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Multiplicity One Theorem in the Orbit Method

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In memory of Professor F. Karpelevič

Abstract. Let $G \supset H$ be Lie groups, $\mathfrak{g} \supset \mathfrak{h}$ their Lie algebras, and $\mathrm{pr} : \mathfrak{g}^* \to \mathfrak{h}^*$ the natural projection. For coadjoint orbits $\mathcal{O}^G \subset \mathfrak{g}^*$ and $\mathcal{O}^H \subset \mathfrak{h}^*$, we denote by $n(\mathcal{O}^G, \mathcal{O}^H)$ the number of H-orbits in the intersection $\mathcal{O}^G \cap \mathrm{pr}^{-1}(\mathcal{O}^H)$, which is known as the Corwin-Greenleaf multiplicity function. In the spirit of the orbit method due to Kirillov and Kostant, one expects that $n(\mathcal{O}^G, \mathcal{O}^H)$ coincides with the multiplicity of $\tau \in \widehat{H}$ occurring in an irreducible unitary representation π of G when restricted to H, if π is 'attached' to $\mathcal{O}^G$ and τ is 'attached' to $\mathcal{O}^H$. Results in this direction have been established for nilpotent Lie groups and certain solvable groups, however, very few attempts have been made so far for semisimple Lie groups.

This paper treats the case where (G, H) is a semisimple symmetric pair. In this setting, the Corwin-Greenleaf multiplicity function $n(\mathcal{O}^G, \mathcal{O}^H)$ may become greater than one, or even worse, may take infinity. We give a sufficient condition on the coadjoint orbit $\mathcal{O}^G$ in $\mathfrak{g}^*$ in order that

$$n(\mathcal{O}^G, \mathcal{O}^H) \le 1 \quad \text{for any coadjoint orbit } \mathcal{O}^H \subset \mathfrak{h}^*.$$

The results here are motivated by a recent *multiplicity-free* theorem of branching laws of unitary representations obtained in [**7**], [**8**] by one of the authors.

1. Introduction

The celebrated **Gindikin-Karpelevič formula** on the c-function gives an explicit Plancherel measure for the Riemannian symmetric space G/K of non-compact type. Implicitly important in this formula is the following:

Fact 1.1. The regular representation on $L^2(G/K)$ decomposes into irreducible unitary representations of G with **multiplicity free**.

In order to explain and enrich Fact 1.1, let us fix some notation. Suppose G is a non-compact semisimple Lie group with maximal compact subgroup K. We write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. We take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, and denote by $\Sigma(\mathfrak{g}, \mathfrak{a})$ the restricted root system. We fix a positive root system Σ^+ and write $\mathfrak{a}_+^*$ for the dominant Weyl

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chamber. Let m_α be the dimension of the root space $\mathfrak{g}(\mathfrak{a};\alpha)$ for each $\alpha \in \Sigma(\mathfrak{g},\mathfrak{a})$, and we define $\Sigma_0 := \{\alpha \in \Sigma(\mathfrak{g},\mathfrak{a}) : \frac{\alpha}{2} \notin \Sigma(\mathfrak{g},\mathfrak{a})\}$.

Spherical unitary principal series representations of G are parametrized by $\lambda \in \mathfrak{a}^*_+$, which we shall denote by $\pi_\lambda\,(\in \widehat{G})$. Then a qualitative refinement of Fact 1.1 (multiplicity free theorem) is given by the following direct integral decomposition into irreducible unitary representations:

$$L^2(G/K) \simeq \int_{\mathfrak{a}^*_+}^{\oplus} \pi_\lambda \, d\lambda \qquad \text{(an abstract Plancherel formula).} \tag{1.1}$$

A further refinement of (1.1) is the **Gindikin-Karpelevič formula** on the c-function ([**2**], see also [**3**]),

$$c(\lambda) = c_0 \prod_{\alpha\in\Sigma_0^+} \frac{2^{-\frac{\langle i\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}}\Gamma\left(\frac{\langle i\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\right)}{\Gamma\left(\frac{1}{2}\left(\frac{1}{2}m_\alpha + 1 + \frac{\langle i\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\right)\right)\Gamma\left(\frac{1}{2}\left(\frac{1}{2}m_\alpha + m_{2\alpha} + \frac{\langle i\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\right)\right)},$$

which enriches formula (1.1) with quantitative result, namely, an explicit Plancherel density for (1.1) with respect to the spherical Fourier transform. Here c_0 is a normalized constant.

On the other hand, one can also enrich Fact 1.1 (multiplicity free theorem) and the Plancherel formula (1.1) from another viewpoint, namely, with **geometry of coadjoint orbits**, motivated by the philosophy of the orbit method due to Kirillov. One way to formulate this is based on an interpretation of the unitary representation of G on $L^2(G/K)$ as an induced representation (see [**9**], Example 5). Another somewhat unusual way is to find a "hidden symmetry" of an "overgroup" $\widetilde{G}$ of G. We shall take the latter viewpoint, which seems interesting not only in unitary representation theory but also in the geometry of coadjoint orbits because it leads us naturally to much wider topological settings.

This paper is organized as follows. First, we recall a multiplicity free theorem ([**7**], [**8**]) in the branching problem of unitary representations in §2, which contains Fact 1.1 as a special case (for classical groups and some other few cases). Its predicted counterpart in the orbital geometry is formulated in §3, and turns out to be true (Theorems A and B). We illustrate these results in an elementary way by a number of figures of lower dimensional examples in §4. A detailed proof of Theorems A and B will be given elsewhere.

2. Multiplicity-one decomposition and branching laws

There are several different approaches to prove Fact 1.1 (a multiplicity free result). A classical approach due to Gelfand is based on the commutativity of the convolution algebra $L^1(K\backslash G/K)$.

Another approach is based on the restriction of a representation of an overgroup $\widetilde{G}$. For instance, consider a semisimple symmetric pair

$$(\widetilde{G}, G) = (\mathrm{Sp}(n,\mathbb{R}), \mathrm{GL}(n,\mathbb{R})).$$

Then we have a natural injective map between two homogeneous spaces $G/K \to \widetilde{G}/\widetilde{K}$, namely,

$$\mathrm{GL}(n,\mathbb{R})/\,\mathrm{O}(n) \hookrightarrow \mathrm{Sp}(n,\mathbb{R})/\,\mathrm{U}(n). \tag{2.1}$$

Via the embedding (2.1), G/K becomes a totally real submanifold in the complex manifold $\widetilde{G}/\widetilde{K}$. Let π be a holomorphic discrete series representation of scalar type of $\widetilde{G}$. Then π is realized in the space of holomorphic sections of a $\widetilde{G}$-equivariant holomorphic line bundle over $\widetilde{G}/\widetilde{K}$. The restriction of the representation π with respect to the subgroup G factors through the representation of G realized on the space of continuous sections for a complex line bundle on the totally real submanifold G/K, and its abstract Plancherel formula coincides with that of $L^2(G/K)$ (see [**4**], [**6**], [**10**], [**11**]). This representation $\pi|_G$ is essentially known as a *canonical representation* in the sense of Vershik-Gelfand-Graev ([**12**]). Thus, the multiplicity one property in Fact 1.1 can be formulated in a more general framework of the **branching laws**, namely, irreducible decompositions of the restrictions of irreducible unitary representations to subgroups.

In this direction, the following theorems have been recently proved (see [**7**], [**8**]) (we shall use slightly different notation: the above pair $(\widetilde{G}, G)$ replaced by (G, H) below): Suppose G is a semisimple Lie group such that G/K is a Hermitian symmetric space of non-compact type. $G = \mathrm{SU}(p,q), \mathrm{Sp}(n,\mathbb{R}), \mathrm{SO}^*(4n), \mathrm{SO}(n,2)$ are typical examples. Then

Theorem 2.1. ([**7**], Theorem B) *Let π be an irreducible unitary highest weight representation of scalar type of G, and (G, H) an arbitrary symmetric pair. Then the restriction $\pi|_H$ decomposes into irreducible representations of H with multiplicity free.*

Theorem 2.2. ([**7**], Theorem A) *Let π_1, π_2 be unitary highest (or lowest) weight representations of scalar type. Then the tensor product representation $\pi_1 \otimes \pi_2$ decomposes with multiplicity free.*

3. Multiplicity-one theorem in the orbit method

The object of this paper is to provide a 'predicted' result in the orbit philosophy corresponding to Theorems 2.1 and 2.2.

For this, let us recall an idea of the orbit method due to Kirillov and Kostant in unitary representation theory of Lie groups.

Let $\mathfrak{g}$ be the Lie algebra of G, and $\mathfrak{g}^*$ the linear dual of $\mathfrak{g}$. Let us consider the contragradient representation

$$\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*)$$

of the adjoint representation of G, $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. This non-unitary finite dimensional representation often has a surprisingly intimate relation with the unitary dual $\widehat{G}$, the set of equivalence classes of irreducible unitary representations, most of which are infinite dimensional if G is non-compact.

For example, let us first consider the case where G is a connected and simply connected nilpotent Lie group. Then Kirillov ([**5**]) proved that the unitary dual $\widehat{G}$ is parametrized by $\mathfrak{g}^*/G$, the set of coadjoint orbits. The bijection

$$\widehat{G} \simeq \mathfrak{g}^*/G$$

is called the Kirillov correspondence. We shall write π_λ for the unitary representation corresponding to a coadjoint orbit $\mathcal{O}^G_\lambda := \mathrm{Ad}^*(G)\lambda \subset \mathfrak{g}^*$.

Let H be a subgroup of G. Then the restriction $\pi|_H$ is decomposed into a direct integral of irreducible unitary representations of H:

$$\pi|_H \simeq \int_{\widehat{H}}^{\oplus} m_\pi(\tau)\tau d\mu(\tau) \qquad \text{(branching law)}, \tag{3.1}$$

where $d\mu$ is a Borel measure on the unitary dual $\widehat{H}$. Then, Corwin and Greenleaf ([**1**]) proved that the multiplicity $m_\pi(\tau)$ in (3.1) coincides almost everywhere with the 'mod H' intersection number $n(\mathcal{O}^G, \mathcal{O}^H)$ defined as follows:

$$n(\mathcal{O}^G, \mathcal{O}^H) := \sharp\Big((\mathcal{O}^G \cap \mathrm{pr}^{-1}(\mathcal{O}^H))/H \Big). \tag{3.2}$$

Here, $\mathcal{O}^G \subset \mathfrak{g}^*$ and $\mathcal{O}^H \subset \mathfrak{h}^*$ are the coadjoint orbits corresponding to $\pi \in \widehat{G}$ and $\tau \in \widehat{H}$, respectively, under the Kirillov correspondence $\widehat{G} \simeq \mathfrak{g}^*/G$ and $\widehat{H} \simeq \mathfrak{h}^*/H$, and

$$\mathrm{pr} : \mathfrak{g}^* \to \mathfrak{h}^*$$

is the natural projection. The function

$$n : \mathfrak{g}^*/G \times \mathfrak{h}^*/H \to \mathbb{N} \cup \{\infty\}, \qquad (\mathcal{O}^G, \mathcal{O}^H) \mapsto n(\mathcal{O}^G, \mathcal{O}^H)$$

is sometimes referred as the **Corwin-Greenleaf multiplicity function**.

Contrary to nilpotent Lie groups, it has been observed by many specialists that the orbit method does not work very well for non-compact semisimple Lie groups (e.g. [**13**]); there is no reasonable bijection between $\widehat{G}$ and (a subset of) $\mathfrak{g}^*/G$. Thus, an analogous statement of Corwin-Greenleaf's theorem does not make sense for semisimple Lie groups G in general.

However, for a semisimple Lie group G, the orbit method still gives a fairly good approximation of the unitary dual $\widehat{G}$. For example, to an 'integral' elliptic coadjoint orbit

$$\mathcal{O}^G_\lambda := \mathrm{Ad}^*(G)\lambda \subset \mathfrak{g}^*,$$

one can associate a unitary representation, denoted by π_λ, of G. This fact was proved by Schmid and Wong as a generalization of the Borel-Weil-Bott theorem, and combined with a unitarization theorem of Zuckerman's derived functor modules due to Vogan and Wallach. Furthermore, π_λ is nonzero and irreducible for 'most' λ (see a survey paper [**6**] for a precise statement and references therein). Namely, to such a coadjoint orbit $\mathcal{O}^G_\lambda$, one can naturally attach an irreducible unitary representation $\pi_\lambda \in \widehat{G}$.

In particular, if G/K is a Hermitian symmetric space, attached to an (integral) coadjoint orbit $\mathcal{O}^G_\lambda$ such that $\lambda \in ([\mathfrak{k}, \mathfrak{k}] + \mathfrak{p})^\perp$ ($\subset \mathfrak{g}^*$), one could obtain an irreducible unitary representation π_λ which is a highest weight module of scalar type.

From now on, we shall identify $\mathfrak{g}^*$ with $\mathfrak{g}$. Then, the one dimensional subspace $([\mathfrak{k}, \mathfrak{k}] + \mathfrak{p})^\perp$ of $\mathfrak{g}^*$ is identified with the center of $\mathfrak{k}$. Therefore, the above coadjoint orbit corresponds to the adjoint orbit

$$\mathcal{O}^G_z := \mathrm{Ad}(G) \cdot z \subset \mathfrak{g},$$

going through a central element z of $\mathfrak{k}$.

Then in the spirit of the Kirillov-Kostant orbit method, Theorem 2.1 may predict that the Corwin-Greenleaf multiplicity function $n(\mathcal{O}^G_z, \mathcal{O}^H)$ is either 0 or 1 for any coadjoint orbit $\mathcal{O}^H$ in $\mathfrak{h}^*$. Since not all coadjoint orbit $\mathcal{O}^H$ "corresponds to" an irreducible unitary representation of a reductive Lie group H, this prediction might be too optimistic. However, it turns out to be true:

THEOREM A. *Let $\mathcal{O}_z^G$ be an adjoint orbit that goes through a central element z of $\mathfrak{k}$ as above. If (G,H) is a symmetric pair, then the intersection*

$$\mathcal{O}_z^G \cap \mathrm{pr}^{-1}(\mathcal{O}^H)$$

is a single H-orbit for any adjoint orbit $\mathcal{O}^H \subset \mathfrak{h}$, whenever the intersection is non-empty. In particular, the intersection is connected. Here, $\mathrm{pr} : \mathfrak{g} \to \mathfrak{h}$ *is a projection with respect to the Killing form.*

Correspondingly to Theorem 2.2 in the tensor product representation ([**7**], [**8**]), we may expect a geometric result in the (co)adjoint orbits. In order to formulate an analogous statement to Theorem A, let us define the projection by

$$\mathrm{pr} : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}, \quad (X,Y) \mapsto \frac{1}{2}(X+Y).$$

We shall write

$$\mathcal{O}_{(x,y)}^{G\times G} = \mathrm{Ad}(G \times G)(x,y) \subset \mathfrak{g} \oplus \mathfrak{g}$$

for the adjoint orbit of the direct product group $G \times G$ that goes through $(x,y) \in \mathfrak{g} \oplus \mathfrak{g}$. Then the following theorem also holds:

THEOREM B. *Let z be a central element of $\mathfrak{k}$.*

1) *The intersection*

$$\mathcal{O}_{(z,z)}^{G\times G} \cap \mathrm{pr}^{-1}(\mathcal{O}^G)$$

is a single G-orbit for any adjoint orbit $\mathcal{O}^G \subset \mathfrak{g}$, whenever the intersection is non-empty.

2) *The intersection*

$$\mathcal{O}_{(z,-z)}^{G\times G} \cap \mathrm{pr}^{-1}(\mathcal{O}^G)$$

is a single G-orbit for any adjoint orbit $\mathcal{O}^G \subset \mathfrak{g}$, whenever the intersection is non-empty.

In particular, the intersections are connected.

4. Examples and Remarks

Let us illustrate our main results (Theorems A and B) by a number of examples of lower dimensions.

First, let $G := \mathrm{SU}(2)$, and we identify $\mathfrak{g}^*$ with

$$\mathfrak{g} \simeq \mathfrak{su}(2) = \{X = \begin{pmatrix} ix_1 & ix_2 - x_3 \\ ix_2 + x_3 & -ix_1 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R}\}.$$

Then the adjoint representation $\mathrm{Ad}(g) : X \mapsto gXg^{-1}$ preserves the determinant of X, that is, $x_1^2 + x_2^2 + x_3^2$. In fact, by an easy computation, any adjoint orbit $\mathcal{O}_X^G$ is identified with a sphere

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = n^2\}$$

of radius n for some $n \in \mathbb{R}_{\geq 0}$. For each integer n, the orbit method 'attaches' an irreducible $(n+1)$-dimensional representation π_n of G to this sphere. As is well-known, the restriction of π_n to a subgroup $K := \mathrm{SO}(2)$ decomposes into a direct sum of irreducible representations:

$$\pi_n|_{\mathrm{SO}(2)} \simeq \bigoplus_{\substack{m=-n \\ m \equiv n \bmod 2}}^{n} \chi_m. \tag{4.1}$$

Here, each one dimensional representation χ_m of $\mathrm{SO}(2)$ occurs with multiplicity free.

In the corresponding orbit picture, the projection $\mathrm{pr} : \mathfrak{g}^* \to \mathfrak{h}^*$ is identified with the map:

$$\mathbb{R}^3 \to \mathbb{R}, \quad (x_1, x_2, x_3) \mapsto x_3.$$

We also note that each coadjoint orbit in $\mathfrak{h}^*$ is a singleton, say $\{m\}$, because H is abelian. Then the intersection of $\mathcal{O}_X^G$ with $\mathrm{pr}^{-1}(\{m\})$ is given by

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = n^2\} \cap \{(x_1, x_2, m) : x_1, x_2 \in \mathbb{R}\}.$$

This becomes empty if $|m| > n$, but it is a circle as in Figure 4.1 if $|m| \leq n$, which is obviously a single orbit of K. This geometry of coadjoint orbits reflects the multiplicity one property of the branching law (4.1).

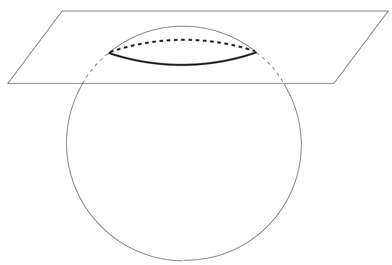

Figure 4.1

In Figure 4.2 (which does not come from any representation of SU(2)), the intersection may consist of two disconnected parts. Such a figure does not arise in the setting of our theorems.

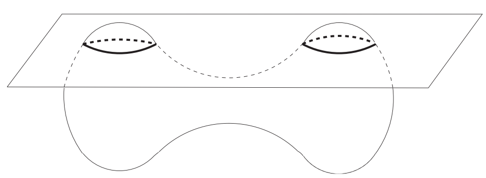

Figure 4.2

Next, let us consider infinite dimensional representations, with which our main concern is. Suppose $G := \mathrm{SL}(2, \mathbb{R})$ and $K = \mathrm{SO}(2)$. We identify $\mathfrak{g}^*$ with

$$\mathfrak{g} \simeq \mathfrak{sl}(2, \mathbb{R}) = \{ \begin{pmatrix} x_1 & x_2 - x_3 \\ x_2 + x_3 & -x_1 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R}\}.$$

A holomorphic discrete series representation π_n^+ $(n = 2, 3, 4, \cdots)$ is an irreducible unitary representation of G realized in the space of square integrable and holomorphic sections of a G-equivariant holomorphic line bundle $G \times_K \chi_n \to G/K$. We put

$$z := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which lies in the center of $\mathfrak{k} = \mathfrak{so}(2)$ (in this case, $\mathfrak{k}$ itself is the center of $\mathfrak{k}$). The representation π_n^+ is a unitary highest weight module of scalar type, and is supposed to be attached to the coadjoint orbit

$$\mathcal{O}_{nz}^G = \mathrm{Ad}^*(G)(nz).$$

We have

$$\begin{aligned} \mathcal{O}_{nz}^G =& \{X = \begin{pmatrix} x_1 & x_2 - x_3 \\ x_2 + x_3 & -x_1 \end{pmatrix} : \det X = \det \begin{pmatrix} 0 & -n \\ n & 0 \end{pmatrix},\ x_3 > 0\} \\ \simeq& \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 - x_3^2 = -n^2, x_3 > 0\}, \end{aligned} \tag{4.2}$$

a connected component of a hyperboloid of two sheets.

We put

$$y := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.3}$$

and $A := \exp \mathbb{R}y$. Let us use the identifications $\widehat{K} \simeq \mathbb{Z}$, $\chi_n \leftrightarrow n$; and $\widehat{A} \simeq \mathbb{R}$, $\chi_\xi \leftrightarrow \xi$. Then, the branching laws of $\pi_n^+ \in \widehat{G}$ with respect to the one dimensional subgroups $K \simeq \mathrm{SO}(2)$ and $A \simeq \mathbb{R}$ are given, respectively, by the following (abstract) Plancherel formulae:

$$\pi_n^+|_K \simeq \sum_{k=0}^{\infty}{}^{\oplus} \chi_{n+2k}, \tag{4.4}$$

$$\pi_n^+|_A \simeq \int_{\mathbb{R}}^{\oplus} \chi_\xi \, d\xi. \tag{4.5}$$

The first formula (4.4) is discretely decomposable, while the second one (4.5) consists only of continuous spectrum. But in both branching laws, the multiplicity is free (this is a special case of Theorem 2.1). In the corresponding orbit pictures, the intersection of the hyperboloid (4.2) with a hyperplane, $x_3 =$ constant, is a circle, which is a single orbit of $K = \mathrm{SO}(2)$ (see Figure 4.3); while that with another hyperplane, $x_1 =$ constant, is a hyperbolic curve, which is a single orbit of $A \simeq \mathbb{R}$ (see Figure 4.4).

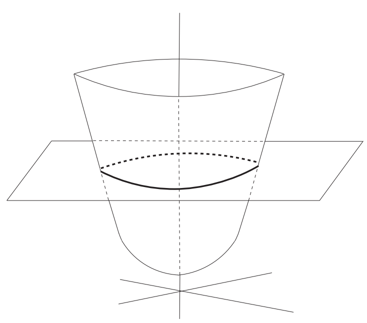

Figure 4.3

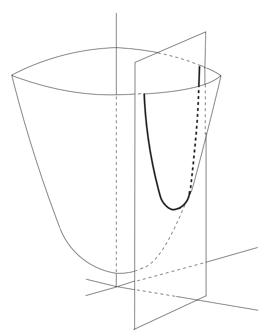

Figure 4.4

These features are exactly what Theorem A asserts in this specific setting.

Finally, let us mention other representations which are **not** treated in our main theorems. For instance, let us consider a spherical unitary principal series representation, denoted by π_λ, of $G = \mathrm{SL}(2, \mathbb{R})$, which is 'attached' to a coadjoint orbit

$$\mathcal{O}^G_{\lambda y} = \mathrm{Ad}^*(G)(\lambda y) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 - x_3^2 = \lambda^2\},$$

by a real polarization, where y is as in (4.3). The coadjoint orbit $\mathcal{O}^G_{\lambda y}$ $(\lambda \neq 0)$ is a hyperboloid of one sheet. We note that the representation π_λ is not the one treated in Theorems 2.1 and 2.2.

The branching laws of π_λ when restricted to $K = \mathrm{SO}(2)$ and $A \simeq \mathbb{R}$ are given, respectively, by the following (abstract) Plancherel formulae:

$$\pi_\lambda\big|_K \simeq \sum_{n \in \mathbb{Z}}^{\oplus} \chi_{2n}, \tag{4.6}$$

$$\pi_\lambda\big|_A \simeq \int_{\mathbb{R}}^{\oplus} 2\chi_\xi \, d\xi. \tag{4.7}$$

It happens that the first formula (4.6) is multiplicity free, and this property is reflected by the corresponding orbit picture (Figure 4.5), namely, the intersection is a circle which is a **single** orbit of $K = \mathrm{SO}(2)$. On the other hand, the multiplicity in (4.7) is **two**, and this property is reflected by the orbit picture (Figure 4.6), namely, the intersection consists of two hyperbolic curves on which A acts with two orbits.

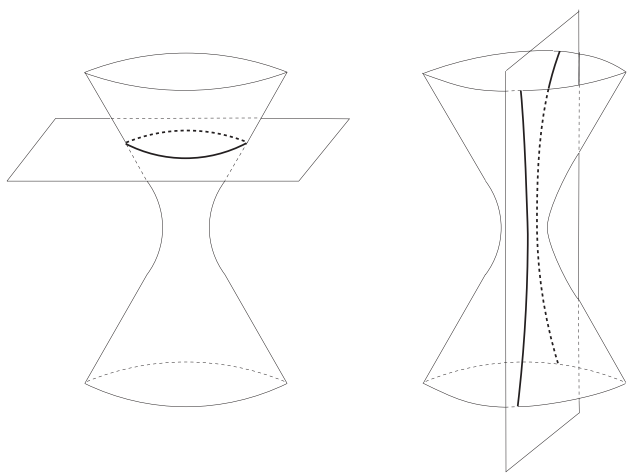

Figure 4.5 Figure 4.6

If we consider a higher dimensional generalization of the last two examples, the intersection $\mathcal{O}^G_\lambda \cap \mathrm{pr}^{-1}(\mathcal{O}^H)$ may consist of infinitely many H-orbits, that is, $n(\mathcal{O}^G_\lambda, \mathcal{O}^H)$ can be infinite for some coadjoint orbit $\mathcal{O}^H \subset \mathfrak{h}^*$. This is the case if $G = \mathrm{SL}(n, \mathbb{R})$ and $H = \mathrm{SO}(n)$ $(n \geq 3)$ and if

$$\lambda = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \in \mathfrak{sl}(n, \mathbb{R}), \quad \Big(\sum_{i=1}^n \lambda_i = 0,\ \lambda_i \neq \lambda_j\ (i \neq j)\Big).$$

In this case, the intersection $\mathcal{O}_\lambda^G \cap \mathrm{pr}^{-1}(\mathcal{O}^H)$ may have a larger dimension than $\dim H$, and consequently, may contain infinitely many H-orbits. (The orbit method attaches $\mathcal{O}_\lambda^G$ to a spherical principal series representation of G by a real polarization, namely, by a usual parabolic induction.) We note that the orbit $\mathcal{O}_\lambda^G$ in this example does not go through even $\mathfrak{k}$, to say nothing of its center. This counterexample indicates certain meaning of our assumption in Theorems A and B on the coadjoint orbit $\mathcal{O}_z^G$, that is, the assumption that $\mathcal{O}_z^G$ goes through the center of $\mathfrak{k}$.

To end this paper, we pin down some questions for further research:

1) Generalize Theorems A and B, of which a counterpart in unitary representation theory has not been known.
2) Find a feedback of (1) to unitary representation theory, namely, prove new multiplicity free results (e.g. Theorems 2.1 and 2.2) of branching laws of unitary representations which may be predicted by the orbit method.
3) Find a refinement of Theorems A and B in the orbit method corresponding to the explicit Plancherel measure (description of its support and the Plancherel density).

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