s.t.**

$$\begin{cases} (G^\tau)_0 \subset H \subset G^\tau \\ \tau \curvearrowright G/K \text{ holomorphic} \end{cases}$$

Eg. $\tau = \theta,\quad H = K$

Eg. $G = Sp(n, \mathbb{R})$ $\quad (n=p+q)$

$H = Sp(p, \mathbb{R}) \times Sp(q, \mathbb{R})$

$H = U(p, q)$

$H = U(n)\ (= K)$

cf. $H = GL(n, \mathbb{R})$ **not holo. type**

Theorem

$$\begin{cases} \pi^G(\mu) \in \widehat{G} : \text{holo. disc. series, scalar type} \\ (G, H) : \text{symmetric pair, holomorphic type} \end{cases}$$

$\Rightarrow$

$$\pi^G(\mu)^H \simeq \sum_{\substack{a_1 \geq \cdots \geq a_k \geq 0 \\ a_j \in \mathbb{N}}}^{\oplus} \pi^H\left(\mu\big|_{\mathfrak{t}^\tau} - \sum_{j=1}^{k} a_j \nu_j\right)$$

$H = K$ $\cdots$ **Schmid (1969)**

$$GL(n, \mathbb{C}) \curvearrowright \mathcal{N} \underset{\text{nilpotent orbit}}{\subset} M(n, \mathbb{C})$$

Eg.21 (Panyushev 1994)

$\mathcal{O}(\mathcal{N})$ **is MF as** $GL(n, \mathbb{C})$**-module**

$\Leftrightarrow \mathcal{N} = \mathcal{N}_{(2^p, 1^{n-2p})}$ $(0 \leq {}^\exists p \leq \frac{n}{2})$

$$\mathcal{N}_{(2^p, 1^{n-2p})} \ni \begin{pmatrix} \overbrace{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \; \ddots \; \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}}^{p} & \\ & \overbrace{\begin{matrix} 0 & & \\ & \ddots & \\ & & 0 \end{matrix}}^{n-2p} \end{pmatrix}$$

Eg. $0 \le 2p \le n$

1) $U(n) \curvearrowright \mathcal{N}_{(2^p, 1^{n-2p})}$ **is visible.**

2) **(Panyushev)** $\mathcal{O}(\mathcal{N}_{2^p, 1^{n-2p}})$ **is MF**

$$\mathcal{N}_{(2^p,1^{n-2p})} \ni \begin{pmatrix} \overbrace{\begin{matrix} 0 & 1 & & & \\ 0 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & 0 & 0 \end{matrix}}^{p} & \\ & \overbrace{\begin{matrix} 0 & & \\ & \ddots & \\ & & 0 \end{matrix}}^{n-2p} \end{pmatrix}$$

Sketch of proof

$$G = U(n)$$

$$H = U(p) \times U(q) \qquad (p + q = n)$$

$$G \underset{H}{\times} M(p, q; \mathbb{C}) \to \overline{\mathcal{N}_{(2^p,1^{n-2p})}}$$

	$H \curvearrowright M(p, q; \mathbb{C})$	**visible**
$\Rightarrow$	$G \curvearrowright \mathcal{N}_{(2^p,1^{n-2p})}$	**visible**

More examples of visible actions

Eg. $n_1 + n_2 + n_3 = p + q = n$

$G_{r_p}(\mathbb{R}^n)$ **meets every orbit of**
$U(n_1) \times U(n_2) \times U(n_3) \curvearrowright G_{r_p}(\mathbb{C}^n)$
($\Rightarrow$ visible action)

$\Leftrightarrow (U(n_1) \times U(n_2) \times U(n_3)) \times O(n) \times (U(p) \times U(q)) \twoheadrightarrow U(n)$

$\Leftrightarrow \min(n_1 + 1, n_2 + 1, n_3 + 1, p, q) \leqq 2$

$\otimes$-product rep. (continued)

$$\lambda = (\underbrace{a, \cdots, a}_{p}, \ \underbrace{b, \cdots, b}_{q}) \in \mathbb{Z}^n, \ \ a \geq b$$

Eg. 8. (Stembridge 2001, K–)
$\pi_\lambda \otimes \pi_\nu$ is MF as a $GL_n(\mathbb{C})$ -module if

1) min$(a-b, p, q) = 1$ (and ν is any),

or

2) min$(a-b, p, q) = 2$ and ν is of the form ⋆

$$\nu = (\underbrace{x \cdots x}_{n_1} \ \underbrace{y \cdots y}_{n_2} \ \underbrace{z \cdots z}_{n_3}) \quad (x \geq y \geq z),$$

or

3) min$(a-b, p, q) \geq 3$, ⋆ &
min$(x-y, \ y-z, n_1, n_2, n_3) = 1$.

$\Downarrow$ **Theorem**

MF property of the following

- $GL_m \times GL_n \curvearrowright S(\mathbb{C}^{mn})$ **Eg.11**
- $GL_{m-1} \times GL_n \curvearrowright S(\mathbb{C}^{mn})$ **Eg.12**
- **the Stembridge list of** $\pi_\lambda \otimes \pi_\nu$ **Eg.8**
- $GL_n \downarrow (GL_p \times GL_q)$ **Eg.22**
- $GL_n \downarrow (GL_{n_1} \times GL_{n_2} \times GL_{n_3})$ **Eg.23**
- ∞**-dimensional versions**

. . .

Eg.22. $(GL_n \downarrow (GL_p \times GL_q)) \quad n = p + q$

$$\pi^{GL_n}_{(\underbrace{x,\dots,x}_{n_1},\underbrace{y,\dots,y}_{n_2},\underbrace{z,\dots,z}_{n_3})}|_{GL_p \times GL_q}$$ **is MF**

if $\min(p,q) \leq 2$

or

if $\min(n_1, n_2, n_3, x-y, y-z) \leq 1$

(Kostant $n_3 = 0$; Krattenthaler 1998)

Eg.23. $(GL_n \downarrow (GL_{n_1} \times GL_{n_2} \times GL_{n_3}))$

$n = n_1 + n_2 + n_3$

$$\pi^{GL_n}_{(\underbrace{a,\dots,a}_{p},\underbrace{b,\dots,b}_{q})}|_{GL_{n_1} \times GL_{n_2} \times GL_{n_3}}$$ **is MF**

if $\min(n_1, n_2, n_3) \leq 1$

or

if $\min(p, q, a-b) \leq 2$

Orbit method

$$\begin{array}{ccccc} G & \overset{\text{Ad}}{\curvearrowright} & \mathfrak{g}^* & \supset & \mathcal{O}^G \\ \cup & & \downarrow \text{pr} & & \\ H & \overset{\text{Ad}^*}{\curvearrowright} & \mathfrak{h}^* & \supset & \mathcal{O}^H \end{array}$$

$H \underset{\text{symm. pair}}{\subset} G \underset{\text{Hermitian symm. pair}}{\supset} K$

$\mathfrak{g} \supset \mathfrak{k} \supset$ center $\Leftrightarrow \mathbb{R} \cdot z \subset \mathfrak{g}^*$

center $\neq 0$

Theorem (Nasrin & K-)

If $\mathcal{O}^G \cap \mathbb{R} \cdot z \neq \emptyset$, **then**

$\#(\mathcal{O}^G \cap \text{pr}^{-1}(\mathcal{O}^H))/H \leqq 1$

for any $\mathcal{O}^H$.