Invariant multipliers and $O(p,q)$-action

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Abstract

In this paper we consider multipliers satisfying some invariance conditions coming from $\mathrm{O}(p,q)$. We will also investigate $\mathbf{L}^p$-boundedness for some of the operators.

1 Introduction

In [S], Stein gave a characterization of the Riesz transforms(and more generally the higher Riesz transforms) in terms of certain invariance conditions under the group $\mathbf{R}_+ \times \mathrm{O}(n)$. In [KN] we showed that this characterization can be viewed as a special case, see example 1 below, of the following general framework: Let H be subgroup of $\mathrm{GL}(n,\mathbf{R})$ and (π, V) a finite dimensional irreducible representation of H. We will assume that there exists a finite set of open orbits, $\mathcal{O}_1, \dots, \mathcal{O}_N$, for the contragradient action $\lambda \to (h^t)^{-1}$, such that their union is conull in the character group $\hat{\mathbf{R}}^n$. Let $\mathcal{C}_{bdd}(\mathcal{O}_j)$ denote the complex vector space consisting of bounded continuous functions on $\mathcal{O}_j$, on which the group H acts by pullback of functions. Let $\mathcal{B}_H(\mathbf{L}^2(\mathbf{R}^n), V \otimes \mathbf{L}^2(\mathbf{R}^n))$ be the vector space of bounded, translation invariant operators $T : \mathbf{L}^2(\mathbf{R}^n) \to V \otimes \mathbf{L}^2(\mathbf{R}^n)$ satisfying

$$\begin{tikzcd} \mathbf{L}^2(\mathbf{R}^n) \arrow[r, "T"] \arrow[d, "l_g"'] & V \otimes \mathbf{L}^2(\mathbf{R}^n) \arrow[d, "\pi(g)\otimes l_g"] \\ \mathbf{L}^2(\mathbf{R}^n) \arrow[r, "T"] & V \otimes \mathbf{L}^2(\mathbf{R}^n), \end{tikzcd} \tag{1}$$

for all $g \in H$, where l_g is defined by $(l_g f)(t) = f(g^{-1}t)$ for $g \in \mathrm{GL}(n,\mathbf{R})$ and $f \in \mathbf{L}^2(\mathbf{R}^n)$. Then we have

Theorem 1 ([KN] Theorem 1 and Example 1).

$$\mathcal{B}_H(\mathbf{L}^2(\mathbf{R}^n), V \otimes \mathbf{L}^2(\mathbf{R}^n)) \cong \bigoplus_{j=1}^{N} \mathrm{Hom}_H(V^*, \mathcal{C}_{bdd}(\mathcal{O}_j))$$

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as vector spaces. Furthermore, if an orbit, $\mathcal{O}_j$ is a reductive symmetric space for H then

$$\dim(\mathrm{Hom}_H(V^*, \mathcal{C}(\mathcal{O}_j)) \leq 1.$$

Hence, if all the orbits are symmetric spaces we obtain

$$\dim \mathcal{B}_H(\mathbf{L}^2(\mathbf{R}^n), V \otimes \mathbf{L}^2(\mathbf{R}^n)) \leq N.$$

Example 1. *Stein's characterization of the Hilbert and Riesz transforms(see [S] sect III.1 Prop 2) can be explained in the framework of Theorem 1 where $H = \mathbf{R}_+ \times \mathrm{O}(n)$, $N = 1$, $\mathcal{O}_1 = \mathbf{R}^n \setminus \{0\}$, a reductive symmetric space, and π is the tensor product of the trivial representation with a spherical representation.*

Example 2. *Consider the action on $\mathbf{R}^n$ by the group $H = \mathbf{R}_+ \times \mathrm{O}(p, q)(pq \neq 0)$. Let π be the standard representation of $\mathrm{O}(p, q)$ on $V := \mathbf{C}^{p+q}$ extended trivially to $\mathbf{R}_+$. In this case there are two open orbits, namely, $\mathcal{O}_1 = \mathbf{R}_+ \times \mathrm{O}(p, q)/\mathrm{O}(p-1, q)$ and $\mathcal{O}_2 = \mathbf{R}_+ \times \mathrm{O}(p, q)/\mathrm{O}(p, q-1)$. Both quotients are reductive symmetric spaces and the representation π appears in $\mathcal{C}(\mathcal{O}_1)$ as well as $\mathcal{C}(\mathcal{O}_2)$. Hence, the Fact 1 tells us that $\dim \mathrm{Hom}_H(V, \mathcal{C}(\mathcal{O}_1) \oplus \mathcal{C}(\mathcal{O}_2)) = 2$. However, in this case, the space $\mathcal{B}_H(\mathbf{L}^2(\mathbf{R}^n), V \otimes \mathbf{L}^2(\mathbf{R}^n))$ is in fact trivial.*

This is explained by the following result

Theorem 2 ([KN] Prop 1). *$\mathcal{B}_H(\mathbf{L}^2(\mathbf{R}^n), V \otimes \mathbf{L}^2(\mathbf{R}^n)) = \{0\}$ if (π, V) is a non-unitarizable representation of a reductive Lie group H.*

In particular this is the case if H is a simple connected non-compact Lie group and (π, V) is a finite dimensional representation.

In [KN] we employed two Strategies to give examples and to characterize multipliers

Strategy 1 (Characterization of multipliers). *Given a multiplier or a family of multipliers.*

Step 1. Find a (maximal) group of relative invariance for that multiplier.

Step 2. Find the dimension of the space of solutions to the equations obtained in Step 1.

Strategy 2 (Finding nice multipliers). *Given an invariance condition by means of a subgroup of $\mathrm{Aff}(\mathbf{R}^n)$.*

Step 1. Solve the equations and find explicit forms of the solutions.

Step 2. Choose the solutions that yield $\mathbf{L}^2$-bounded (or $\mathbf{L}^p$-bounded) operators.

However, as we have seen in Fact 2 they cannot be used when the group is non-compact. In this paper we will instead use another strategy and show how multipliers can be given an "infinitesimal characterization" in terms of differential equations.

Let $T : \mathbf{L}^2(\mathbf{R}^n) \to \mathbf{L}^2(\mathbf{R}^n)$ be a bounded translation invariant operator. Then there exists a distribution $\mathcal{T} \in \mathcal{S}'(\mathbf{R}^n)$ such that

$$Tf(x) = \int_{\mathbf{R}^n} \mathcal{T}(x-y)f(y)\,dy (= \langle \mathcal{T}(x - .), f\rangle.$$

By an infinitesimal characterization of T, we shall mean(see Strategy 3 in the Introduction)

to find differential equations that are satisfied by $\mathcal{T}$ and conversely,

to prove uniqueness, in a suitable sense, of the solutions of the system of differential equations.

Strategy 3. *Starting from a system of differential or functional equations or a multiplier operator, we apply Strategy 1 or Strategy 2 respectively.*

2 O(p,q)-action and multipliers

In this section, following Strategy 3 in the Introduction, we describe another approach to find "nice multiplier operators" by using differential equations that arise from infinite dimensional representations of non-compact Lie groups. We shall illustrate this approach by the example of infinite dimensional representations of $\mathrm{O}(p,q)$ which are realized on the pseudo-Riemannian symmetric space $\mathrm{O}(p,q)/\mathrm{O}(p,q-1)$. We begin with an alternate characterization of the Riesz transforms on $\mathbf{R}^n$ by means of differential equations. Note that this is not the infinitesimal version of Example 1 but a new type of characterization in fact the differential operators in Theorem 4 commute with the vector fields generated by the natural action of $\mathbf{R}_+ \times \mathrm{O}(p,q)$. Since there are plenty of irreducible infinite dimensional representations of $\mathrm{O}(p,q)$ where matrix coefficients are bounded functions, we shall see that there are bounded translation invariant operators that satisfy some nice invariance conditions coming from $\mathrm{O}(p,q)$.

2.1 Infinitesimal characterization of the Riesz transforms for $\mathbf{R}^n$

From this viewpoint, let us examine the (higher) Riesz transforms, i.e. elements in the algebra generated by the Riesz transforms, see [S] sections III.3 and III.4.8. Let us denote the homogenized Laplacian by $D_x := |x|^2\Delta_{\mathbf{R}^n}$ and the Euler operator $x \cdot \nabla$ by E_x.

Theorem 3. *The system of differential equations*

$$\begin{cases} E_x\mathcal{T} = -n\mathcal{T} \\ D_x\mathcal{T} = \nu\mathcal{T}, \end{cases} \tag{2}$$

considered in the sense of distributions, with $\mathcal{T} \in \mathcal{S}'(\mathbf{R}^n)$ has non-trivial solutions if and only if ν is of the form

$$\nu = -(k-2)(k+n), \tag{3}$$

for some $k \in \mathbf{N}$. Then the dimension of the solution space is

$$a(n,k) := \binom{n+k-1}{k} - \binom{n+k-3}{k-2} \tag{4}$$

If $\mathcal{T}$ is a solution to (2) and T is the operator defined by convolution with $\mathcal{T}$ then T is bounded on $\mathbf{L}(\mathbf{R}^n)$. Furthermore, if ν satisfies (3) then T is a higher Riesz transform of degree k, see [S] section 4.8. In particular, when $k = 1$ we have the equation

$$|x|^2 \Delta_{\mathbf{R}^n} \mathcal{T} = (n+1)\mathcal{T}.$$

The solution space is n-dimensional, and the Riesz transforms R_j $(1 \leq j \leq n)$ forms its linear basis.

Sketch of proof. Let

$$C_x := E_x^2 + (n-2)E_x - D_x = -\sum_{1 \leq i < j \leq n} \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2.$$

Then the second equation in (2) is equivalent to

$$C_x \mathcal{T} = -\mu \mathcal{T}$$

where $\mu = (\nu - 2n)$. Hence, $\mu = k(2-k-n)$ if $\nu = -(k-2)(k+n)$. The operator C_x is induced by the Casimir operator for the group $\mathrm{O}(n)$. Taking the Fourier transform of the system (2) gives the following system of differential equations in the tempered distribution sense for the corresponding multiplier m

$$\begin{cases} E_\lambda m = 0 \\ C_\lambda m = -\mu m \end{cases}, \tag{5}$$

where E_λ are the corresponding operators in the λ-variables. Since $C_\lambda = E_\lambda^2 + (n-2)E_\lambda - |\lambda|^2 \Delta_{\mathbf{R}^n}$. and $|\lambda|^2 \Delta_{\mathbf{R}^n}$ is elliptic on $\mathbf{R}^n \setminus \{0\}$, we find that the solution, m, must be analytic on the set $\mathbf{R}^n \setminus \{0\}$. Hence, as the differential operator in the second condition acts tangentially, we may restrict to spheres. The differential operator C_λ is induced from the Casimir operator, and therefore it is generated by the vector fields of the natural action of $\mathrm{O}(n)$. Since the Casimir operator induces the Laplacian $\Delta_{S^{n-1}}$ on S^{n-1}, and since $\mathrm{O}(n)$ leaves S^{n-1} stable,we have that

$$C_\lambda m|_{S^{n-1}} = \Delta_{S^{n-1}}(m|_{S^{n-1}}).$$

The second equation then says that m is an eigenfunction for the Laplace operator on the sphere, namely spherical harmonics of degree k, which shows that it is non-zero if and only if ν satisfies the condition (3). It is also well-known that the dimension of the solution space has the indicated form, see [SW] Section IV.2. So far we have only considered solutions in the set $\mathbf{R}^n \setminus \{0\}$ now we want to show that they extend to solutions in the set $\mathbf{R}^n$ and that there are no new solutions appearing. Any homogeneous distribution of degree zero can be extended from $\mathbf{R}^n \setminus \{0\}$ to $\mathbf{R}^n$. Furthermore, since $C_\lambda - \mu$ is a homogeneous differential operator of degree zero, we see that the extension will satisfy the system (5) in all of $\mathbf{R}^n$. Finally, any solution of (5) can differ from the extended solutions we found only at the origin. But there is no distribution supported at the origin which is homogeneous of degree zero. Hence, we have found all solutions. $\square$

2.2 Infinitesimal characterization of certain multipliers for $\mathbf{R}^{p,q}$

We shall generalize the infinitesimal characterization of the Riesz transforms for $\mathbf{R}^n$(see Theorem 3) by taking an ultrahyperbolic differential operator

$$\Delta_{\mathbf{R}^p} - \Delta_{\mathbf{R}^q} = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_p^2} - \ldots - \frac{\partial^2}{\partial x_{p+1}^2} - \ldots - \frac{\partial^2}{\partial x_q^2},$$

where $n = p+q$, $p, q \geq 1$, instead of $\Delta_{\mathbf{R}^n}$. Then we shall see that there will appear naturally matrix coefficients of certain infinite dimensional representations of $\mathrm{O}(p,q)$, which can be regarded as a generalization of spherical harmonics of $\mathrm{O}(n)$. Let

$$D_x := (|x'|^2 - |x''|^2)(\Delta_{\mathbf{R}^p} - \Delta_{\mathbf{R}^q}).$$

Moreover we denote by

$$C_x^{(1)} := -\sum_{1 \leq i < j \leq p} \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2$$

and

$$C_x^{(2)} := -\sum_{1 \leq i < j \leq q} \left(x_{i+p} \frac{\partial}{\partial x_{j+p}} - x_{j+p} \frac{\partial}{\partial x_{i+p}} \right)^2.$$

The operators $C_x^{(1)}$ and $C_x^{(2)}$ are the Casimir operators, up to a scalar multiple, for the compact subgroups $\mathrm{O}(p) \times \{1\}$ and $\{1\} \times \mathrm{O}(q)$ respectively.

Theorem 4. *Let $n = p+q$ and $\nu_1, \nu_2, \mu \in \mathbf{C}$. Consider the system of differential equations*

$$\mathcal{N}_{\nu_1,\nu_2,\mu} : \begin{cases} E_x \mathcal{T} = -n\mathcal{T} \\ C_x^{(1)} \mathcal{T} = -\nu_1 \mathcal{T} \\ C_x^{(2)} \mathcal{T} = -\nu_2 \mathcal{T} \\ D_x \mathcal{T} = (2n+\mu)\mathcal{T} \end{cases} \tag{6}$$

for elements $\mathcal{T} \in \mathcal{S}'(\mathbf{R}^n)$.

1. *There exist non-trivial solutions $\mathcal{T}$ if and only if ν_1 is of the form $\nu_1 = k(2-k-p)$ for some $k \in \mathbf{N}$ and ν_2 is of the form $\nu_2 = l(2-l-q)$ for some $l \in \mathbf{N}$. The dimension of the corresponding space of bounded translation invariant operators, given by convolution with $\mathcal{T}$, does not exceed*

$$2a(p,k)a(q,l)$$

where $a(p,k)$ is defined in (4).

2. *Furthermore, if μ satisfies the inequality $-4\rho^2 \operatorname{Re}\mu \geq (\operatorname{Im}\mu)^2$, where $\rho = \frac{n-2}{2}$ then the dimension of the space of bounded translation invariant operators satisfying $\mathcal{N}(\nu_1, \nu_2, \mu)$ is precisely $2a(p,k)a(q,l)$. We also have some exceptional cases when $\mu = d(2\rho + d)$, where $d \in \mathbf{Z}$ with $d > 0$. They occur if either $p > 1$, $k - l = q + d + 2j$ for some non-negative integer j, or if $q > 1$ and $l - k = p + d + 2j$, where j is a non-negative integer. The dimension is then $a(p,k)a(q,l)$*

2.3 Proof of Theorem 4

Let $X_{p,q} = \{|x'|^2 - |x''|^2 = 1\} \subset \mathbf{R}^n \cong \mathrm{O}(p,q)/\mathrm{O}(p-1,q)$ with the induced pseudo-Riemannian structure and $\Delta_{X^{p,q}}$ the corresponding Laplace-Beltrami operator. We also observe that $X_{q,p} \cong \mathrm{O}(q,p)/\mathrm{O}(q-1,p) \cong \mathrm{O}(p,q)/\mathrm{O}(p,q-1)$. Let us introduce the notation

$$\mathcal{B}(\mathcal{N}_{\nu_1,\nu_2,\mu}, \mathbf{L}^2(\mathbf{R}^n)) = \{T : \mathbf{L}^2(\mathbf{R}^n) \to \mathbf{L}^2(\mathbf{R}^n) \text{ is a bounded translation invariant operator ; The distribution kernel } \mathcal{T} \text{ satisfies } \mathcal{N}_{\nu_1,\nu_2,\mu}\}.$$

The system of equations (6) gives the following system of differential equations in the distribution sense, on the Fourier transform side

$$\mathcal{M}_{\nu_1,\nu_2,\mu} : \begin{cases} E_\lambda m(\lambda) = 0 \\ C_\lambda^{(1)} m(\lambda) = -\nu_1 m(\lambda) \\ C_\lambda^{(2)} m(\lambda) = -\nu_2 m(\lambda) \\ C_\lambda m(\lambda) = -\mu m(\lambda) \end{cases}, \tag{7}$$

where the operators E_λ, $C_\lambda^{(1)}$ and $C_\lambda^{(2)}$ have the same form as E_x, $C_x^{(1)}$, and $C_x^{(2)}$, respectively, but in the λ-variables and C_λ is given by

$$C_\lambda = E_\lambda^2 + (n-2)E_\lambda - D_\lambda,$$

where D_λ is the same as D_x but in the λ-variables. From the general theory of multipliers it is clear that

$$\mathcal{B}(\mathcal{N}_{\nu_1,\nu_2,\mu}, \mathbf{L}^2(\mathbf{R}^n)) \cong \{m \in \mathbf{L}^\infty(\mathbf{R}^n); m \text{ satisfies } \mathcal{M}_{\nu_1,\nu_2,\mu} \text{ as a distribution }\}.$$

We begin by solving the system $\mathcal{M}(\nu_1, \nu_2, \mu)$ and then check which solutions that are bounded. The first observation is that the solutions have to be real analytic in a large open set

Lemma 1. *The operator* $-2C_\lambda^{(1)} - 2C_\lambda^{(2)} + C_\lambda + E_\lambda^2$ *is elliptic outside the set* $|\lambda'|^2 = |\lambda''|^2$*, so any solution to the system has to be real analytic outside that set.*

We will now study the system (7) by reducing to an ordinary differential equation. We begin by introducing some new coordinates: Let

$$\begin{aligned}
\mathcal{U} &= \{x \in \mathbf{R}^n; \sum_{i=1}^{p} x_i^2 \neq 0, \sum_{j=1}^{q} x_{j+p}^2 \neq 0\}, \\
\mathcal{V} &= (-1,1) \times \mathbf{R}_{>0} \times S^{p-1} \times S^{q-1}
\end{aligned}$$

Let T be the transformation

$$\begin{aligned}
(x_1, \ldots, x_n) &\rightarrow (s, t, \omega_1, \ldots, \omega_p, \eta_1, \ldots, \eta_q) \\
\mathcal{U} &\rightarrow \mathcal{V}
\end{aligned}$$

where

$$s = \frac{\sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{q} x_{j+p}^2}{\sum_{i=1}^{p} x_i^2 + \sum_{j=1}^{q} x_j^2}, \qquad t = \sum_{i=1}^{p} x_i^2 + \sum_{j=1}^{q} x_{j+p}^2,$$

$$\omega_i = \frac{x_i}{\sqrt{\sum_{i=1}^{p} x_i^2}}, \quad 1 \leq i \leq p, \qquad \eta_j = \frac{x_{j+p}}{\sqrt{\sum_{j=1}^{q} x_{j+p}^2}}, \quad 1 \leq j \leq q.$$

Then T is a diffeomorphism from $\mathcal{U}$ onto $\mathcal{V}$. In these coordinates the system $M(\nu_1, \nu_2, \mu)$ becomes the system $V(\nu_1, \nu_2, \mu)$ below

$$\begin{cases}
4(s^2-1)s^2\frac{\partial^2 F}{\partial s^2} + (8s^2 - 2(p-q)s + (2n-8))s\frac{\partial F}{\partial s} \\
\qquad\qquad\qquad + (\frac{2s}{s+1}\nu_1 - \frac{2s}{1-s}\nu_2 - \mu)F = 0 \\
t\frac{\partial F}{\partial t} = 0 \\
\Delta_{S^{p-1}} F = \nu_1 \\
\Delta_{S^{q-1}} F = \nu_2
\end{cases}$$

Lemma 2. *The solutions of* $W(p,q;\mu)$ *in* $\mathcal{D}'(\mathcal{U})$ *are in* $1-1$ *correspondence with the solutions to* $V(p,q;\mu)$ *in* $\mathcal{D}'(\mathcal{V})$*.*

Lemma 3. *The system* $V(p,q,\mu)$ *is equivalent to the ordinary differential equation*

$$\begin{aligned}
4(s^2-1)s^2u''(s) + (8s^2 - 2(p-q)s + (2n-8))su'(s) + \\
\frac{2s}{1+s}\nu_1 u(s) - \frac{2s}{1-s}\nu_2 u(s) - \quad \mu u(s) = 0
\end{aligned} \tag{8}$$

in the sense that any solution $u \in \mathcal{D}'((-1,1))$ *to (8) can be extended to a solution* $F \in \mathcal{D}'(\mathcal{V})$ *of* $V(p,q;\mu)$*, and any solution to* $V(p,q;\mu)$ *in* $\mathcal{D}'(\mathcal{V})$ *can be restricted to a solution of (8) in* $\mathcal{D}'((-1,1))$*.*

Lemma 4. *There is a bijection between the solutions of $\mathcal{M}(\nu_1, \nu_2, \mu)$ in $\mathcal{D}'(\mathcal{U})$ and the solutions in $\mathcal{D}'((-1,1))$ of (8).*

The solutions of (8) in $\mathcal{D}'((-1,1))$ are the following

$$\begin{array}{rl}
\mu = -\rho^2, : & s_{\pm}^{\frac{\rho}{2}} f(s), \\
\lambda = 0 & c(s^{\frac{\rho}{2}} \ln s)_{\pm} f(s) + s_{\pm}^{\frac{\rho}{2}} g(s) \\
\mu = j^2 - \rho^2, : & s_{\pm}^{\frac{\rho}{2}+\frac{j}{2}} f(s), \\
\lambda = \pm j & c(s^{\frac{\rho}{2}+\frac{j}{2}} \ln s)_{\pm} f(s) + s_{\pm}^{\frac{\rho}{2}-\frac{j}{2}} g(s) \\
\mu = 2k(2k+2\rho), : & s_{\pm}^{k+\rho} f(s), \\
\lambda = \pm(\rho + 2k) & (s_{+}^{-k} - s_{-}^{-k}) g(s), \\
 & \sum_{j=0}^{k-1} b_j \delta_0^{(j)} \\
\mu = \lambda^2 - \rho^2, : & s_{\pm}^{\frac{\rho}{2}+\frac{\lambda}{2}} f(s), \\
\lambda \text{ not as above} & s_{\pm}^{\frac{\rho}{2}-\frac{\lambda}{2}} g(s),
\end{array}$$

where f and g are real analytic in the interval $(-1,1)$ and uniquely determined up to a constant multiple and the constants b_j are determined by b_{k-1}. Thus we observe that the space of solutions is 2 dimensional at every point. Next we have to consider which of these solutions that correspond to solutions in $\mathcal{U}$ that extend to solutions in the set $\mathbf{R}^n \setminus \{0\}$. We know that such solution must be real analytic outside the set $\{|x'| = |x''|\}$. The equation (8) has three regular singular points. Except zero we also have -1 and 1. Close to $s = 1$ the solutions have the form, if q is odd

$$u_1(1-s) = (1-s)^{\frac{l}{2}} v(1-s), \quad u_2(1-s) = (1-s)^{\frac{2-q-l}{2}} w(1-s)$$

and if q is even

$$\begin{aligned}
u_1(1-s) &= (1-s)^{\frac{l}{2}} v(1-s), \\
u_2(1-s) &= c(1-s)^{\frac{l}{2}} \ln(1-s)\, v(1-s) + (1-s)^{\frac{2-q-l}{2}} w(1-s),
\end{aligned}$$

where the functions v and w are real analytic in the interval $(-1,1)$. By the correspondence these solutions extend to solutions on $\mathcal{U}$ by taking

$$f(x) = A_k(x')A_l(x'')|x'|^{-k}|x''|^{-l} u\left(\frac{2|x''|^2}{|x|^2}\right),$$

where $A_k(x')$ and $A_l(x'')$ are homogeneous extensions to $\mathbf{R}^k$ respectively $\mathbf{R}^l$ of degree k resp. l of spherical harmonics of degree k and l respectively. Thus they become analytic close to points in $\mathcal{U} \cap \{|x''| = 0\}$ if and only if u has the form $(1-s)^{\frac{l}{2}} v(1-s)$, where v is real analytic close to $s = 1$.

Close to $s=-1$) the solutions have the following form

$$u_1(1+s) = (1+s)^{\frac{k}{2}} v(1+s), \quad u_2(1+s) = (1+s)^{\frac{2-p-k}{2}} w(1+s)$$

if p is odd, and

$$\begin{aligned} u_1(1+s) &= (1+s)^{\frac{k}{2}} v(1+s), \\ u_2(1+s) &= c(1+s)^{\frac{k}{2}} \ln(1+s)\, v(1+s) + (1+s)^{\frac{2-p-k}{2}} w(1+s) \end{aligned}$$

if p is even, where v and w are real analytic in the interval $(-1,1)$. These solutions correspond to solutions of $W(p,q;\mu)$ in $\mathcal{U}$ by taking

$$f(x) = A_k(x')A_l(x'')|x'|^{-k}|x''|^{-l} u\left(\frac{2|x'|^2}{|x|^2}\right),$$

where $A_k(x')$ and $A_l(x'')$ are homogeneous extensions to $\mathbf{R}^k$ respectively $\mathbf{R}^l$ of degree k resp l of spherical harmonics of degree k resp l. Thus we see that they extend to solutions of $W(p,q;\mu)$, real analytic in a neighbourhood of points in $\mathcal{U} \cap \{|x'|=0\}$, if and only if u has the form $(1+s)^{\frac{k}{2}} v(1+s)$, where v is real analytic close to the point $s=-1$.

To conclude, we have proved

Lemma 5. *The space of solutions of $\mathcal{M}(\nu_1,\nu_2,\mu)$ in $\mathcal{D}'(\mathbf{R}^n \backslash \{0\})$ has dimension $2a(p,k)a(q,l)$.*

We now recall two facts from distribution theory

Fact 1. *Any homogeneous distribution of degree zero in $\mathbf{R}^n \setminus \{0\}$ has a unique extension to $\mathbf{R}^n$ which is homogeneous of degree zero.*

Fact 2. *Any homogeneous distribution of degree zero is in fact a tempered distribution.*

Proposition 1. *The space of solutions to $\mathcal{M}(\nu_1,\nu_2,\mu)$ in $\mathcal{S}'(\mathbf{R}^n)$ has dimension $2a(p,k)a(p,l)$.*

Proof of proposition. Since the equations in $\mathcal{M}(\nu_1,\nu_2,\mu)$ are homogenous the homogeneous extension also satisfies the same system. Any new solution would give a solution supported at the origin. But distributions supported at the origin cannot be homogeneous of degree zero. □

Thus, the first statement of Theorem 4 is proved. To prove the second part, let m be a solution to the system of equations $\mathcal{M}_{\nu_1,\nu_2,\mu}$, with μ satisfying the condition $-4\rho^2 \operatorname{Re}\mu \geq (\operatorname{Im}\mu)^2$. Note that if μ is real then the condition says that $\mu \leq 0$. In particular this means that the solutions supported on the cone are not in $\mathcal{M}(\nu_1,\nu_2,\mu)$. Since m is homogeneous of degree zero, it is determined by its restrictions $f_1 = m|_{X_{p,q}}$ and $f_2 = m|_{X_{q,p}}$(as we have seen in lemma 1 any solution is real analytic on the set $\{(\lambda',\lambda'') \in \mathbf{R}^n; |\lambda'|^2 \neq |\lambda''|^2\}$, so taking the restriction to $X_{p,q}$ or $X_{q,p}$ makes sense) Since C_λ is induced from the Casimir operator for the group $\mathrm{O}(p,q)$ its restriction to $X_{p,q}$ must be the Laplacian $\Delta_{X_{p,q}}$. Now, let η be such that $\eta^2 = \rho^2 + \mu$. Since m is an eigenfunction of the Laplace operator with eigenvalue $\mu = \eta^2 - \rho^2$. The condition $-4\rho^2 \operatorname{Re}\mu \geq (\operatorname{Im}\mu)^2$ implies that $|\operatorname{Re}\eta| \leq \rho$. By [O] Corollary 4.3, these functions are bounded when $|\operatorname{Re}\eta| \leq \rho$ and so, under that assumption, they are multipliers for $\mathbf{L}^2$. We can also see this directly from our list of solutions. In some exceptional cases the solution can be bounded even though $\eta > \rho$ this happens when f_1 respectively f_2 comes from the discrete series for $X_{p,q}$ resp. $X_{q,p}$, which is characterized by the conditions on the parameters as in the statement of the Theorem 4, (except that d can be negative as long as it $> -\rho$ but when $-\rho < d \leq 0$ we have $\eta \leq \rho$, which we have already covered), see [Sch] Theorem 6.4 or [St] Theorem 2.

2.4 L^p results

In this section we want to consider $\mathbf{L}^p(\mathbf{R}^n)$-boundedness for some of the operators considered in the previous section. The operators we will consider are the ones coming from the discrete series for $X_{p,q}$ and $X_{q,p}$ respectively. These appear when $\mu = d(n-2+d)$, $\eta = d+\rho$, for some $d \in \mathbf{Z}$ with $d > -\rho$, if $p > 1$ $k - l = q + d + 2j$,(if $q = 1$ then $l = 0$ or $l = 1$), j a non-negative integer or if $q > 1$ and $l - k = p + d + 2j$,(if $p = 1$ then $l = 0$ or $l = 1$), where j is a non-negative integer(recall that $\nu_1 = k(2-k-p)$ and $\nu_2 = l(2-l-q)$), see [Sch] Theorem 6.4 and [St] Theorem 2. The point is that these solutions are supported inside, respectively outside the cone, and satisfies the inequality

$$m(\lambda', \lambda'') \leq C \left(\frac{|\lambda'|^2 - |\lambda''|^2}{|\lambda'|^2 + |\lambda''|^2} \right)^{\rho + \frac{d}{2}}$$

Theorem 5. *Let m be a solution to $\mathcal{M}(\nu_1, \nu_2, \mu)$ coming from the discrete series. If n is even and $d \geq 4$ or n is odd and $d \geq 3$ then the corresponding translation invariant operator $T$is bounded on $\mathbf{L}^p(\mathbf{R}^n)$.*

Proof. A sufficient condition for a function m to be a $\mathbf{L}^p$-multiplier, $1 < p < \infty$ is given by the Hörmander-Michlin condition:

Fact 3. *Let $m \in \mathcal{C}^{[\frac{n}{2}]+1}(\mathbf{R}^n \setminus \{0\})$ and assume that it satisfies*

$$\left| \frac{\partial^\alpha m(x)}{\partial x_i^\alpha} \right| \leq \frac{C}{|x|^{|\alpha|}}, \tag{9}$$

for all multi-indices α with $|\alpha| \leq [\frac{n}{2}] + 1$. Then m is a $\mathbf{L}^p$-multiplier for $1 < p < \infty$.

We may assume that $|\lambda'| \geq |\lambda''|$. Let us introduce the coordinates

$$\begin{array}{rcl} S^{p-1} \times \mathbf{R} \times \mathbf{R}^q & \to & \mathbf{R}^{p+q} \\ (\omega, r, \lambda'') & \mapsto & (r\omega, \lambda'') \end{array}$$

where $r = |\lambda'|$ and $\omega_i = \frac{\lambda'_i}{|\lambda'|}$. We denote

$$\mathbf{R}^{p,q}_{+,-} = \{(\lambda', \lambda'') \in \mathbf{R}^{p+q} : |\lambda'| > |\lambda''|\}.$$

Fix $\kappa \in \mathbf{R}_+$ and let

$$\mathcal{V}_\kappa := \left\{ f \in \mathcal{C}^\infty(\mathbf{R}^{p,q}_{+-}); \begin{array}{ll} 1) & f \text{ is homogeneous of degree } 0 \\ 2) & |f(\lambda', \lambda'')| \leq C \left(\frac{|\lambda'|^2 - |\lambda''|^2}{|\lambda'|^2 + |\lambda''|^2} \right)^m \end{array} \right\}$$

The natural action of $\mathrm{O}(p,q)$ on $\mathbf{R}^{p,q}$ preserves $\mathbf{R}^{p,q}_{+,-}$, and hence acts also on $\mathcal{C}^\infty(\mathbf{R}^{p,q}_{+,-})$. We will denote this action with π and its differential $d\pi$.

$$\begin{aligned} \mathcal{V}^\infty_\kappa := \{ & f \in \mathcal{V}_\kappa; d\pi(X_1) \cdot \ldots \cdot d\pi(X_k) f \in \mathcal{V}_\kappa, \\ & \text{for any } k = 0, 1, \ldots \text{ and } X_1, \ldots, X_k \in \mathfrak{o}(p,q)\} \end{aligned}$$

We observe that our function m belongs to $\mathcal{V}^\infty_{\rho + \frac{d}{2}}$. To take care of factors appearing from differentiation we introduce the spaces

$$\begin{aligned} H_{a,b,c} := \Big\{ & g \in \mathcal{C}^\infty(\mathbf{R}^{p,q}_{+,-}); g \text{ is a linear combination} \\ & \text{of terms of the form } \frac{A(\omega) P_a(\lambda'')}{|\lambda'|^b (|\lambda'|^2 - |\lambda''|^2)^c} \Big\}, \end{aligned}$$

where $A \in \mathcal{C}^\infty(S^{p-1})$ and P_a is a homogeneous polynomial on $\mathbf{R}^q$ of degree a. We will also use the space

$$H_N := \bigoplus_{\substack{c \leq a \leq 3N, c \leq N, \\ b - a + 2c = N}} H_{a,b,c}$$

For simplicity, we denote by $H_N \mathcal{V}^\infty_\kappa$ the subspace of $\mathcal{C}^\infty(\mathbf{R}^{p,q}_{+,-})$ consisting of finite linear combinations of products of elements from H_N and $\mathcal{V}^\infty_\kappa$. We want to show that

Proposition 2. *If $f \in \mathcal{V}^\infty_\kappa$ then, for α with $|\alpha| \leq \kappa$, we have $|\frac{\partial^\alpha}{\partial \lambda^\alpha} f| \leq \frac{C}{(|\lambda'|^2 + |\lambda''|^2)^{\frac{|\alpha|}{2}}}$ on $\mathbf{R}^{p,q}_{+,-}$.*

The first observation is that

Lemma 6. *Let $f \in \mathcal{V}_\kappa$ and $g \in H_{a,b,c}$, where $c \leq \kappa$. Then*

$$|fg| \leq \frac{C}{(|\lambda'|^2 + |\lambda''|^2)^{\frac{b-a+2c}{2}}}$$

in $\mathbf{R}^{p,q}_{+,-}$.

Hence Proposition 2 follows from this lemma and

Proposition 3. $\frac{\partial^\alpha}{\partial \lambda_i^\alpha}\mathcal{V}_\kappa^\infty \subset H_{|\alpha|} \cdot \mathcal{V}_\kappa^\infty$.

To prove Proposition 3 we need a lemma

Lemma 7.
$$H_N\left(\frac{\partial}{\partial \lambda_i}\mathcal{V}_\kappa^\infty\right) \subset H_{N+1}\mathcal{V}_\kappa^\infty.$$
and
$$\frac{\partial}{\partial \lambda_i}H_N \subset H_{N+1}.$$

Proposition 3 then follows from lemma 7 by induction. Proposition 2 shows that the conditions (3) are satisfied on $\mathbf{R}^{p,q}_{+,-}$ but we need this to be true also for the extension by zero to all of $\mathbf{R}^n$. The functions in $\mathcal{V}_\kappa$ are of polynomial degree $\kappa - c$ when we approach the boundary $|\lambda'| = |\lambda''|$. Hence, as long as $\kappa - c \geq 0$ the extension will ha continuous derivatives. Thus, if we take $\kappa = \rho + \frac{d}{2}$ then $\kappa \geq [\frac{n}{2}] + 1$ if d satisfies the conditions in the statement of Theorem 5. Hence, all the assumptions in Fact 3 are satisfied so the operator will be bounded on $\mathbf{L}^p(\mathbf{R}^n)$. □

References

[H2] Helgason, S. *Groups and Geometric Analysis*, Academic Press (1984)

[KN] Kobayashi, T. and Nilsson, A. *Invariant multipliers*, preprint

[O] Oshima, T. *Asymptotic Behaviour of Spherical Functions on Semisimple Symmetric Spaces*, Adv. Stud. in Pure Math. **14** (1988) 561-601

[Sch] Schlichtkrull, H. *Eigenspaces of the Laplacian on Hyperbolic Spaces: Composition Series and Integral Transforms* J. Funct. Anal. **70** (1987) 194-219

[S] Stein, E. M. *Singular integrals and differentiability properties of functions*, Princeton University Press (1970)

[SW] Stein, E. M. and Weiss, G. *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press (1971)

[St] Strichartz, Robert S. *Harmonic Analysis on Hyperboloids*, J. Funct. Anal. **12** (1973), 341-383