Multiplicity-free theorems of the Restrictions of Unitary Highest Weight Modules with respect to Reductive Symmetric Pairs

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Summary. The complex analytic methods have found a wide range of applications in the study of multiplicity-free representations. This article discusses, in particular, its applications to the question of restricting highest weight modules with respect to reductive symmetric pairs. We present a number of multiplicity-free branching theorems that include the multiplicity-free property of some of known results such as the Clebsh–Gordan–Pieri formula for tensor products, the Plancherel theorem for Hermitian symmetric spaces (also for line bundle cases), the Hua–Kostant–Schmid K-type formula, and the canonical representations in the sense of Vershik–Gelfand–Graev. Our method works in a uniform manner for both finite and infinite dimensional cases, for both discrete and continuous spectra, and for both classical and exceptional cases.

Key words: multiplicity-free representation, branching rule, symmetric pair, highest weight module, Hermitian symmetric space, reproducing kernel, semisimple Lie group.
Subject Classifications: Primary 22E46, Secondary 32A37, 05E15, 20G05, 53C35.

Contents

1 Introduction and statement of main results

The purpose of this article is to give a quite detailed account of the theory of multiplicity-free representations based on a non-standard method (*visible actions* on complex manifolds) through its application to branching problems. More precisely, we address the question of restricting irreducible highest weight representations π of reductive Lie groups G with respect to symmetric pairs (G, H). Then, our main goal is to give a simple and sufficient condition on the triple (G, H, π) such that the restriction $\pi|_H$ is multiplicity-free. We shall see that our method works in a uniform way for both infinite and finite dimensional representations, for both classical and exceptional cases, and for both continuous and discrete spectra.

This article is an outgrowth of the manuscript [44] which I did not publish, but which has been circulated as a preprint. From then onwards, we have extended the theory, in particular, to the following three directions:
1) the generalization of our main machinery (Theorem 2.2) to the vector bundle case ([49]),
2) the theory of 'visible actions' on complex manifolds ([50, 51, 52]),
3) 'multiplicity-free geometry' for coadjoint orbits ([53]).

We refer the reader to our paper [47] for a precise statement of the general results and an exposition of the related topics that have recently developed.

In this article, we confine ourselves to the line bundle case. On the one hand, this is sufficiently general to produce many interesting consequences, some of which are new and some others may be regarded as prototypes of various multiplicity-free branching theorems (e.g. [5, 10, 46, 54, 58, 66, 68, 81, 90, 92]). On the other hand, the line bundle case is sufficiently simple, so that we can illustrate the essence of our main ideas without going into technical details. Thus, keeping the spirit of [44], we have included here the proof of our method (Theorem 2.2), its applications to multiplicity-free theorems (Theorems A–F), and the explicit formulae (Theorems 8.3, 8.4, and 8.11), except that we referred to another paper [50] for the proof of some algebraic lemmas on the triple of involutions of Lie algebras (Lemmas 3.6 and 7.5).

1.1 Definition of multiplicity-free representations

Let us begin by recalling the concept of the multiplicity-free decomposition of a unitary representation.

Suppose H is a Lie group of type I in the sense of von Neumann algebras. Any reductive Lie group is of type I as well as any algebraic group. We denote

by $\widehat{H}$ the unitary dual of H, that is, the set of equivalence classes of irreducible unitary representations of H. The unitary dual $\widehat{H}$ is endowed with the Fell topology.

Suppose that $(\pi, \mathcal{H})$ is a unitary representation of H defined on a (second countable) Hilbert space $\mathcal{H}$. By a theorem of Mautner, π is decomposed uniquely into irreducible unitary representations of H in terms of the direct integral of Hilbert spaces:

$$\pi \simeq \int_{\widehat{H}} m_\pi(\mu)\mu \, d\sigma(\mu) \,, \tag{1.1.1}$$

where $d\sigma(\mu)$ is a Borel measure on $\widehat{H}$, and the multiplicity function $m_\pi : \widehat{H} \to \mathbb{N} \cup \{\infty\}$ is uniquely defined almost everywhere with respect to the measure $d\sigma$.

Let $\mathrm{End}(\mathcal{H})$ be the ring of continuous operators on $\mathcal{H}$, and $\mathrm{End}_H(\mathcal{H})$ the subring of H-intertwining operators, that is, the commutant of $\{\pi(g) : g \in H\}$ in $\mathrm{End}(\mathcal{H})$.

Definition 1.1. We say that the unitary representation $(\pi, \mathcal{H})$ is *multiplicity-free* if the ring $\mathrm{End}_H(\mathcal{H})$ is commutative.

It is not difficult to see that this definition is equivalent to the following property:

$$m_\pi(\mu) \leq 1 \text{ for almost all } \mu \in \widehat{H} \text{ with respect to the measure } d\sigma(\mu)$$

by Schur's lemma for unitary representations. In particular, it implies that any irreducible unitary representation μ of H occurs at most once as a sub-representation of π.

1.2 Multiplicities for inductions and restrictions

With regard to the question of finding irreducible decompositions of unitary representations, there are two fundamental settings: one is the induced representation from smaller groups (e.g. harmonic analysis on homogeneous spaces), and the other is the restriction from larger groups (e.g. tensor product representations).

To be more rigorous, suppose G is a Lie group, and H is a closed subgroup of G. The G-irreducible decomposition of the induced representation $L^2\text{-}\mathrm{Ind}_H^G \tau$ ($\tau \in \widehat{H}$) is called the *Plancherel formula*, while the H-irreducible decomposition of the restriction $\pi|_H$ ($\pi \in \widehat{G}$) is referred to as the *branching law*.

This subsection examines multiplicities in the irreducible decomposition of the induction and the restriction for reductive symmetric pairs (G, H) (see Subsection 3.1 for definition).

Let us start with the induced representation. Van den Ban [2] proved that the multiplicity in the Plancherel formula for $L^2\text{-}\mathrm{Ind}_H^G \tau$ is finite as far as

$\dim \tau < \infty$. In particular, this is the case if τ is the trivial representation **1**. Over the past several decades, the induced representation L^2-$\mathrm{Ind}_H^G \mathbf{1}$ has developed its own identity (harmonic analysis on reductive symmetric spaces G/H) as a rich and meaningful part of mathematics.

In contrast, the multiplicities of the branching law of the restriction $\pi|_H$ ($\pi \in \widehat{G}$) are usually infinite. For instance, we saw in [36] that this is the case if $(G, H) = (GL(p+q, \mathbb{R}), GL(p, \mathbb{R}) \times GL(q, \mathbb{R}))$ where $\min(p, q) \geq 2$, for any tempered representation π of G. In this article, we illuminate by Example 6.3 this wild behavior.

In light of such a wild phenomenon of branching laws for reductive symmetric pairs (G, H) with H non-compact, we proposed in [38, 40] to seek for a 'nice' class of the triple (G, H, π) in which a systematic study of the restriction $\pi|_H$ could be launched.

Finiteness of multiplicities is a natural requirement for this program. By also imposing discrete decomposability on the restriction $\pi|_H$, we established the general theory for *admissible restriction* in [38, 40, 41] and found that there exist fairly rich triples (G, H, π) that enjoy this nice property. It is noteworthy that new interesting directions of research in the framework of admissible restrictions have been recently developed by M. Duflo, D. Gross, J.-S. Huang, J.-S. Li, S.-T. Lee, H.-Y. Loke, T. Oda, P. Pandžić, G. Savin, B. Speh, J. Vargas, D. Vogan, and N. Wallach (see [45, 48] and references therein).

Multiplicity-freeness is another ideal situation, in which we may expect an especially simple and detailed study of the branching law of $\pi|_H$. Thus, we aim for principles that lead us to abundant family of multiplicity-free cases. Among them, a well-known one is the dual pair correspondence, which has given fruitful examples in infinite dimensional theory in the following setting:

  a) G is the metaplectic group, and π is the Weil representation.
  b) $H = H_1 \cdot H_2$ forms a dual pair, that is, H_1 is the commutant of H_2 in G, and vice versa.

This paper uses a new principle that generates multiplicity-free representations. The general theory discussed in Section 2 brings us to uniformly bounded multiplicity theorems (Theorems B and D) and multiplicity-free theorems (Theorems A, C, E and F) in the following setting:

  a) π is a unitary highest weight representation of G (see Subsection 1.3),
  b) (G, H) is a symmetric pair (see Subsection 1.4).

We note that we allow the case where continuous spectra occur in the branching law, and consequently, irreducible summands are not always highest weight representations.

We remark that our bounded multiplicity theorems for the restriction $\pi|_H$ (π: highest weight module) may be regarded as the counterpart of the bounded multiplicity theorem for the induction L^2-$\mathrm{Ind}_H^G \tau$ (τ: finite dimensional representation) due to van den Ban.

1.3 Unitary highest weight modules

Let us recall the basic notion of highest weight modules.

Let G be a non-compact simple Lie group, θ a Cartan involution of G, and $K := \{g \in G : \theta g = g\}$. We write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, corresponding to the Cartan involution θ.

We assume that G is of *Hermitian type*, that is, the Riemannian symmetric space G/K carries the structure of a Hermitian symmetric space, or equivalently, the center $\mathfrak{c}(\mathfrak{k})$ of $\mathfrak{k}$ is non-trivial. The classification of simple Lie algebras $\mathfrak{g}$ of Hermitian type is given as follows:

$$\mathfrak{su}(p,q)\,,\ \mathfrak{sp}(n,\mathbb{R})\,,\ \mathfrak{so}(m,2)\ (m \neq 2)\,,\ \mathfrak{e}_{6(-14)}\,,\ \mathfrak{e}_{7(-25)}\,.$$

Such a Lie algebras $\mathfrak{g}$ satisfies the rank condition:

$$\operatorname{rank} G = \operatorname{rank} K\,, \tag{1.3.1}$$

or equivalently, a Cartan subalgebra of $\mathfrak{k}$ becomes a Cartan subalgebra of $\mathfrak{g}$. By a theorem of Harish-Chandra, the rank condition (1.3.1) is equivalent to the existence of (relative) discrete series representations of G. Here, an irreducible unitary representation $(\pi, \mathcal{H})$ is called a *(relative) discrete series representation* of G if the matrix coefficient $g \mapsto (\pi(g)u, v)$ is square integrable on G (modulo its center) for any $u, v \in \mathcal{H}$.

If $\mathfrak{g}$ is a simple Lie algebra of Hermitian type, then there exists a characteristic element $Z \in \mathfrak{c}(\mathfrak{k})$ such that

$$\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_- \tag{1.3.2}$$

is the eigenspace decomposition of $\operatorname{ad}(Z)$ with eigenvalues 0, $\sqrt{-1}$ and $-\sqrt{-1}$, respectively. We note that $\dim \mathfrak{c}(\mathfrak{k}) = 1$ if $\mathfrak{g}$ is a simple Lie algebra of Hermitian type, and therefore $\mathfrak{c}(\mathfrak{k}) = \mathbb{R}Z$.

Suppose V is an irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$-module. We set

$$V^{\mathfrak{p}_+} := \{v \in V : Yv = 0 \ \text{ for any } Y \in \mathfrak{p}_+\}\,. \tag{1.3.3}$$

Since K normalizes $\mathfrak{p}_+$, $V^{\mathfrak{p}_+}$ is a K-submodule. Further, $V^{\mathfrak{p}_+}$ is either zero or an irreducible finite dimensional representation of K. We say V is a *highest weight module* if $V^{\mathfrak{p}_+} \neq \{0\}$.

Definition 1.3. Suppose π is an irreducible unitary representation of G on a Hilbert space $\mathcal{H}$. We set $\mathcal{H}_K := \{v \in \mathcal{H} : \dim_{\mathbb{C}} \mathbb{C}\text{-span}\{\pi(k)v : k \in K\} < \infty\}$. Then, $\mathcal{H}_K$ is a dense subspace of $\mathcal{H}$, on which the differential action $d\pi$ of the Lie algebra $\mathfrak{g}$ (and consequently that of its complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$) and the action of the compact subgroup K is well-defined. We say $\mathcal{H}_K$ is *the underlying $(\mathfrak{g}_{\mathbb{C}}, K)$-module of $(\pi, \mathcal{H})$*. We say $(\pi, \mathcal{H})$ is a *unitary highest weight representation* of G if $\mathcal{H}_K^{\mathfrak{p}_+} \neq \{0\}$. Then, π is *of scalar type* (or *of scalar minimal K-type*) if $\mathcal{H}_K^{\mathfrak{p}_+}$ is one dimensional; π is a *(relative) holomorphic discrete*

series representation for G if the matrix coefficient $g \mapsto (\pi(g)u, v)$ is square integrable on G modulo its center for any $u, v \in \mathcal{H}$. Lowest weight modules and anti-holomorphic discrete series representations are defined similarly with $\mathfrak{p}_+$ replaced by $\mathfrak{p}_-$.

This definition also applies to G which is not simple (see Subsection 8.1).

The classification of irreducible unitary highest weight representations was accomplished by Enright–Howe–Wallach [12] and H. Jakobsen [30] independently; see also [13]. There always exist infinitely many (relative) holomorphic discrete series representations of scalar type for any non-compact simple Lie group of Hermitian type.

1.4 Involutions on Hermitian symmetric spaces

Suppose G is a non-compact simple Lie group of Hermitian type. Let τ be an involutive automorphism of G commuting with the Cartan involution θ. We use the same letter τ to denote its differential. Then τ stabilizes $\mathfrak{k}$ and also $\mathfrak{c}(\mathfrak{k})$. Because $\tau^2 = \mathrm{id}$ and $\mathfrak{c}(\mathfrak{k}) = \mathbb{R}Z$, we have the following two possibilities:

$$\tau Z = Z, \tag{1.4.1}$$

$$\tau Z = -Z. \tag{1.4.2}$$

Geometric meanings of these conditions become clear in the context of the embedding $G^\tau/K^\tau \hookrightarrow G/K$, where $G^\tau := \{g \in G : \tau g = g\}$ and $K^\tau := G^\tau \cap K$ (see [14, 27, 28, 35]). The condition (1.4.1) implies:

1-a) τ acts **holomorphically** on the Hermitian symmetric space G/K,

1-b) $G^\tau/K^\tau \hookrightarrow G/K$ defines a complex submanifold,

whereas the condition (1.4.2) implies:

2-a) τ acts **anti-holomorphically** on G/K,

2-b) $G^\tau/K^\tau \hookrightarrow G/K$ defines a totally real submanifold.

Definition 1.4. We say the involutive automorphism τ is *of holomorphic type* if (1.4.1) is satisfied, and is of *anti-holomorphic type* if (1.4.2) is satisfied. The same terminology will be applied also to the symmetric pair (G, H) (or its Lie algebras $(\mathfrak{g}, \mathfrak{h})$) corresponding to the involution τ.

Here, we recall that (G, H) is called a *symmetric pair* corresponding to τ if H is an open subgroup of G^τ (see Subsections 3.1 and 3.2). We note that the Lie algebra $\mathfrak{h}$ of H is equal to $\mathfrak{g}^\tau := \{X \in \mathfrak{g} : \tau X = X\}$. The classification of symmetric pairs $(\mathfrak{g}, \mathfrak{g}^\tau)$ for simple Lie algebras $\mathfrak{g}$ was accomplished by M. Berger [6]. The classification of symmetric pairs $(\mathfrak{g}, \mathfrak{g}^\tau)$ of holomorphic type (respectively, of anti-holomorphic type) is regarded as a subset of Berger's list, and will be presented in Table 3.4.1 (respectively, Table 3.4.2).

1.5 Multiplicity-free restrictions — infinite dimensional case

We are ready to state our main results. Let G be a non-compact simple Lie group of Hermitian type, and (G, H) a symmetric pair.

Theorem A (multiplicity-free restriction). *If π is an irreducible unitary highest weight representation of scalar type of G, then the restriction $\pi|_H$ is multiplicity-free.*

The branching law of the restriction $\pi|_H$ may and may not contain discrete spectra in Theorem A. If (G, H) is of holomorphic type then the restriction $\pi|_H$ is discretely decomposable (i.e. there is no continuous spectrum in the branching law); see Fact 5.1. Besides, the following theorem asserts that the multiplicities are still uniformly bounded even if we drop the assumption that π is of scalar type.

Theorem B (uniformly bounded multiplicities). *We assume that the symmetric pair (G, H) is of holomorphic type. Let π be an irreducible unitary highest weight representation of G.*
1) *The restriction $\pi|_H$ splits into a discrete Hilbert sum of irreducible unitary representations of H:*

$$\pi|_H \simeq \sum_{\mu \in \widehat{H}}^{\oplus} m_\pi(\mu)\mu \,,$$

and the multiplicities are uniformly bounded:

$$C(\pi) := \sup_{\mu \in \widehat{H}} m_\pi(\mu) < \infty \,.$$

2) *$C(\pi) = 1$ if π is of scalar type.*

The second statement is a direct consequence of Theorems A and B (1). As we shall see in Section 6, such uniform boundedness theorem does not hold in general if π is not a highest weight representation (see Examples 6.2 and 6.3).

Here are multiplicity-free theorems for the decomposition of tensor products, which are parallel to Theorems A and B:

Theorem C (multiplicity-free tensor product). *Let π_1 and π_2 be irreducible unitary highest (or lowest) weight representations of scalar type. Then the tensor product $\pi_1 \widehat{\otimes} \pi_2$ is multiplicity-free as a representation of G.*

Here, $\pi_1 \widehat{\otimes} \pi_2$ stands for the tensor product representation of two unitary representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ realized on the completion $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$ of the pre-Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$. (We do not need to take the completion if at least one of $\mathcal{H}_1$ or $\mathcal{H}_2$ is finite dimensional.) Theorem C asserts that multiplicities in the direct integral of the irreducible decomposition are not greater than

one in both discrete and continuous spectra. We note that continuous spectra appear in the irreducible decomposition of the tensor product representation $\pi_1 \widehat{\otimes} \pi_2$ only if

$$\begin{cases} \pi_1 \text{ is a highest weight representation, and} \\ \pi_2 \text{ is a lowest weight representation,} \end{cases}$$

or in reverse order.

If π_1 and π_2 are simultaneously highest weight representations (or simultaneously lowest weight representations), then the tensor product $\pi_1 \widehat{\otimes} \pi_2$ decomposes discretely. Dropping the assumption of "scalar type", we have still a uniform estimate of multiplicities:

Theorem D (uniformly bounded multiplicities). *Let π_1 and π_2 be two irreducible unitary highest weight representations of G.*
1) *The tensor product $\pi_1 \widehat{\otimes} \pi_2$ splits into a discrete Hilbert sum of irreducible unitary representations of G:*

$$\pi_1 \widehat{\otimes} \pi_2 \simeq \sum_{\mu \in \widehat{G}}^{\oplus} m_{\pi_1,\pi_2}(\mu)\mu \,,$$

and the multiplicities $m_{\pi_1,\pi_2}(\mu)$ are uniformly bounded:

$$C(\pi_1, \pi_2) := \sup_{\mu \in \widehat{G}} m_{\pi_1,\pi_2}(\mu) < \infty \,.$$

2) *$C(\pi_1, \pi_2) = 1$ if both π_1 and π_2 are of scalar type.*

Remark 1.5. For classical groups, we can relate the constants $C(\pi)$ and $C(\pi_1, \pi_2)$ to the *stable constants* of branching coefficients of finite dimensional representations in the sense of F. Sato [77] by using the see-saw dual pair correspondence due to R. Howe [23].

Our machinery that gives the above multiplicity-free theorems is built on complex geometry, and we shall explicate the general theory for the line bundle case in Section 2. The key idea is to transfer properties on representations (e.g. unitarity, multiplicity-freeness) into the corresponding properties of reproducing kernels, which we analyze by geometric methods.

1.6 Multiplicity-free restrictions — finite dimensional case

Our method yields multiplicity-free theorems not only for infinite dimensional representations but also for finite dimensional representations.

This subsection presents multiplicity-free theorems that are regarded as 'finite dimensional version' of Theorems A and C. They give a unified explanation of the multiplicity-free property of previously known branching formulae

obtained by combinatorial methods such as the Littlewood–Richardson rule, Koike–Terada's Young diagrammatic methods, Littelmann's path method, minor summation formulae, etc. (see [25, 55, 62, 68, 73, 80] and references therein). They also contain some 'new' cases, for which there are, to the best of our knowledge, no explicit branching formulae in the literature.

To state the theorems, let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra, and $\mathfrak{j}$ a Cartan subalgebra. We fix a positive root system $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j})$, and write $\alpha_1, \ldots, \alpha_n$ for the simple roots. Let $\omega_1, \ldots, \omega_n$ be the corresponding fundamental weights. We denote by $\pi_\lambda \equiv \pi_\lambda^{\mathfrak{g}_{\mathbb{C}}}$ the irreducible finite dimensional representation of $\mathfrak{g}_{\mathbb{C}}$ with highest weight λ.

We say π_λ is of **pan type** if λ is a scalar multiple of some ω_i such that the nilradical of the maximal parabolic subalgebra corresponding to α_i is abelian (see Lemma 7.3.1 for equivalent definitions).

Theorem E (multiplicity-free restriction — finite dimensional case). *Let π be an arbitrary irreducible finite dimensional representation of $\mathfrak{g}_{\mathbb{C}}$ of pan type, and $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ be any symmetric pair. Then, the restriction $\pi|_{\mathfrak{h}_{\mathbb{C}}}$ is multiplicity-free.*

Theorem F (multiplicity-free tensor product — finite dimensional case). *The tensor product $\pi_1 \otimes \pi_2$ of any two irreducible finite dimensional representations π_1 and π_2 of pan type is multiplicity-free.*

Theorems E and F are the counterpart to Theorems A and C for finite dimensional representations. The main machinery of the proof is again Theorem 2.2.

Alternatively, one could verify Theorems E and F by a classical technique: finding an open orbit of a Borel subgroup. For example, Littelmann [61] and Panyushev independently classified the pair of maximal parabolic subalgebras $(\mathfrak{p}_1, \mathfrak{p}_2)$ such that the diagonal action of a Borel subgroup B of a complex simple Lie group $G_{\mathbb{C}}$ on $G_{\mathbb{C}}/P_1 \times G_{\mathbb{C}}/P_2$ has an open orbit. Here, P_1, P_2 are the corresponding maximal parabolic subgroups of $G_{\mathbb{C}}$. This gives another proof of Theorem F.

The advantage of our method is that it enables us to understand (or even to discover) the multiplicity-free property simultaneously, for both infinite and finite dimensional representations, for both continuous and discrete spectra, and for both classical and exceptional cases by the single principle. This is because our main machinery (Theorem 2.2) uses only a *local* geometric assumption (see Remark 2.3.2 (2)). Thus, we can verify it at the same time for both compact and non-compact complex manifolds, and in turn get finite and infinite dimensional results, respectively.

Once we tell a priori that a representation is multiplicity-free, we may be tempted to find explicitly its irreducible decomposition. Recently, S. Okada [68] found explicit branching laws for some classical cases that arise in Theorems E and F by using minor summation formulae, and H. Alikawa [1] for $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{e}_6, \mathfrak{f}_4)$ corresponding to Theorem E. We note that the concept of pan

type representations includes *rectangular-shaped* representations of classical groups (see [58, 68]).

There are also some few cases where $\pi_1 \otimes \pi_2$ is multiplicity-free even though neither π_1 nor π_2 is of pan type. See the recent papers [46] or [81] for the complete list of such pairs (π_1, π_2) for $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$. The method in [46] to find all such pairs is geometric and based on the 'vector bundle version' of Theorem 2.2 proved in [49], whereas the method in [81] is combinatorial and based on case-by-case argument.

We refer the reader to our papers [50, 51, 52] for some further results relevant to Theorems E and F along the same line of argument here.

1.7 SL_2 examples

We illustrate the above theorems by SL_2 examples.

Example 1.7. 1) We denote by π_n the holomorphic discrete series representation of $G = SL(2, \mathbb{R})$ with minimal K-type χ_n ($n \geq 2$), where we write χ_n for the character of $K = SO(2)$ parametrized by $n \in \mathbb{Z}$. Likewise π_{-n} denotes the anti-holomorphic discrete series representation of $SL(2, \mathbb{R})$ with minimal K-type χ_{-n} ($n \geq 2$). We note that any holomorphic discrete series of $SL(2, \mathbb{R})$ is of scalar type.

We write $\pi^{\varepsilon}_{\sqrt{-1}\nu}$ ($\varepsilon = \pm 1, \nu \in \mathbb{R}$) for the unitary principal series representations of $SL(2, \mathbb{R})$. We have a unitary equivalence $\pi^{\varepsilon}_{\sqrt{-1}\nu} \simeq \pi^{\varepsilon}_{-\sqrt{-1}\nu}$. We write χ_{ζ} for the unitary character of $SO_0(1, 1) \simeq \mathbb{R}$ parametrized by $\zeta \in \mathbb{R}$.

Let $m \geq n \geq 2$. Then, the following branching formulae hold. All of them are multiplicity-free, as is 'predicted' by Theorems A and C:

$$\pi_n|_{SO_0(1,1)} \simeq \int_{-\infty}^{\infty} \chi_{\zeta} \, d\zeta \,, \tag{1.7.1 (a)}$$

$$\pi_n|_{SO(2)} \simeq \sum_{k \in \mathbb{N}}^{\oplus} \chi_{n+2k} \,, \tag{1.7.1 (b)}$$

$$\pi_m \widehat{\otimes} \pi_{-n} \simeq \int_0^{\infty} \pi^{(-1)^{m-n}}_{\sqrt{-1}\nu} \, d\nu \quad \oplus \sum_{\substack{k \in \mathbb{N} \\ 0 \leq 2k \leq m-n-2}} \pi_{m-n-2k} \,, \tag{1.7.1 (c)}$$

$$\pi_m \widehat{\otimes} \pi_n \simeq \sum_{k \in \mathbb{N}}^{\oplus} \pi_{m+n+2k} \,. \tag{1.7.1 (d)}$$

The key assumption of our main machinery (Theorem 2.2) that leads us to Theorems A and C is illustrated by the following geometric results in this SL_2 case:

i) Given any element z in the Poincaré disk D, there exists $\varphi \in \mathbb{R}$ such that $e^{\sqrt{-1}\varphi} z = \bar{z}$. In fact, one can take $\varphi = -2 \arg z$. This is the geometry that explains the multiplicity-free property of (1.7.1) (b).

ii) Given any two elements $z, w \in D$, there exists a linear fractional transform T on D such that $T(z) = \bar{z}$ and $T(w) = \bar{w}$. This is the geometry for (1.7.1) (d).

These are examples of the geometric view point that we pursued in [50] for symmetric pairs.

2)   Here is a "finite dimensional version" of the above example. Let π_n be the irreducible $n + 1$-dimensional representation of $SU(2)$. Then we have the following branching formulae: For $m, n \in \mathbb{N}$,

$$\pi_n|_{SO(2)} \simeq \chi_n \oplus \chi_{n-2} \oplus \cdots \oplus \chi_{-n} \,, \qquad\qquad (1.7.1) \text{ (e)}$$

$$\pi_m \otimes \pi_n \simeq \pi_{n+m} \oplus \pi_{n+m-2} \oplus \cdots \oplus \pi_{|n-m|} \,. \qquad\qquad (1.7.1) \text{ (f)}$$

The formula (1.7.1) (e) corresponds to the character formula, whereas (1.7.1) (f) is known as the Clebsch–Gordan formula. The multiplicity-free property of these formulae is the simplest example of Theorems E and F.

1.8 Analysis on multiplicity-free representations

Multiplicity-free property arouses our interest in developing beautiful analysis on such representations, as we discussed in Subsection 1.6 for finite dimensional cases. This subsection picks up some recent topics about detailed analysis on multiplicity-free representations for infinite dimensional cases.

Let G be a connected, simple non-compact Lie group of Hermitian type. We begin with branching laws without continuous spectra, and then discuss branching laws with continuous spectra.

1) (Discretely decomposable case) Let (G, H) be a symmetric pair of holomorphic type. Then, any unitary highest weight representation π of G decomposes discretely when restricted to H (Fact 5.1).

1-a) Suppose now that π is a holomorphic discrete series representation. L.-K. Hua [26], B. Kostant, W. Schmid [78] and K. Johnson [32] found an explicit formula of the restriction $\pi|_K$ (K-type formula). This turns out to be multiplicity-free. Alternatively, the special case of Theorem B (2) by setting $H = K$ gives a new proof of this multiplicity-free property.

1-b) Furthermore, we consider a generalization of the Hua–Kostant–Schmid formula from compact H to noncompact H, for which Theorem B (2) still ensures that the generalization will be multiplicity-free. This generalized formula is stated in Theorem 8.3, which was originally given in [39, Theorem C]. In Section 8, we give a full account of its proof. W. Bertram and J. Hilgert [7] obtained some special cases independently, and Ben Saïd [5] studied a quantative estimate of this multiplicity-free H-type formula (see also [90, 91] for some *singular* cases).

1-c) The branching formulae of the restriction of *singular* highest weight representations π are also interesting. For instance, the restriction of the Segal–Shale–Weil representation ϖ of $Mp(n, \mathbb{R})$ with respect to $U(p, n - p)$ (more precisely, its double covering) decomposes discretely into a multiplicity-free sum of the so called *ladder representations* of $U(p, n - p)$ (e.g. [33, Introduction]). This multiplicity-free property is a special case of Howe's correspondence because $(U(p, n-p), U(1))$ forms a dual pair in $Mp(n, \mathbb{R})$, and also

is a special case of Theorem A because $(\mathfrak{sp}(n,\mathbb{R}),\mathfrak{u}(p,n-p))$ forms a symmetric pair. Explicit branching laws for most of classical cases corresponding to Theorems B (2) and D (2) (see Theorems 8.3, 8.4, 8.11) can be obtained by using the "see-saw dual pair", which we hope to report in another paper.

2) (Branching laws with continuous spectra) Suppose π_1 is a highest weight module and π_2 is a lowest weight module, and both being of scalar type.

2-a) If both π_1 and π_2 are discrete series representations in addition, then the tensor product $\pi_1 \widehat{\otimes} \pi_2$ is unitarily equivalent to the regular representation on $L^2(G/K,\chi)$, the Hilbert space of L^2-sections of the G-equivalent line bundle $G \times_K \mathbb{C}_\chi \to G/K$ associated to some unitary character χ of K (R. Howe [23], J. Repka [74]). In particular, Theorem C gives a new proof of the multiplicity-free property of the Plancherel formula for $L^2(G/K,\chi)$. Yet another proof of the multiplicity-free property of $L^2(G/K,\chi)$ was given in [47, Theorem 21] by still applying Theorem 2.2 to the *crown domain* (equivalently, the Akhiezer–Gindikin domain) of the Riemannian symmetric space G/K. The explicit decomposition of $L^2(G/K,\chi)$ was found by J. Heckman [20] and N. Shimeno [79] that generalizes the work of Harish-Chandra, S. Helgason, and S. Gindikin–F. Karpelevich for the trivial bundle case.

In contrast to Riemannian symmetric spaces, it is known that "multiplicity-free property" in the Plancherel formula fails for (non-Riemannian) symmetric spaces G/H in general (see [3, 8] for the description of the multiplicity of the most continuous series representations for G/H in terms of Weyl groups).

2-b) Similarly to the case 2-a), the restriction $\pi|_H$ for a symmetric pair (G,H) of non-holomorphic type is multiplicity-free and is decomposed into only continuous spectra if π is a holomorphic discrete series of scalar type. This case was studied by G. Ólafsson–B. Ørsted ([69]).

2-c) Theorem C applied to non-discrete series representations π_1 and π_2 (i.e. tensor products of *singular* unitary highest weight representations) provides new settings of multiplicity-free branching laws. They might be interesting from the view point of representation theory because they construct "small" representations as discrete summands. (We note that irreducible unitary representations of reductive Lie groups have not been classified even in the spherical case. See [4] for the split case.) They might be interesting also from the view point of spectral theory and harmonic analysis which is relevant to the *canonical representation* in the sense of Vershik–Gelfand–Graev. Once we know the branching law is a priori multiplicity-free, it is promising to obtain its explicit formula. Some special cases have been worked on in this direction so far, for $G = SL(2,\mathbb{R})$ by V. F. Molchanov [64]; for $G = SU(2,2)$ by B. Ørsted and G. Zhang [70]; for $G = SU(n,1)$ by G. van Dijk and S. Hille [10]; for $G = SU(p,q)$ by Y. Neretin and G. Ol'shanskiĭ [66, 67]. See also G. van Dijk–M. Pevzner [11], M. Pevzner [72] and G. Zhang [92]. Their results show that a different family of irreducible unitary representations (sometimes, spherical complementary series representations) can occur in the same branching laws and each multiplicity is not greater than one.

1.9 Organization of this article

This paper is organized as follows: In Section 2, we give a proof of an abstract multiplicity-free theorem (Theorem 2.2) in the line bundle setting. This is an extension of a theorem of Faraut–Thomas [15], whose idea may go back to Gelfand's proof [17] of the commutativity of the Hecke algebra $L^1(K\backslash G/K)$. Theorem 2.2 is a main method in this article to find various multiplicity-free theorems. In Section 3, we use Theorem 2.2 to give a proof of Theorem A. The key idea is the reduction of the geometric condition (2.2.3) (*strongly visible action* in the sense of [47]) to the existence problem of a "nice" involutive automorphism σ of G satisfying a certain rank condition. Section 4 considers the multiplicity-free theorem for the tensor product representations of two irreducible highest (or lowest) weight modules and gives a proof of Theorem C. Sections 5 and 6 examine our assumptions in our multiplicity-free theorems (Theorems A and C). That is, we drop the assumption of 'scalar type' in Section 5 and prove that multiplicities are still uniformly bounded (Theorems B and D). We note that multiplicities can be greater than one in this generality. In Section 6, we leave unchanged the assumption that (G, H) is a symmetric pair, and relax the assumption that π is a highest weight module. We illustrate by examples a wild behavior of multiplicities without this assumption. In Section 7, analogous results of Theorems A and C are proved for finite dimensional representations of compact groups. In Section 8, we present explicit branching laws that are assured a priori to be multiplicity-free by Theorems A and C. Theorem 8.4 generalizes the Hua–Kostant–Schmid formula. In Section 9 (Appendix) we present some basic results on homogeneous line bundles for the convenience of the reader, which give a sufficient condition for the assumption (2.2.2) in Theorem 2.2.

2 Main machinery from complex geometry

J. Faraut and E. Thomas [15], in the case of trivial twisting parameter, gave a sufficient condition for the commutativity of $\mathrm{End}_H(\mathcal{H})$ by using the theory of reproducing kernels, which we extend to the general, twisted case in this preliminary section. The proof parallels to theirs, except that we need just find an additional condition (2.2.2) when we formalize Theorem 2.2 in the line bundle setting.

2.1 Basic operations on holomorphic line bundles

Let $\mathcal{L} \to D$ be a holomorphic line bundle over a complex manifold D. We denote by $\mathcal{O}(\mathcal{L}) \equiv \mathcal{O}(D, \mathcal{L})$ the space of holomorphic sections of $\mathcal{L} \to D$. Then $\mathcal{O}(\mathcal{L})$ carries a Fréchet topology by the uniform convergence on compact sets. If a Lie group H acts holomorphically and equivariantly on the holomorphic

line bundle $\mathcal{L} \to D$, then H defines a (continuous) representation on $\mathcal{O}(\mathcal{L})$ by the pull-back of sections.

Let $\{U_\alpha\}$ be trivializing neighborhoods of D, and $g_{\alpha\beta} \in \mathcal{O}^\times(U_\alpha \cap U_\beta)$ the transition functions of the holomorphic line bundle $\mathcal{L} \to D$. Then an anti-holomorphic line bundle $\overline{\mathcal{L}} \to D$ is a complex line bundle with the transition functions $\overline{g_{\alpha\beta}}$. We denote by $\overline{\mathcal{O}(\mathcal{L})}$ the space of anti-holomorphic sections for $\overline{\mathcal{L}} \to D$.

Suppose σ is an anti-holomorphic diffeomorphism of D. Then the pull-back $\sigma^*\mathcal{L} \to D$ is an anti-holomorphic line bundle over D. In turn, $\overline{\sigma^*\mathcal{L}} \to D$ is a holomorphic line bundle over D (see Appendix for more details).

2.2 Abstract multiplicity-free theorem

Here is the main machinery to prove various multiplicity-free theorems of branching laws including Theorems A and C (infinite dimensional representations) and Theorems E and F (finite dimensional representations).

Theorem 2.2. *Let $(\pi, \mathcal{H})$ be a unitary representation of a Lie group H. Assume that there exist an H-equivariant holomorphic line bundle $\mathcal{L} \to D$ and an anti-holomorphic involutive diffeomorphism σ of D with the following three conditions:*
(2.2.1)   *There is an injective (continuous) H-intertwining map $\mathcal{H} \to \mathcal{O}(\mathcal{L})$.*
(2.2.2)   *There exists an isomorphism of H-equivariant holomorphic line bundles $\Psi : \mathcal{L} \xrightarrow{\sim} \overline{\sigma^*\mathcal{L}}$.*
(2.2.3)   *Given $x \in D$, there exists $g \in H$ such that $\sigma(x) = g \cdot x$.*
    Then, the ring $\mathrm{End}_H(\mathcal{H})$ of continuous H-intertwining operators on $\mathcal{H}$ is commutative. Consequently, $(\pi, \mathcal{H})$ is multiplicity-free (see Definition 1.1).

2.3 Remarks on Theorem 2.2

This subsection gives brief comments on Theorem 2.2. First, we consider a special case, and also a generalization.

Remark 2.3.1 (specialization and generalization). 1) Suppose $\mathcal{L} \to D$ is the trivial line bundle. Then, the condition (2.2.2) is automatically satisfied. In this case, Theorem 2.2 was proved in [15].
2) An extension of Theorem 2.2 to the equivariant vector bundle $\mathcal{V} \to D$ is the main subject of [49], where a more general multiplicity-free theorem is obtained under an additional condition that the isotropy representation of $H_x = \{h \in H : h \cdot x = x\}$ on the fiber $\mathcal{V}_x$ is multiplicity-free for generic $x \in D$. Obviously, the H_x-action on $\mathcal{V}_x$ is multiplicity-free for the case $\dim \mathcal{V}_x = 1$, namely, for the line bundle case.

Next, we examine the conditions (2.2.2) and (2.2.3).

Remark 2.3.2. 1) In many cases, the condition (2.2.2) is naturally satisfied. We shall explicate how to construct the bundle isomorphism Ψ in Lemma 9.4 for a Hermitian symmetric space D.

2) As the proof below shows, Theorem 2.2 still holds if we replace D by an H-invariant open subset D'. Thus, the condition (2.2.3) is **local**. The concept of '**visible action**' (see [46, 49, 51]) arises from the condition (2.2.3) on the base space D.

3) The condition (2.2.3) is automatically satisfied if H acts transitively on D. But we are interested in a more general setting where each H-orbit has a positive codimension in D. We find in Lemma 3.3 a sufficient condition for (2.2.3) in terms of rank condition for a symmetric space D.

2.4 Reproducing kernel

This subsection gives a quick summary for the reproducing kernel of a Hilbert space $\mathcal{H}$ realized in the space $\mathcal{O}(\mathcal{L})$ of holomorphic sections for a holomorphic line bundle $\mathcal{L}$ (see [49] for a generalization to the vector bundle case). Since the reproducing kernel $K_{\mathcal{H}}$ contains all the information on the Hilbert space $\mathcal{H}$, our strategy is to make use of $K_{\mathcal{H}}$ in order to prove Theorem 2.2.

Suppose that there is an injective and continuous map for a Hilbert space $\mathcal{H}$ into the Fréchet space $\mathcal{O}(\mathcal{L})$. Then, the point evaluation map

$$\mathcal{O}(\mathcal{L}) \supset \mathcal{H} \to \mathcal{L}_z \simeq \mathbb{C}, \quad f \mapsto f(z)$$

is continuous with respect to the Hilbert topology on $\mathcal{H}$.

Let $\{\varphi_\nu\}$ be an orthonormal basis of $\mathcal{H}$. We define

$$K_{\mathcal{H}}(x,y) \equiv K(x,y) := \sum_\nu \varphi_\nu(x)\overline{\varphi_\nu(y)} \in \mathcal{O}(\mathcal{L})\widehat{\otimes}\overline{\mathcal{O}}(\overline{\mathcal{L}}).$$

Then, $K(x,y)$ is well-defined as a holomorphic section of $\mathcal{L} \to D$ for the first variable, and as an anti-holomorphic section of $\overline{\mathcal{L}} \to D$ for the second variable. The definition is independent of the choice of an orthonormal basis $\{\varphi_\nu\}$. $K(x,y)$ is called the reproducing kernel of $\mathcal{H}$.

Lemma 2.4. 1) *For each* $y \in D$, $K(\cdot,y) \in \mathcal{H}\otimes\overline{\mathcal{L}_y}$ $(\simeq \mathcal{H})$ *and* $(f(\cdot), K(\cdot,y))_{\mathcal{H}} = f(y)$ *for any* $f \in \mathcal{H}$.
2) *Let* $K_i(x,y)$ *be the reproducing kernels of Hilbert spaces* $\mathcal{H}_i \subset \mathcal{O}(\mathcal{L})$ *with inner products* $(,)_{\mathcal{H}_i}$, *respectively, for* $i = 1, 2$. *If* $K_1 \equiv K_2$, *then* $\mathcal{H}_1 = \mathcal{H}_2$ *and* $(,)_{\mathcal{H}_1} = (,)_{\mathcal{H}_2}$.
3) *If* $K_1(x,x) = K_2(x,x)$ *for any* $x \in D$, *then* $K_1 \equiv K_2$.

Proof. (1) and (2) are standard. We review only the way how to recover $\mathcal{H}$ together with its inner product from a given reproducing kernel. For each $y \in D$, we fix an isomorphism $\mathcal{L}_y \simeq \mathbb{C}$. Through this isomorphism, we can regard $K(\cdot,y) \in \mathcal{H} \otimes \overline{\mathcal{L}_y}$ as an element of $\mathcal{H}$. The Hilbert space $\mathcal{H}$ is the completion of the $\mathbb{C}$-span of $\{K(\cdot,y) : y \in D\}$ with pre-Hilbert structure

$$(K(\cdot, y_1), K(\cdot, y_2))_{\mathcal{H}} := K(y_2, y_1) \in \mathcal{L}_{y_2} \otimes \overline{\mathcal{L}_{y_1}} \ (\simeq \mathbb{C}). \qquad (2.4.1)$$

This procedure is independent of the choice of the isomorphism $\mathcal{L}_y \simeq \mathbb{C}$. Hence, the Hilbert space $\mathcal{H}$ together with its inner product is recovered.

3)   We denote by $\overline{D}$ the complex manifold endowed with the conjugate complex structure on D. Then, $\overline{\mathcal{L}} \to \overline{D}$ is a holomorphic line bundle, and $K(\cdot, \cdot) \equiv K_{\mathcal{H}}(\cdot, \cdot)$ is a holomorphic section of the holomorphic line bundle $\mathcal{L} \boxtimes \overline{\mathcal{L}} \to D \times \overline{D}$. As the diagonal embedding $\iota : D \to D \times \overline{D}, z \mapsto (z, z)$ is totally real, $(K_1 - K_2)|_{\iota(D)} \equiv 0$ implies $K_1 - K_2 \equiv 0$ by the unicity theorem of holomorphic functions. $\square$

2.5 Construction of J

Suppose we are in the setting of Theorem 2.2. We define an anti-linear map

$$J : \mathcal{O}(\mathcal{L}) \to \mathcal{O}(\mathcal{L}), \quad f \mapsto Jf$$

by $Jf(z) := \overline{f(\sigma(z))}$ $(z \in D)$. Jf is regarded as an element of $\mathcal{O}(\mathcal{L})$ through the isomorphism $\Psi_* : \mathcal{O}(\mathcal{L}) \simeq \mathcal{O}(\overline{\sigma^*\mathcal{L}})$ (see (2.2.2)).

Lemma 2.5. *In the setting of Theorem 2.2, we identify $\mathcal{H}$ with a subspace of $\mathcal{O}(\mathcal{L})$. Then, the anti-linear map J is an isometry from $\mathcal{H}$ onto $\mathcal{H}$.*

Proof. We put $\widetilde{\mathcal{H}} := J(\mathcal{H})$, equipped with the inner product

$$(Jf_1, Jf_2)_{\widetilde{\mathcal{H}}} := (f_2, f_1)_{\mathcal{H}} \quad \text{for } f_1, f_2 \in \mathcal{H}. \qquad (2.5.1)$$

If $\{\varphi_\nu\}$ is an orthonormal basis of $\mathcal{H}$, then $\widetilde{\mathcal{H}}$ is a Hilbert space with orthonormal basis $\{J\varphi_\nu\}$. Hence, the reproducing kernel of $\widetilde{\mathcal{H}}$ is given by $K_{\widetilde{\mathcal{H}}}(x, y) = K_{\mathcal{H}}(\sigma(y), \sigma(x))$ because

$$K_{\widetilde{\mathcal{H}}}(x, y) = \sum_\nu J\varphi_\nu(x)\overline{J\varphi_\nu(y)} = \sum_\nu \overline{\varphi_\nu(\sigma(x))} \ \overline{\overline{\varphi_\nu(\sigma(y))}} = K_{\mathcal{H}}(\sigma(y), \sigma(x)).$$
$$(2.5.2)$$

We fix $x \in D$ and take $g \in H$ such that $\sigma(x) = g \cdot x$ (see (2.2.3)). Substituting x for y in (2.5.2), we have

$$K_{\widetilde{\mathcal{H}}}(x, x) = K_{\mathcal{H}}(\sigma(x), \sigma(x)) = K_{\mathcal{H}}(g \cdot x, g \cdot x) = K_{\mathcal{H}}(x, x).$$

Here, the last equality holds because $\{\varphi_\nu(g \cdot)\}$ is also an orthonormal basis of $\mathcal{H}$ as $(\pi, \mathcal{H})$ is a unitary representation of H. Then, by Lemma 2.4, the Hilbert space $\widetilde{\mathcal{H}}$ coincides with $\mathcal{H}$ and

$$(Jf_1, Jf_2)_{\mathcal{H}} = (f_2, f_1)_{\mathcal{H}} \quad \text{for } f_1, f_2 \in \mathcal{H}. \qquad (2.5.3)$$

This is what we wanted to prove. $\square$

2.6 Proof of $A^* = JAJ^{-1}$

Lemma 2.6 (see [15]). *Suppose $A \in \mathrm{End}_H(\mathcal{H})$. Then the adjoint operator A^* of A is given by*

$$A^* = JAJ^{-1}. \tag{2.6.1}$$

Proof. We divide the proof into three steps.

Step 1 (positive self-adjoint case): Assume $A \in \mathrm{End}_H(\mathcal{H})$ is a positive self-adjoint operator. Let $\mathcal{H}_A$ be the Hilbert completion of $\mathcal{H}$ by the pre-Hilbert structure

$$(f_1, f_2)_{\mathcal{H}_A} := (Af_1, f_2)_{\mathcal{H}} \quad \text{for } f_1, f_2 \in \mathcal{H}. \tag{2.6.2}$$

If $f_1, f_2 \in \mathcal{H}$ and $g \in H$, then

$$\begin{aligned}
(\pi(g)f_1, \pi(g)f_2)_{\mathcal{H}_A} &= (A\pi(g)f_1, \pi(g)f_2)_{\mathcal{H}} \\
&= (\pi(g)Af_1, \pi(g)f_2)_{\mathcal{H}} = (Af_1, f_2)_{\mathcal{H}} = (f_1, f_2)_{\mathcal{H}_A}.
\end{aligned}$$

Therefore, $(\pi, \mathcal{H})$ extends to a unitary representation on $\mathcal{H}_A$. Applying (2.5.3) to both $\mathcal{H}_A$ and $\mathcal{H}$, we have

$$\begin{aligned}
(Af_1, f_2)_{\mathcal{H}} &= (f_1, f_2)_{\mathcal{H}_A} = (Jf_2, Jf_1)_{\mathcal{H}_A} = (AJf_2, Jf_1)_{\mathcal{H}} \\
&= (Jf_2, A^*Jf_1)_{\mathcal{H}} = (Jf_2, JJ^{-1}A^*Jf_1)_{\mathcal{H}} = (J^{-1}A^*Jf_1, f_2)_{\mathcal{H}}.
\end{aligned}$$

Hence, $A = J^{-1}A^*J$, and (2.6.1) follows.

Step 2 (self-adjoint case): Assume $A \in \mathrm{End}_H(\mathcal{H})$ is a self-adjoint operator. Let $A = \int \lambda dE_\lambda$ be the spectral decomposition of A. Then every projection operator $E_\lambda \in \mathrm{End}(\mathcal{H})$ also commutes with $\pi(g)$ for all $g \in H$, namely, $E_\lambda \in \mathrm{End}_H(\mathcal{H})$. We define

$$A_+ := \int_{\lambda \geq 0} \lambda dE_\lambda, \qquad A_- := \int_{\lambda < 0} \lambda dE_\lambda.$$

Then $A = A_+ + A_-$. Let I be the identity operator on $\mathcal{H}$. As a positive self-adjoint operator $A_+ + I$ is an element of $\mathrm{End}_H(\mathcal{H})$, we have $(A_+ + I)^* = J(A_+ + I)J^{-1}$ by Step 1, whence $A_+^* = JA_+J^{-1}$. Applying Step 1 again to $-A_-$, we have $A_-^* = JA_-J^{-1}$. Thus,

$$A^* = A_+^* + A_-^* = JA_+J^{-1} + JA_-J^{-1} = J(A_+ + A_-)J^{-1} = JAJ^{-1}.$$

Step 3 (general case): Suppose $A \in \mathrm{End}_H(\mathcal{H})$. Then A^* also commutes with $\pi(g)$ $(g \in H)$ because π is unitary. We put $B := \frac{1}{2}(A + A^*)$ and $C := \frac{\sqrt{-1}}{2}(A^* - A)$. Then, both B and C are self-adjoint operators commuting with $\pi(g)$ $(g \in H)$. It follows from Step 2 that $B^* = JBJ^{-1}$ and $C^* = JCJ^{-1}$. As J is an anti-linear map, we have

$$(\sqrt{-1}\,C)^* = -\sqrt{-1}\,C^* = -\sqrt{-1}\,JCJ^{-1} = J(\sqrt{-1}\,C)J^{-1}.$$

Hence, $A = B + \sqrt{-1}\,C$ also satisfies $A^* = JAJ^{-1}$. $\quad \square$

2.7 Proof of Theorem 2.2

We are now ready to complete the proof of Theorem 2.2. Let $A, B \in \mathrm{End}_H(\mathcal{H})$. By Lemma 2.6, we have

$$AB = J^{-1}(AB)^*J = (J^{-1}B^*J)(J^{-1}A^*J) = BA \,.$$

Therefore, $\mathrm{End}_H(\mathcal{H})$ is commutative.   □

3 Proof of Theorem A

This section gives a proof of Theorem A by using Theorem 2.2. The core of the proof is to reduce the geometric condition (2.2.3) to an algebraic condition (the existence of a certain involution of the Lie algebra). This reduction is stated in Lemma 3.3. The reader who is familiar with symmetric pairs can skip Subsections 3.1, 3.2, 3.4 and 3.5.

3.1 Reductive symmetric pairs

Let G be a Lie group. Suppose that τ is an involutive automorphism of G. We write

$$G^\tau := \{g \in G : \tau g = g\}$$

for the fixed point subgroup of τ, and denote by G_0^τ its connected component containing the unit element. The pair (G, H) (or the pair $(\mathfrak{g}, \mathfrak{h})$ of their Lie algebras) is called a *symmetric pair* if the subgroup H is an open subgroup of G^τ, that is, if H satisfies

$$G_0^\tau \subset H \subset G^\tau \,.$$

It is called a *reductive symmetric pair* if G is a reductive Lie group; a *semisimple symmetric pair* if G is a semisimple Lie group. Obviously, a semisimple symmetric pair is a reductive symmetric pair.

We shall use the same letter τ to denote the differential of τ. We set

$$\mathfrak{g}^{\pm\tau} := \{Y \in \mathfrak{g} : \tau Y = \pm Y\} \,.$$

Then, it follows from $\tau^2 = \mathrm{id}$ that we have a direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}^\tau \oplus \mathfrak{g}^{-\tau}.$$

Suppose now that G is a semisimple Lie group. It is known that there exists a Cartan involution θ of G commuting with τ. Take such θ, and we write $K := G^\theta = \{g \in G : \theta g = g\}$. Then, K is compact if G is a linear Lie group. The direct sum decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \equiv \mathfrak{g}^\theta \oplus \mathfrak{g}^{-\theta}$$

is called a Cartan decomposition. Later, we shall allow G to be non-linear, in particular, K is not necessarily compact. The *real rank* of $\mathfrak{g}$, denoted by $\mathbb{R}$-$\operatorname{rank}\mathfrak{g}$, is defined to be the dimension of a maximal abelian subspace of $\mathfrak{g}^{-\theta}$.

As $(\tau\theta)^2 = \operatorname{id}$, the pair $(\mathfrak{g}, \mathfrak{g}^{\tau\theta})$ also forms a symmetric pair. The Lie group

$$G^{\tau\theta} = \{g \in G : (\tau\theta)(g) = g\}$$

is a reductive Lie group with Cartan involution $\theta|_{G^{\tau\theta}}$, and its Lie algebra $\mathfrak{g}^{\tau\theta}$ is reductive with Cartan decomposition

$$\mathfrak{g}^{\tau\theta} = \mathfrak{g}^{\tau\theta,\theta} \oplus \mathfrak{g}^{\tau\theta,-\theta} = \mathfrak{g}^{\tau,\theta} \oplus \mathfrak{g}^{-\tau,-\theta} . \tag{3.1.1}$$

Here, we have used the notation $\mathfrak{g}^{-\tau,-\theta}$ and alike, defined as follows:

$$\mathfrak{g}^{-\tau,-\theta} := \{Y \in \mathfrak{g} : (-\tau)Y = (-\theta)Y = Y\} .$$

Then, the dimension of a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{g}^{-\tau,-\theta}$ is equal to the real rank of $\mathfrak{g}^{\tau\theta}$, which is referred to as the *split rank* of the semisimple symmetric space G/H. We shall write $\mathbb{R}$-$\operatorname{rank}G/H$ or $\mathbb{R}$-$\operatorname{rank}\mathfrak{g}/\mathfrak{g}^{\tau}$ for this dimension. Thus,

$$\mathbb{R}\text{-}\operatorname{rank}\mathfrak{g}^{\theta\tau} = \mathbb{R}\text{-}\operatorname{rank}\mathfrak{g}/\mathfrak{g}^{\tau} . \tag{3.1.2}$$

In particular, we have $\mathbb{R}$-$\operatorname{rank}\mathfrak{g} = \mathbb{R}$-$\operatorname{rank}\mathfrak{g}/\mathfrak{k}$ if we take τ to be θ.

The Killing form on the Lie algebra $\mathfrak{g}$ is non-degenerate on $\mathfrak{g}$, and is also non-degenerate when restricted to $\mathfrak{h}$. Then, it induces an $\operatorname{Ad}(H)$-invariant non-degenerate bilinear form on $\mathfrak{g}/\mathfrak{h}$, and therefore a G-invariant pseudo-Riemannian structure on the homogeneous space G/H, so that G/H becomes a symmetric space with respect to the Levi–Civita connection and is called a *semisimple symmetric space*. In this context, the subspace $\mathfrak{a}$ has the following geometric meaning: Let $A := \exp(\mathfrak{a})$, the connected abelian subgroup of G with Lie algebra $\mathfrak{a}$. Then, the orbit $A \cdot o$ through $o := eH \in G/H$ becomes a flat, totally geodesic submanifold in G/H. Furthermore, we have a (generalized) Cartan decomposition:

Fact 3.1 (see [16, Section 2])**.** $G = KAH$.

Sketch of Proof. The direct sum decomposition of the Lie algebra

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{g}^{-\tau,-\theta} \oplus \mathfrak{g}^{\tau,-\theta}$$

lifts to a diffeomorphism:

$$\mathfrak{g}^{-\tau,-\theta} + \mathfrak{g}^{\tau,-\theta} \overset{\sim}{\to} K\backslash G, \quad (X,Y) \mapsto Ke^X e^Y .$$

Since $\exp(\mathfrak{g}^{\tau,-\theta}) \subset H$, the decomposition $G = KAH$ follows if we show

$$\operatorname{Ad}(H \cap K)\mathfrak{a} = \mathfrak{g}^{-\tau,-\theta}. \tag{3.1.3}$$

The equation (3.1.3) is well-known as the key ingredient of the original Cartan decomposition $G^{\tau\theta} = K^{\tau}AK^{\tau}$ in light of (3.1.1). $\square$

Furthermore, suppose that σ is an involutive automorphism of G such that σ, τ and θ commute with one another. We set

$$G^{\sigma,\tau} := G^\sigma \cap G^\tau = \{g \in G : \sigma g = \tau g = g\}.$$

Then $(G^\sigma, G^{\sigma,\tau})$ forms a reductive symmetric pair, because σ and τ commute. The commutativity of σ and θ implies that the automorphism $\sigma : G \to G$ stabilizes K and induces a diffeomorphism of G/K, for which we use the same letter σ.

3.2 Examples of symmetric pairs

This subsection presents some basic examples of semisimple (and therefore, reductive) symmetric pairs.

Example 3.2.1 (group manifold). Let G' be a semisimple Lie group, and $G := G' \times G'$. We define an involutive automorphism τ of G by $\tau(x,y) := (y,x)$. Then, $G^\tau = \{(g,g) : g \in G'\}$ is the diagonal subgroup, denoted by $\mathrm{diag}(G')$, which is isomorphic to G'. Thus, $(G' \times G', \mathrm{diag}(G'))$ forms a semisimple symmetric pair.

We set

$$I_{p,q} := \begin{pmatrix} 1 & & & & & & \\ & \ddots & & p & & 0 & \\ & & 1 & & & & \\ & & & -1 & & & \\ & 0 & & & -1 & q & \\ & & & & & \ddots & \\ & & & & & & -1 \end{pmatrix}$$

$$J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

Example 3.2.2. Let $G = SL(n, \mathbb{C})$, and fix p, q such that $p + q = n$. Then,

$$\tau(g) := I_{p,q}\, g^* I_{p,q} \quad (g \in G)$$

defines an involutive automorphism of G, and $G^\tau = SU(p,q)$ (the indefinite unitary group). Thus, $(SL(n, \mathbb{C}), SU(p,q))$ forms a semisimple symmetric pair.

Example 3.2.3. Let $G = SL(n, \mathbb{C})$, and $\sigma(g) := \overline{g}$. Then σ is an involutive automorphism of G, and $G^\sigma = SL(n, \mathbb{R})$. We note that σ commutes with the involution τ in the previous example, and

$$\begin{aligned} G^{\sigma,\tau} &= \{g \in SL(n, \mathbb{C}) : \overline{g} = g = I_{p,q}\, {}^t\overline{g}\, I_{p,q}\} \\ &= SO(p,q). \end{aligned}$$

Thus, $(SL(n, \mathbb{C}), SL(n, \mathbb{R}))$, $(SU(p,q), SO(p,q))$, $(SL(n, \mathbb{R}), SO(p,q))$ are examples of semisimple symmetric pairs.

Example 3.2.4. Let $G := SL(2n, \mathbb{R})$, and $\tau(g) := J\,{}^t g^{-1} J^{-1}$. Then, $G^\tau = Sp(n, \mathbb{R})$ (the real symplectic group). Thus, $(SL(2n, \mathbb{R}), Sp(n, \mathbb{R}))$ forms a semisimple symmetric pair.

3.3 Reduction of visibility to real rank condition

The following lemma gives a sufficient condition for (2.2.3). Then, it plays a key role when we apply Theorem 2.2 to the branching problem for the restriction from G to G^τ (with the notation of Theorem 2.2, $D = G/K$ and $H = G_0^\tau$). This lemma is also used in reducing 'visibility' of an action to an algebraic condition ([50, Lemma 2.2]).

Lemma 3.3. *Let σ and τ be involutive automorphisms of G. We assume that the pair (σ, τ) satisfies the following two conditions:*
(3.3.1)   *σ, τ and θ commute with one another.*
(3.3.2)   *$\mathbb{R}$-$\mathrm{rank}\,\mathfrak{g}^{\tau\theta} = \mathbb{R}$-$\mathrm{rank}\,\mathfrak{g}^{\sigma,\tau\theta}$.*
    Then for any $x \in G/K$, there exists $g \in G_0^\tau$ such that $\sigma(x) = g \cdot x$.

Proof. It follows from the condition (3.3.1) that $\theta|_{G^\sigma}$ is a Cartan involution of a reductive Lie group G^σ and that $\tau|_{G^\sigma}$ is an involutive automorphism of G^σ commuting with $\theta|_{G^\sigma}$. Take a maximal abelian subspace $\mathfrak{a}$ in

$$\mathfrak{g}^{-\theta,\sigma,\tau\theta} := \{Y \in \mathfrak{g} : (-\theta)Y = \sigma Y = \tau\theta Y = Y\}.$$

From definition, we have $\dim \mathfrak{a} = \mathbb{R}$-$\mathrm{rank}\,\mathfrak{g}^{\sigma,\tau\theta}$, which in turn equals $\mathbb{R}$-$\mathrm{rank}\,\mathfrak{g}^{\tau\theta}$ by the condition (3.3.2). This means that $\mathfrak{a}$ is also a maximal abelian subspace in

$$\mathfrak{g}^{-\theta,\tau\theta} = \{Y \in \mathfrak{g} : (-\theta)Y = \tau\theta Y = Y\}.$$

Let $A = \exp(\mathfrak{a})$. Then it follows from Fact 3.1 that we have a generalized Cartan decomposition

$$G = G_0^\tau A K. \tag{3.3.3}$$

Let $o := eK \in G/K$. Fix $x \in G/K$. Then, according to the decomposition (3.3.3), we find $h \in G_0^\tau$ and $a \in A$ such that

$$x = ha \cdot o.$$

We set $g := \sigma(h)\,h^{-1}$. We claim $g \in G_0^\tau$. In fact, by using $\sigma\tau = \tau\sigma$ and $\tau h = h$, we have

$$\tau(g) = \tau\sigma(h)\,\tau(h^{-1}) = \sigma\tau(h)\,\tau(h)^{-1} = \sigma(h)\,h^{-1} = g.$$

Hence, $g \in G^\tau$. Moreover, since the image of the continuous map

$$G_0^\tau \to G\,, \quad h \mapsto \sigma(h)\, h^{-1}$$

is connected, we have $g \in G_0^\tau$.

On the other hand, we have $\sigma(a) = a$ because $\mathfrak{a} \subset \mathfrak{g}^{-\theta,\sigma,-\tau} \subset \mathfrak{g}^\sigma$. Therefore we have

$$\sigma(x) = \sigma(h)\, \sigma(a) \cdot o = \sigma(h)\, h^{-1} ha \cdot o = g \cdot x\,,$$

proving the lemma.   □

3.4 Hermitian Symmetric Space G/K

Throughout the rest of this section, we assume that G is a simple, non-compact, Lie group of Hermitian type. We retain the notation of Subsection 1.3.

Let $G_\mathbb{C}$ be a connected complex Lie group with Lie algebra $\mathfrak{g}_\mathbb{C}$, and Q^- the maximal parabolic subgroup of $G_\mathbb{C}$ with Lie algebra $\mathfrak{k}_\mathbb{C} + \mathfrak{p}_-$. Then we have an open embedding $G/K \hookrightarrow G_\mathbb{C}/Q^-$ because $\mathfrak{g}_\mathbb{C} = \mathfrak{g} + (\mathfrak{k}_\mathbb{C} + \mathfrak{p}_-)$. Thus, a G-invariant complex structure on G/K is induced from $G_\mathbb{C}/Q^-$. (We remark that the embedding $G/K \hookrightarrow G_\mathbb{C}/Q^-$ is well-defined, even though G is not necessarily a subgroup of $G_\mathbb{C}$.)

Suppose τ is an involutive automorphism of G commuting with θ. We recall from Subsection 1.4 that we have either

$$\tau Z = Z \qquad \text{(holomorphic type)}, \tag{1.4.1}$$

or

$$\tau Z = -Z \quad \text{(anti-holomorphic type)}. \tag{1.4.2}$$

Here is the classification of semisimple symmetric pairs $(\mathfrak{g}, \mathfrak{g}^\tau)$ with $\mathfrak{g}$ simple such that the pair $(\mathfrak{g}, \mathfrak{g}^\tau)$ satisfies the condition (1.4.1) (respectively, (1.4.2)). Table 3.4.2 is equivalent to the classification of totally real symmetric spaces G^τ/K^τ of the Hermitian symmetric space G/K (see [14, 27, 28, 35]).

Table 3.4.1.

$(\mathfrak{g}, \mathfrak{g}^\tau)$ is of holomorphic type	
$\mathfrak{g}$	$\mathfrak{g}^\tau$
$\mathfrak{su}(p,q)$	$\mathfrak{s}(\mathfrak{u}(i,j) + \mathfrak{u}(p-i, q-j))$
$\mathfrak{su}(n,n)$	$\mathfrak{so}^*(2n)$
$\mathfrak{su}(n,n)$	$\mathfrak{sp}(n, \mathbb{R})$
$\mathfrak{so}^*(2n)$	$\mathfrak{so}^*(2p) + \mathfrak{so}^*(2n - 2p)$
$\mathfrak{so}^*(2n)$	$\mathfrak{u}(p, n-p)$
$\mathfrak{so}(2,n)$	$\mathfrak{so}(2,p) + \mathfrak{so}(n-p)$
$\mathfrak{so}(2, 2n)$	$\mathfrak{u}(1, n)$
$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{u}(p, n-p)$
$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{sp}(p, \mathbb{R}) + \mathfrak{sp}(n-p, \mathbb{R})$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(10) + \mathfrak{so}(2)$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}^*(10) + \mathfrak{so}(2)$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(8,2) + \mathfrak{so}(2)$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{su}(5,1) + \mathfrak{sl}(2, \mathbb{R})$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{su}(4,2) + \mathfrak{su}(2)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-78)} + \mathfrak{so}(2)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-14)} + \mathfrak{so}(2)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{so}(10,2) + \mathfrak{sl}(2, \mathbb{R})$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{so}^*(12) + \mathfrak{su}(2)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{su}(6,2)$

Table 3.4.2.

$(\mathfrak{g}, \mathfrak{g}^\tau)$ is of anti-holomorphic type	
$\mathfrak{g}$	$\mathfrak{g}^\tau$
$\mathfrak{su}(p,q)$	$\mathfrak{so}(p,q)$
$\mathfrak{su}(n,n)$	$\mathfrak{sl}(n, \mathbb{C}) + \mathbb{R}$
$\mathfrak{su}(2p, 2q)$	$\mathfrak{sp}(p, q)$
$\mathfrak{so}^*(2n)$	$\mathfrak{so}(n, \mathbb{C})$
$\mathfrak{so}^*(4n)$	$\mathfrak{su}^*(2n) + \mathbb{R}$
$\mathfrak{so}(2,n)$	$\mathfrak{so}(1,p) + \mathfrak{so}(1, n-p)$
$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{gl}(n, \mathbb{R})$
$\mathfrak{sp}(2n, \mathbb{R})$	$\mathfrak{sp}(n, \mathbb{C})$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{f}_{4(-20)}$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{sp}(2,2)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-26)} + \mathfrak{so}(1,1)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{su}^*(8)$

3.5 Holomorphic realization of highest weight representations

It is well-known that an irreducible highest weight representation π of G can be realized as a subrepresentation of the space of global holomorphic sections of an equivariant holomorphic vector bundle over the Hermitian symmetric space G/K. We supply a proof here for the convenience of the reader in a way that we shall use later.

Lemma 3.5. *Let $(\pi, \mathcal{H})$ be an irreducible unitary highest weight module. We write χ for the representation of K on $U := \mathcal{H}_K^{\mathfrak{p}_+}$ (see Definition 1.3). Let $\mathcal{L} := G \times_K U \to G/K$ be the G-equivariant holomorphic vector bundle associated to χ. Then, there is a natural injective continuous G-homomorphism $\mathcal{H} \to \mathcal{O}(\mathcal{L})$.*

Proof. Let $(\ ,\ )_{\mathcal{H}}$ be a G-invariant inner product on $\mathcal{H}$. We write $(\ ,\ )_U$ for the induced inner product on U. Then, K acts unitarily on $\mathcal{H}$, and in particular on U. We consider the map

$$G \times \mathcal{H} \times U \to \mathbb{C}, \quad (g, v, u) \mapsto (\pi(g)^{-1}v, u)_{\mathcal{H}} = (v, \pi(g)u)_{\mathcal{H}}.$$

For each fixed $g \in G$ and $v \in \mathcal{H}$, the map $U \to \mathbb{C}, \ u \mapsto (\pi(g)^{-1}v, u)_{\mathcal{H}}$ is an anti-linear functional on U. Then there exists a unique element $F_v(g) \in U$ by the Riesz representation theorem for the finite dimensional Hilbert space U such that

$$(F_v(g), u)_U = (\pi(g)^{-1}v, u)_{\mathcal{H}} \quad \text{for any } u \in U.$$

Then it is readily seen that $F_v(gk) = \chi(k)^{-1}F_v(g)$ and $F_{\pi(g')v}(g) = F_v(g'^{-1}g)$ for any $g, g' \in G$, $k \in K$ and $v \in \mathcal{H}$. As u is a smooth vector in $\mathcal{H}$, $(F_v(g), u)_U = (v, \pi(g)u)_{\mathcal{H}}$ is a C^∞-function on G. Then $F_v(g)$ is a C^∞-function on G with value in U for each fixed $v \in \mathcal{H}$. Thus, we have a non-zero G-intertwining operator given by

$$F : \mathcal{H} \to C^\infty(G \times_K U), \quad v \mapsto F_v.$$

As U is annihilated by $\mathfrak{p}_+$, F_v is a holomorphic section of the holomorphic vector bundle $G \times_K U \to G/K$, that is, $F_v \in \mathcal{O}(G \times_K U)$. Then, the non-zero map $F : \mathcal{H} \to \mathcal{O}(G \times_K U)$ is injective because $\mathcal{H}$ is irreducible. Furthermore, F is continuous by the closed graph theorem. Hence, Lemma 3.6 is proved. $\square$

3.6 Reduction to real rank condition

The next Lemma is a stepping-stone to Theorem A. It becomes also a key lemma to the theorem that the action of a subgroup H on the bounded symmetric domain G/K is 'strongly visible' for any symmetric pair (G, H) (see [50]).

Lemma 3.6. *Suppose $\mathfrak{g}$ is a real simple Lie algebra of Hermitian type. Let τ be an involutive automorphism of $\mathfrak{g}$, commuting with a fixed Cartan involution θ. Then there exists an involutive automorphism σ of $\mathfrak{g}$ satisfying the following three conditions:*
(3.6.1)   *σ, τ and θ commute with one another.*
(3.6.2)   *$\mathbb{R}$-$\mathrm{rank}\,\mathfrak{g}^{\tau\theta} = \mathbb{R}$-$\mathrm{rank}\,\mathfrak{g}^{\sigma,\tau\theta}$.*
(3.6.3)   *$\sigma Z = -Z$.*

Proof. We shall give a proof in the special case $\tau = \theta$ in Subsection 4.1. For the general case, see [50, Lemma 3.1] or [44, Lemma 5.1].   □

3.7 Proof of Theorem A

Now, we are ready to complete the proof of Theorem A.

Without loss of generality, we may and do assume that G is simply connected. Let $(\pi, \mathcal{H})$ be an irreducible unitary highest weight representation of scalar type. We define a holomorphic line bundle by $\mathcal{L} := G \times_K \mathcal{H}_K^{\mathfrak{p}_+}$ over the Hermitian symmetric space $D := G/K$. Then, it follows from Lemma 3.5 that there is an injective continuous G-intertwining map $\mathcal{H} \to \mathcal{O}(\mathcal{L})$.

Suppose (G, H) is a symmetric pair. We first note that for an involutive automorphism τ of G, there exists $g \in G$ such that $\tau^g\theta = \theta\tau^g$ if we set

$$\tau^g(x) := g\tau(g^{-1}xg)g^{-1}$$

for $x \in G$. Then, $G^{\tau_g} = gHg^{-1}$ is θ-stable. Since the multiplicity-free property of the restriction $\pi|_H$ is unchanged if we replace H by gHg^{-1}, we may and do assume that $\theta H = H$, in other words, $\theta\tau = \tau\theta$.

Now, by applying Lemma 3.6, we can take σ satisfying (3.6.1), (3.6.2) and (3.6.3). We use the same letter σ to denote its lift to G. It follows from (3.6.3) that the induced involutive diffeomorphism $\sigma : G/K \to G/K$ is anti-holomorphic (see Subsection 1.4). In light of the conditions (3.6.1) and (3.6.2), we can apply Lemma 3.3 to see that for any $x \in D$ there exists $g \in H$ such that $\sigma(x) = g \cdot x$.

Moreover, by using Lemma 9.4 in the Appendix, we have an isomorphism $\overline{\sigma^*\mathcal{L}} \simeq \mathcal{L}$ as G-equivariant holomorphic line bundles over G/K. Therefore, all the assumptions of Theorem 2.2 are satisfied. Thus, we conclude that the restriction $\pi|_H$ is multiplicity-free by Theorem 2.2.   □

4 Proof of Theorem C

In this section we give a proof of Theorem C.

Throughout this section, we may and do assume that G is simply connected so that any automorphism of $\mathfrak{g}$ lifts to G. We divide the proof of Theorem C into the following cases:

Case I. Both π_1 and π_2 are highest weight modules.
Case I′. Both π_1 and π_2 are lowest weight modules.
Case II. π_1 is a highest weight module, and π_2 is a lowest weight module.
Case II′. π_1 is a lowest weight module, and π_2 is a highest weight module.

4.1 Reduction to real rank condition

The following lemma is a special case of Lemma 3.6 with $\tau = \theta$. We shall see that Theorem C in Case I (likewise, Case I′) reduces to this algebraic result.

Lemma 4.1.1. *Suppose $\mathfrak{g}$ is a real simple Lie algebra of Hermitian type. Let θ be a Cartan involution. Then there exists an involutive automorphism σ of $\mathfrak{g}$ satisfying the following three conditions:*
(4.1.1) σ and θ commute.
(4.1.2) $\mathbb{R}$-rank $\mathfrak{g} = \mathbb{R}$-rank $\mathfrak{g}^\sigma$.
(4.1.3) $\sigma Z = -Z$.

Proof. We give a proof of the Lemma based on the classification of simple Lie algebras $\mathfrak{g}$ of Hermitian type.

We recall that for any involutive automorphism σ of G, there exists $g \in G$ such that $\sigma^g \theta = \theta \sigma^g$. Thus, (4.1.1) is always satisfied after replacing σ by some σ^g. The remaining conditions (4.1.2) and (4.1.3) (cf. Table 3.4.2) are satisfied if we choose $\sigma \in \mathrm{Aut}(G)$ in the following Table 4.1.2 for each simple non-compact Lie group G of Hermitian type:

Table 4.1.2.

$(\mathfrak{g}, \mathfrak{g}^\sigma)$ satisfying (4.1.2) and (4.1.3)		
$\mathfrak{g}$	$\mathfrak{g}^\sigma$	$\mathbb{R}$-rank $\mathfrak{g} = \mathbb{R}$-rank $\mathfrak{g}^\sigma$
$\mathfrak{su}(p, q)$	$\mathfrak{so}(p, q)$	$\min(p, q)$
$\mathfrak{so}^*(2n)$	$\mathfrak{so}(n, \mathbb{C})$	$[\frac{1}{2}n]$
$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{gl}(n, \mathbb{R})$	n
$\mathfrak{so}(2, n)$	$\mathfrak{so}(1, n-1) + \mathfrak{so}(1, 1)$	$\min(2, n)$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{sp}(2, 2)$	2
$\mathfrak{e}_{7(-25)}$	$\mathfrak{su}^*(8)$	3

Here, we have proved Lemma.   □

Remark 4.1.3. The choice of σ in Lemma 4.1.1 is not unique. For example, we may choose $\mathfrak{g}^\sigma \simeq \mathfrak{e}_{6(-26)} \oplus \mathbb{R}$ instead of the above choice $\mathfrak{g}^\sigma \simeq \mathfrak{su}^*(8)$ for $\mathfrak{g} = \mathfrak{e}_{7(-25)}$.

4.2 Proof of Theorem C in Case I

Let G be a non-compact simply-connected, simple Lie group such that G/K is a Hermitian symmetric space.

Let $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ be two irreducible unitary highest weight representations of scalar type. By Lemma 3.5, we can realize $(\pi_i, \mathcal{H}_i)$ in the space $\mathcal{O}(\mathcal{L}_i)$ of holomorphic sections of the holomorphic line bundle $\mathcal{L}_i := G \times_K (\mathcal{H}_i)_K^{\mathfrak{p}_+}$ $(i = 1, 2)$ over the Hermitian symmetric space G/K. We now define a holomorphic line bundle $\mathcal{L} := \mathcal{L}_1 \boxtimes \mathcal{L}_2$ over $D := G/K \times G/K$ as the outer tensor product of $\mathcal{L}_1$ and $\mathcal{L}_2$. Then, we have naturally an injective continuous $(G \times G)$-intertwining map $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2 \to \mathcal{O}(\mathcal{L})$.

Let us take an involution σ' of $\mathfrak{g}$ as in Lemma 4.1.1 (but we use the letter σ' instead of σ), and lift it to G. We set $\sigma := \sigma' \times \sigma'$. Then it follows from (4.1.3) that σ' acts anti-holomorphically on G/K, and so does σ on D. Furthermore, we have isomorphisms of holomorphic line bundles $\overline{(\sigma')^* \mathcal{L}_i} \simeq \mathcal{L}_i$ $(i = 1, 2)$ by Lemma 9.4 and thus $\overline{\sigma^* \mathcal{L}} \simeq \mathcal{L}$.

We now introduce another involutive automorphism τ of $G \times G$ by $\tau(g_1, g_2) := (g_2, g_1)$. Then $(G \times G)^\tau = \mathrm{diag}(G) := \{(g, g) : g \in G\}$. We shall use the same letter θ to denote the Cartan involution $\theta \times \theta$ on $G \times G$ (and $\theta \oplus \theta$ on $\mathfrak{g} \oplus \mathfrak{g}$). Then, we observe the following isomorphisms:

$$(\mathfrak{g} \oplus \mathfrak{g})^{\tau\theta} = \{(X, \theta X) : X \in \mathfrak{g}\} \quad \simeq \mathfrak{g},$$
$$(\mathfrak{g} \oplus \mathfrak{g})^{\sigma,\tau\theta} = \{(X, \theta X) : X \in \mathfrak{g}^{\sigma'}\} \simeq \mathfrak{g}^{\sigma'}.$$

Thus, the condition (4.1.2) implies

$$\mathbb{R}\text{-}\mathrm{rank}(\mathfrak{g} \oplus \mathfrak{g})^{\tau\theta} = \mathbb{R}\text{-}\mathrm{rank}(\mathfrak{g} \oplus \mathfrak{g})^{\sigma,\tau\theta}.$$

Therefore, given $(x_1, x_2) \in D \simeq (G \times G)/(K \times K)$, there exists $(g, g) \in (G \times G)^\tau$ satisfying $(g \cdot x_1, g \cdot x_2) = (\sigma'(x_1), \sigma'(x_2))$ $(= \sigma(x_1, x_2))$ by Lemma 3.3.

Let us apply Theorem 2.2 to the setting $(\mathcal{L} \to D, \mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2, \mathrm{diag}(G), \sigma)$. Now that all the assumptions of Theorem 2.2 are satisfied, we conclude that the tensor product $\pi_1 \widehat{\otimes} \pi_2$ is multiplicity-free as a G-module, that is, Theorem C holds in the case I. $\square$

4.3 Proof of Theorem C in Case II

Let us give a proof of Theorem C in the case II. We use the same τ as in Subsection 4.2, that is, $\tau(g_1, g_2) := (g_2, g_1)$ and define a new involution σ by $\sigma := \tau\theta$, that is, $\sigma(g_1, g_2) = (\theta g_2, \theta g_1)$ for $g_1, g_2 \in G$. Obviously, σ, τ and the Cartan involution θ of $G \times G$ all commute.

We write M for the Hermitian symmetric space G/K, and $\overline{M}$ for the conjugate complex manifold. Then σ acts anti-holomorphically on $D := M \times \overline{M}$ because so does τ and because θ acts holomorphically.

By the obvious identity $(\mathfrak{g} \oplus \mathfrak{g})^{\tau\theta} = (\mathfrak{g} \oplus \mathfrak{g})^{\sigma,\tau\theta}$, we have $\mathbb{R}\text{-}\mathrm{rank}(\mathfrak{g} \oplus \mathfrak{g})^{\tau\theta} = \mathbb{R}\text{-}\mathrm{rank}(\mathfrak{g} \oplus \mathfrak{g})^{\sigma,\tau\theta}$ $(= \mathbb{R}\text{-}\mathrm{rank}\,\mathfrak{g})$. Therefore, it follows from Lemma 3.3 that

for any $(x_1, x_2) \in D$ there exists $(g, g) \in (G \times G)^\tau$ such that $\sigma(x_1, x_2) = (g, g) \cdot (x_1, x_2)$.

Suppose π_1 (respectively, π_2) is a unitary highest (respectively, lowest) weight representation of scalar type. We set $\mathcal{L}_1 := G \times_K (\mathcal{H}_1)_K^{\mathfrak{p}_+}$ and $\mathcal{L}_2 := G \times_K (\mathcal{H}_2)_K^{\mathfrak{p}_-}$. Then, $\mathcal{L}_1 \to M$ and $\mathcal{L}_2 \to \overline{M}$ are both holomorphic line bundles, and we can realize π_1 in $\mathcal{O}(M, \mathcal{L}_1)$, and π_2 in $\mathcal{O}(\overline{M}, \mathcal{L}_2)$, respectively. Therefore, the outer tensor product $\pi_1 \boxtimes \pi_2$ is realized in a subspace of holomorphic sections of the holomorphic line bundle $\mathcal{L} := \mathcal{L}_1 \boxtimes \mathcal{L}_2$ over $D = M \times \overline{M}$.

Now, we apply Theorem 2.2 to $(\mathcal{L} \to D, \mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2, \mathrm{diag}(G), \sigma)$. The condition (2.2.2) holds by Lemma 9.4. Hence, all the assumptions of Theorem 2.2 are satisfied, and therefore, Theorem C holds in the case II. □

Hence, Theorem C has been proved.

5 Uniformly bounded multiplicities — Proof of Theorems B and D

This section gives the proof of Theorems B and D. Since the proof of Theorem B parallels to that of Theorem D, we deal mostly with Theorem D here. Without loss of generality, we assume G is a non-compact simple Lie group of Hermitian type.

5.1 General theory of restriction

A unitary representation $(\pi, \mathcal{H})$ of a group L is *discretely decomposable* if π is unitarily equivalent to the discrete Hilbert sum of irreducible unitary representations of L:

$$\pi \simeq \sum_{\mu \in \widehat{L}}^{\oplus} m_\pi(\mu) \mu \,.$$

Furthermore, we say π is *L-admissible* ([38]) if all the multiplicities $m_\pi(\mu)$ are finite. In this definition, we do not require $m_\pi(\mu)$ to be uniformly bounded with respect to μ.

Suppose L' is a subgroup of L. Then, the restriction of π to L' is regarded as a unitary representation of L'. If π is L'-admissible, then π is L-admissible ([38, Theorem 1.2]).

We start with recalling from [42] a discrete decomposability theorem of branching laws in the following settings:

Fact 5.1. 1) *Suppose τ is of holomorphic type (see Definition 1.4) and set $H := G_0^\tau$. If π is an irreducible unitary highest weight representation of G, then π is $(H \cap K)$-admissible. In particular, π is H-admissible. The restriction $\pi|_H$ splits into a discrete Hilbert sum of irreducible unitary highest weight representations of H:*

$$\pi|_H \simeq \sum_{\mu \in \widehat{H}}^{\oplus} m_\pi(\mu)\mu \qquad (discrete\ Hilbert\ sum), \qquad (5.1.1)$$

where the multiplicity $m_\pi(\mu)$ is finite for every μ.

2) Let π_1, π_2 be two irreducible unitary highest weight representations of G. Then the tensor product $\pi_1 \widehat{\otimes} \pi_2$ is K-admissible under the diagonal action. Furthermore, $\pi_1 \widehat{\otimes} \pi_2$ splits into a discrete Hilbert sum of irreducible unitary highest weight representations of G, each occurring with finite multiplicity. Furthermore, if at least one of π_1 or π_2 is a holomorphic discrete series representation for G, then any irreducible summand is a holomorphic discrete series representation.

Proof. See [42, Theorem 7.4] for the proof. The main idea of the proof is taking normal derivatives of holomorphic sections, which goes back to S. Martens [63]. The same idea was also employed in a number of papers including Lipsman ([60, Theorem 4.2]) and Jakobsen–Vergne ([31, Corollary 2.3]).   □

Remark 5.1. Fact 5.1 (1) holds more generally for a closed subgroup H satisfying the following two conditions:

1) H is θ-stable.

2) The Lie algebra $\mathfrak{h}$ of H contains Z.

Here, we recall that Z is the generator of the center of $\mathfrak{k}$. The proof is essentially the same as that of Fact 5.1 (1).

Theorem B (2) follows from Theorem A and Fact 5.1 (1). Likewise, Theorem D (2) follows from Theorem C and Fact 5.1 (2). What remains to show for Theorems B and D is the uniform boundedness of multiplicities.

5.2 Remarks on Fact 5.1

Some remarks on Fact 5.1 are in order.

Remark 5.2.1. A Cartan involution θ is clearly of holomorphic type because $\theta Z = Z$. If $\theta = \tau$ then $H = K$ and any irreducible summand μ is finite dimensional. In this case, the finiteness of $m_\pi(\mu)$ in Fact 5.1 (1) is a special case of Harish-Chandra's admissibility theorem (this holds for any irreducible unitary representation π of G).

Remark 5.2.2. Fact 5.1 asserts in particular that there is no continuous spectrum in the irreducible decomposition formula. The crucial assumption for this is that (G, H) is of holomorphic type. In contrast, the restriction $\pi|_H$ is not discretely decomposable if (G, H) is of anti-holomorphic type and if π is a holomorphic discrete series representation of G ([38, Theorem 5.3]). In this setting, R. Howe, J. Repka, G. Ólafsson, B. Ørsted, van Dijk, S. Hille, M. Pevzner, V. Molchanov, Y. Neretin, G. Zhang and others studied irreducible decompositions of the restriction $\pi|_H$ by means of the L^2-harmonic analysis

on Riemannian symmetric spaces $H/H \cap K$ ([9, 10, 11, 23, 64, 66, 69, 70, 74]). The key idea in Howe and Repka [23, 74] is that a holomorphic function on G/K is uniquely determined by its restriction to the totally real submanifold $H/H \cap K$ (essentially, the unicity theorem of holomorphic functions), and that any function on $H/H \cap K$ can be approximated (in a sense) by holomorphic functions on G/K (essentially, the Weierstrass polynomial approximation theorem).

Remark 5.2.3. A finite multiplicity theorem of the branching law (5.1.1) with respect to semisimple symmetric pairs (G, H) holds for more general π (i.e. π is not a highest weight module), under the assumption that π is discretely decomposable as an $(\mathfrak{h}_\mathbb{C}, H \cap K)$-module (see [41, Corollary 4.3], [45]). However, the multiplicity of the branching law can be infinite if the restriction is not discretely decomposable (see Example 6.3).

Remark 5.2.4. Theorems B and D assert that multiplicities $m_\pi(\mu)$ in Fact 5.1 are **uniformly bounded** when we vary μ. This is a distinguished feature for the restriction of highest weight representations π. A similar statement may fail if π is not a highest weight module (see Example 6.2).

5.3 Reduction to the scalar type case

In order to deduce Theorem D (1) from Theorem D (2), we use the idea of 'coherent family' of representations of reductive Lie groups (for example, see [85]). For this, we prepare the following Lemma 5.3 and Proposition 5.4.1.

Lemma 5.3. *Suppose that $(\pi, \mathcal{H})$ is an irreducible unitary highest weight representation of G. Then there exist an irreducible unitary highest weight representation π' of scalar type and a finite dimensional representation F of G such that the underlying $(\mathfrak{g}_\mathbb{C}, K)$-module π_K occurs as a subquotient of the tensor product $\pi'_K \otimes F$.*

Proof. Without loss of generality, we may and do assume that G is simply connected. Since G is a simple Lie group of Hermitian type, the center $\mathfrak{c}(\mathfrak{k})$ of $\mathfrak{k}$ is one dimensional. We take its generator Z as in Subsection 1.4, and write C for the connected subgroup with Lie algebra $\mathfrak{c}(\mathfrak{k})$. Then, K is isomorphic to the direct product group of C and a semisimple group K'.

As $(\pi, \mathcal{H})$ is an irreducible unitary highest weight representation of G, $\mathcal{H}_K^{\mathfrak{p}^+}$ is an irreducible (finite dimensional) unitary representation of K. The K-module $\mathcal{H}_K^{\mathfrak{p}^+}$ has an expression $\sigma \otimes \chi_0$, where $\sigma \in \widehat{K}$ such that $\sigma|_C$ is trivial and χ_0 is a unitary character of K.

Let χ' be a unitary character of K such that χ' is trivial on the center Z_G of G (namely, χ' is well-defined as a representation of $\mathrm{Ad}_G(K) \simeq K/Z_G$). For later purposes, we take χ' such that $-\sqrt{-1}\,d\chi'(Z) \gg 0$. There exists an irreducible finite dimensional representation F of G such that $F^{\mathfrak{p}^+} \simeq \sigma \otimes \chi'$

as K-modules because $\sigma \otimes \chi'$ is well-defined as an algebraic representation of $\mathrm{Ad}_G(K)$.

We set $\chi := \chi_0 \otimes (\chi')^*$ of K. Because $-\sqrt{-1}\, d\chi(Z) \ll 0$, the irreducible highest weight $(\mathfrak{g}_{\mathbb{C}}, K)$-module V' such that $(V')^{\mathfrak{p}+} \simeq \chi$ is unitarizable. Let $(\pi', \mathcal{H}')$ denote the irreducible unitary representation of G whose underlying $(\mathfrak{g}_{\mathbb{C}}, K)$-module $\mathcal{H}'_K$ is isomorphic to V'. Since $\mathcal{H}'_K$ is an irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$-module, $\mathcal{H}'_K \otimes F$ is a $(\mathfrak{g}_{\mathbb{C}}, K)$-module of finite length. Furthermore, as $\mathcal{H}'_K$ is a highest weight module, so are all subquotient modules of $\mathcal{H}'_K \otimes F$. Then, $\mathcal{H}_K$ arises as a subquotient of $\mathcal{H}'_K \otimes F$ because the K-module $\mathcal{H}_K^{\mathfrak{p}+}$ occurs as a subrepresentation of $(\mathcal{H}'_K \otimes F)^{\mathfrak{p}+}$ in view of

$$\mathcal{H}_K^{\mathfrak{p}+} \simeq \sigma \otimes \chi_0 \simeq \chi \otimes (\sigma \otimes \chi') \simeq (\mathcal{H}'_K)^{\mathfrak{p}+} \otimes F^{\mathfrak{p}+} \subset (\mathcal{H}'_K \otimes F)^{\mathfrak{p}+} \,.$$

Hence, we have shown Lemma 5.3.   □

5.4 Uniform estimate of multiplicities for tensor products

Let (π, X) be a $(\mathfrak{g}_{\mathbb{C}}, K)$-module of finite length. This means that π admits a chain of submodules

$$0 = Y_0 \subset Y_1 \subset \cdots \subset Y_N = X \tag{5.4.1}$$

such that Y_i/Y_{i-1} is irreducible for $i = 1, \ldots, N$. The number N is independent of the choice of the chain (5.4.1), and we will write

$$m(\pi) := N \,.$$

That is, $m(\pi)$ is the number of irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$-modules (counted with multiplicity) occurring as subquotients in π. Here is a uniform estimate of $m(\pi)$ under the operation of tensor products:

Proposition 5.4.1. *Let F be a finite dimensional representation of a real reductive connected Lie group G. Then there exists a constant $C \equiv C(F)$ such that*

$$m(\pi \otimes F) \leq C$$

for any irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$-module π.

Before entering the proof, we fix some terminologies:

Definition 5.4.2. We write $\mathcal{F}(\mathfrak{g}_{\mathbb{C}}, K)$ for the category of $(\mathfrak{g}_{\mathbb{C}}, K)$-modules of finite length. The *Grothendieck group* $\mathcal{V}(\mathfrak{g}_{\mathbb{C}}, K)$ of $\mathcal{F}(\mathfrak{g}_{\mathbb{C}}, K)$ is the abelian group generated by $(\mathfrak{g}_{\mathbb{C}}, K)$-modules of finite length, modulo the equivalence relations

$$X \sim Y + Z$$

whenever there is a short exact sequence

$$0 \to Y \to X \to Z \to 0$$

of $(\mathfrak{g}_{\mathbb{C}}, K)$-modules. Then

$$m : \mathcal{F}(\mathfrak{g}_{\mathbb{C}}, K) \to \mathbb{N}$$

induces a group homomorphism of abelian groups:

$$m : \mathcal{V}(\mathfrak{g}_{\mathbb{C}}, K) \to \mathbb{Z} .$$

The Grothendieck group $\mathcal{V}(\mathfrak{g}_{\mathbb{C}}, K)$ is isomorphic to the free abelian group having irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$-modules as its set of finite generators.

Suppose (π, X) is a $(\mathfrak{g}_{\mathbb{C}}, K)$-module of finite length. Then, in the Grothendieck group $\mathcal{V}(\mathfrak{g}_{\mathbb{C}}, K)$, we have the relation

$$X = \bigoplus_Y m_\pi(Y) Y , \tag{5.4.2}$$

where the sum is taken over irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$-modules. Then we have

$$m(\pi) = \sum_Y m_\pi(Y) . \tag{5.4.3}$$

Suppose (π', X') is also a $(\mathfrak{g}_{\mathbb{C}}, K)$-modules of finite length. We set

$$[\pi : \pi'] := \dim \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}(\bigoplus_Y m_\pi(Y) Y, \bigoplus_Y m_{\pi'}(Y) Y) \tag{5.4.4}$$

$$= \sum_Y m_\pi(Y) m_{\pi'}(Y) . \tag{5.4.5}$$

The definition (5.4.4) makes sense in a more general setting where one of X or X' is not of finite length. To be more precise, we recall from [41, Definition 1.1]:

Definition 5.4.3. Let $\mathcal{A}(\mathfrak{g}_{\mathbb{C}}, K)$ be the category of $(\mathfrak{g}_{\mathbb{C}}, K)$-modules (π, X) having the following properties:
1) (K-admissibility) $\dim \mathrm{Hom}_K(\tau, \pi) < \infty$ for any $\tau \in \widehat{K}$.
2) (discretely decomposability, see [41, Definition 1.1]) X admits an increasing filtration

$$0 = Y_0 \subset Y_1 \subset Y_2 \subset \cdots$$

of $\mathfrak{g}_{\mathbb{C}}$-modules such that Y_i / Y_{i-1} is of finite length and that $X = \bigcup_{i=1}^{\infty} Y_i$.

We refer the reader to [41] for algebraic results on discretely decomposable $(\mathfrak{g}_{\mathbb{C}}, K)$-modules such as:

Lemma 5.4.4. *Suppose $X \in \mathcal{A}(\mathfrak{g}_{\mathbb{C}}, K)$.*
1) *Any submodule or quotient of X is an object of $\mathcal{A}(\mathfrak{g}_{\mathbb{C}}, K)$.*
2) *The tensor product $X \otimes F$ is also an object of $\mathcal{A}(\mathfrak{g}_{\mathbb{C}}, K)$ for any finite dimensional $(\mathfrak{g}_{\mathbb{C}}, K)$-module.*

For $X \in \mathcal{A}(\mathfrak{g}_{\mathbb{C}}, K)$, we can take the filtration $\{Y_i\}$ such that Y_i/Y_{i-1} is irreducible as a $(\mathfrak{g}_{\mathbb{C}}, K)$-module for any i. Then, for any irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$-module,

$$\#\{i : Y_i/Y_{i-1} \text{ is isomorphic to } Y\}$$

is finite and independent of the filtration, which we will denote by $m_\pi(Y)$.

Definition 5.4.5. Suppose $X \in \mathcal{A}(\mathfrak{g}_{\mathbb{C}}, K)$. We say the $(\mathfrak{g}_{\mathbb{C}}, K)$-module X is *multiplicity-free* if

$$m_\pi(Y) \leq 1 \quad \text{for any irreducible } (\mathfrak{g}_{\mathbb{C}}, K)\text{-module } Y.$$

This concept coincides with Definition 1.1 if X is the underlying $(\mathfrak{g}_{\mathbb{C}}, K)$-module of a unitary representation of G. The point of Definition 5.4.5 is that we allow the case where X is not unitarizable.

Generalizing (5.4.5), we set

$$[\pi : \pi'] := \sum_Y m_\pi(Y) m_{\pi'}(Y)$$

for $\pi, \pi' \in \mathcal{A}(\mathfrak{g}_{\mathbb{C}}, K)$. Here are immediate results from the definition:

Lemma 5.4.6. *Let* $\pi, \pi' \in \mathcal{A}(\mathfrak{g}_{\mathbb{C}}, K)$.
1) $[\pi : \pi'] < \infty$ *if at least one of* π *and* π' *belongs to* $\mathcal{F}(\mathfrak{g}_{\mathbb{C}}, K)$.
2) $\dim \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}(\pi, \pi') \leq [\pi : \pi']$.
3) $[\pi : \pi'] = [\pi' : \pi]$.
4) $m_\pi(Y) = [\pi : Y]$ *if* Y *is an irreducible* $(\mathfrak{g}_{\mathbb{C}}, K)$-*module.*
5) $[\pi : \pi'] \leq m(\pi)$ *if* π' *is multiplicity-free.*

Now, we return to Proposition 5.4.1.

Proof of Proposition 5.4.1. We divide the proof into three steps:
Step 1 (π is a finite dimensional representation):   We shall prove

$$m(\pi \otimes F) \leq \dim F \tag{5.4.6}$$

for any finite dimensional representation π of G.

Let $\mathfrak{b} = \mathfrak{t} + \mathfrak{u}$ be a Borel subalgebra of $\mathfrak{g}_{\mathbb{C}}$ with $\mathfrak{u}$ nilradical. We denote by $H^j(\mathfrak{u}, V)$ the jth cohomology group of the Lie algebra $\mathfrak{u}$ with coefficients in a $\mathfrak{u}$-module V. Since the Lie algebra $\mathfrak{b}$ is solvable, we can choose a $\mathfrak{b}$-stable filtration

$$F = F_k \supset F_{k-1} \supset \cdots \supset F_0 = \{0\}$$

such that $\dim F_i/F_{i-1} = 1$.

Let us show by induction on i that

$$\dim H^0(\mathfrak{u}, \pi \otimes F_i) \leq i. \tag{5.4.7}$$

This will imply $m(\pi \otimes F) = \dim H^0(\mathfrak{u}, \pi \otimes F) \leq k = \dim F$.

The inequality (5.4.7) is trivial if $i = 0$. Suppose (5.4.7) holds for $i - 1$. The short exact sequence of $\mathfrak{b}$-modules

$$0 \to \pi \otimes F_{i-1} \to \pi \otimes F_i \to \pi \otimes (F_i/F_{i-1}) \to 0$$

gives rise to a long exact sequence

$$0 \to H^0(\mathfrak{u}, \pi \otimes F_{i-1}) \to H^0(\mathfrak{u}, \pi \otimes F_i) \to H^0(\mathfrak{u}, \pi \otimes (F_i/F_{i-1}))$$
$$\to H^1(\mathfrak{u}, \pi \otimes F_{i-1}) \to \dots$$

of $\mathfrak{t}$-modules. In particular, we have

$$\dim H^0(\mathfrak{u}, \pi \otimes F_i) \leq \dim H^0(\mathfrak{u}, \pi \otimes F_{i-1}) + \dim H^0(\mathfrak{u}, \pi \otimes (F_i/F_{i-1})) . \quad (5.4.8)$$

Because F_i/F_{i-1} is trivial as a $\mathfrak{u}$-module, we have

$$H^0(\mathfrak{u}, \pi \otimes (F_i/F_{i-1})) = H^0(\mathfrak{u}, \pi) \otimes (F_i/F_{i-1}) . \quad (5.4.9)$$

By definition $H^0(\mathfrak{u}, \pi)$ is the space of highest weight vectors, and therefore the dimension of the right-hand side of (5.4.9) is one. Now, the inductive assumption combined with (5.4.8) implies $\dim H^0(\mathfrak{u}, \pi \otimes F_i) \leq i$, as desired.

Step 2 (π is a principal series representation): In this step, we consider the case where π is a principal series representation. We note that π may be reducible here.

Let $P = LN$ be a Levi decomposition of a minimal parabolic subgroup P of G, W an irreducible (finite dimensional) representation of L, and $\mathrm{Ind}_P^G(W)$ the underlying $(\mathfrak{g}_{\mathbb{C}}, K)$-module of a principal series representation induced from the representation $W \boxtimes \mathbf{1}$ of $P = LN$ (without ρ-shift). Then, the socle filtration is unchanged so far as the parameter lies in the equisingular set, and thus, there are only finitely many possibilities of the socle filtration of $\mathrm{Ind}_P^G(W)$ for irreducible representations W of L. We denote by $m(G)$ the maximum of $m(\mathrm{Ind}_P^G(W))$ for irreducible representations W of L.

Let F be a finite dimensional representation of G. Then we have an isomorphism of $(\mathfrak{g}_{\mathbb{C}}, K)$-modules

$$\mathrm{Ind}_P^G(W) \otimes F \simeq \mathrm{Ind}_P^G(W \otimes F) ,$$

where F is regarded as a P-module on the right-hand side. We take a P-stable filtration

$$W_n := W \otimes F \supset W_{n-1} \supset \dots \supset W_0 = \{0\}$$

such that each W_i/W_{i-1} is irreducible as a P-module. We notice that $n \leq \dim F$ by applying Step 1 to the L-module $F|_L$. As $\mathrm{Ind}_P^G(W \otimes F)$ is isomorphic to $\bigoplus_{i=1}^n \mathrm{Ind}_P^G(W_i/W_{i-1})$ in the Grothendieck group $\mathcal{V}(\mathfrak{g}_{\mathbb{C}}, K)$, we have shown that

$$m(\mathrm{Ind}_P^G(W) \otimes F) \leq n\, m(G) \leq (\dim F)\, m(G)$$

for any irreducible finite dimensional representation W of L.

<u>Step 3</u> (general case): By Casselman's subrepresentation theorem (see [87, Chapter 3]), any irreducible $(\mathfrak{g}_\mathbb{C}, K)$-module π is realized as a subrepresentation of some induced representation $\mathrm{Ind}_P^G(W)$. Then

$$m(\pi \otimes F) \leq m(\pi \otimes \mathrm{Ind}_P^G(W)) \leq C$$

by step 2. Thus, Proposition 5.4.1 is proved. □

5.5 Proof of Theorem D

Now let us complete the proof of Theorem D.

Let $\pi = \pi_1 \boxtimes \pi_2$ be an irreducible unitary highest weight representation of $G' := G \times G$. It follows from Lemma 5.3 that there exist an irreducible unitary highest weight representation $\pi' = \pi_1' \boxtimes \pi_2'$ of scalar type and a finite dimensional representation F of G' such that π_K occurs as a subquotient of $\pi_K' \otimes F$.

By using the notation (5.4.4), we set $[V_1 : V_2] := [(V_1)_K : (V_2)_K]$ for G-modules V_1 and V_2 of finite length. Then, for $\mu \in \widehat{G}$, we have

$$\begin{aligned}
m_{\pi_1, \pi_2}(\mu) = \dim \mathrm{Hom}_G(\mu, \pi|_{\mathrm{diag}(G)}) \\
\leq \left[\mu : \pi|_{\mathrm{diag}(G)}\right] \\
\leq \left[\mu : (\pi' \otimes F)|_{\mathrm{diag}(G)}\right] \\
= \left[\mu \otimes (F^*|_{\mathrm{diag}(G)}) : \pi'|_{\mathrm{diag}(G)}\right] \\
\leq m(\mu \otimes (F^*|_{\mathrm{diag}(G)})) \qquad (5.5.1) \\
\leq C(F^*).
\end{aligned}$$

Here the inequality (5.5.1) follows from Lemma 5.4.6 (5) because $\pi'|_{\mathrm{diag}(G)} \simeq \pi_1' \widehat{\otimes} \pi_2'$ is multiplicity-free (see Theorem D (2)). In the last inequality, $C(F^*)$ is the constant in Proposition 5.4.1. This completes the proof of Theorem D (1). □

Remark 5.5. The argument in Subsections 8.8 and 8.9 gives a different and more straightforward proof of Theorem D.

6 Counter examples

In this section, we analyze the assumptions in Theorems A and B by counterexamples, that is, how the conclusions fail if we relax the assumptions on the representation π.

Let (G, H) be a reductive symmetric pair corresponding to an involutive automorphism τ of G, and π an irreducible unitary representation of G. We shall see that the multiplicity of an irreducible summand occurring in the

restriction $\pi|_H$ can be:

1) **greater than one** if π is not of scalar type (but we still assume that π is a highest weight module);

2) **finite but not uniformly bounded** if π is not a highest weight module (but we still assume that $\pi|_H$ decomposes discretely);

3) **infinite** if $\pi|_H$ contains continuous spectra.

Although our concern in this paper is mainly with a non-compact subgroup H, we can construct such examples for (1) and (2) even for $H = K$ (a maximal compact subgroup modulo the center of G).

Case (1) will be discussed in Subsection 6.1, (2) in Subsection 6.2, and (3) in Subsection 6.3, respectively. To construct an example for (3), we use those for (1) and (2).

6.1 Failure of multiplicity-free property

Let $G = Sp(2, \mathbb{R})$. Then, the maximal compact subgroup K is isomorphic to $U(2)$. We take a compact Cartan subalgebra $\mathfrak{t}$. Let $\{f_1, f_2\}$ be the standard basis of $\sqrt{-1}\,\mathfrak{t}^*$ such that $\Delta(\mathfrak{g}, \mathfrak{t}) = \{\pm f_1 \pm f_2, \pm 2f_1, \pm 2f_2\}$, and we fix a positive system $\Delta^+(\mathfrak{k}, \mathfrak{t}) := \{f_1 - f_2\}$. In what follows, we shall use the notation (λ_1, λ_2) to denote the character $\lambda_1 f_1 + \lambda_2 f_2$ of $\mathfrak{t}$.

Given $(p, q) \in \mathbb{Z}^2$ with $p \geq q$, we denote by $\pi^{U(2)}_{(p,q)}$ the irreducible representation of $U(2)$ with highest weight $(p, q) = pf_1 + qf_2$. Then $\dim \pi^{U(2)}_{(p,q)} = p - q + 1$.

The set of holomorphic discrete series representations of G is parametrized by $\lambda := (\lambda_1, \lambda_2) \in \mathbb{N}^2$ with $\lambda_1 > \lambda_2 > 0$. We set $\mu \equiv (\mu_1, \mu_2) := (\lambda_1 + 1, \lambda_2 + 2)$ and denote by $\pi^G_\mu \equiv \pi^{Sp(2,\mathbb{R})}_{(\mu_1, \mu_2)}$ the holomorphic discrete series representation of G characterized by

$$Z(\mathfrak{g})\text{-infinitesimal character } = (\lambda_1, \lambda_2) \quad \text{(Harish-Chandra parameter)},$$

$$\text{minimal } K\text{-type } = \pi^{U(2)}_{(\mu_1, \mu_2)} \qquad \qquad \text{(Blattner parameter)}.$$

We note that π^G_μ is of scalar type if and only if $\mu_1 = \mu_2$.

We know from Theorem B that multiplicities of K-type τ occurring in π^G_μ are uniformly bounded for fixed $\mu = (\mu_1, \mu_2)$. Here is the formula:

Example 6.1 (upper bound of K-multiplicities of holomorphic discrete series).

$$\sup_{\tau \in \widehat{K}} \dim \mathrm{Hom}_K(\tau, \pi^G_\mu|_K) = \left[\frac{\mu_1 - \mu_2 + 2}{2} \right]. \qquad (6.1.1)$$

The right side of (6.1.1) $= 1$ if and only if either of the following two cases holds:

$$\mu_1 = \mu_2 \qquad \text{(i.e. } \pi^G_\mu \text{ is of scalar type)}, \qquad \qquad (6.1.2)\ (a)$$

$$\mu_1 = \mu_2 + 1 \quad \text{(i.e. } \pi^G_\mu \text{ is of two dimensional minimal } K\text{-type)}. \quad (6.1.2)\ (b)$$

Thus, the branching law of the restriction $\pi_\mu^G|_K$ is multiplicity-free if and only if $\mu_1 = \mu_2$ or $\mu_1 = \mu_2 + 1$. The multiplicity-free property for $\mu_1 = \mu_2$ (i.e. for π_μ^G of scalar type) follows from Theorem A. The multiplicity-free property for $\mu_1 = \mu_2 + 1$ is outside of the scope of this paper, but can be explained in the general framework of the 'vector bundle version' of Theorem 2.2 (see [47, Theorem 2], [49]).

Proof. It follows from the Blattner formula for a holomorphic discrete series representation ([32], [78]) that the K-type formula of π_μ^G is given by

$$
\begin{aligned}
\pi_\mu^G|_K &\simeq \pi_{(\mu_1,\mu_2)}^{U(2)} \otimes S(\mathbb{C}^3) \\
&= \pi_{(\mu_1,\mu_2)}^{U(2)} \otimes \bigoplus_{\substack{a \geq b \geq 0 \\ (a,b) \in \mathbb{N}^2}} \pi_{(2a,2b)}^{U(2)}\,,
\end{aligned}
\tag{6.1.3}
$$

where $K = U(2)$ acts on $\mathbb{C}^3 \simeq S^2(\mathbb{C}^2)$ as the symmetric tensor of the natural representation. We write $n_\mu(p,q)$ for the multiplicity of the K-type $\pi_{(p,q)}^{U(2)}$ occurring in $\pi_\mu^G \equiv \pi_{(\mu_1,\mu_2)}^{Sp(2,\mathbb{R})}$, that is,

$$
n_\mu(p,q) := \dim \operatorname{Hom}_K(\pi_{(p,q)}^K, \pi_\mu^G|_K)\,.
$$

Then, applying the Clebsch–Gordan formula (1.7.1) (f) to (6.1.3), we obtain

$$
n_\mu(p,q) = \#\{(a,b) \in \mathbb{N}^2 : (a,b) \text{ satisfies } a \geq b \geq 0, \ (6.1.4) \text{ and } (6.1.5)\},
$$

where

$$
p + q = \mu_1 + \mu_2 + 2a + 2b\,,
\tag{6.1.4}
$$

$$
\max(2a + \mu_2, 2b + \mu_1) \leq p \leq 2a + \mu_1\,.
\tag{6.1.5}
$$

In particular, for fixed (μ_1, μ_2) and (p,q), the integer b is determined by a from (6.1.4), whereas the integer a satisfies the inequalities $p - \mu_1 \leq 2a \leq p - \mu_2$. Therefore,

$$
n_\mu(p,q) \leq \left[\frac{(p - \mu_2) - (p - \mu_1)}{2}\right] + 1 = \left[\frac{\mu_1 - \mu_2 + 2}{2}\right]\,. \qquad \square
$$

6.2 Failure of uniform boundedness

We continue the setting of Subsection 6.1. Let B be a Borel subgroup of $G_\mathbb{C} \simeq Sp(2,\mathbb{C})$. Then, there exist 4 closed orbits of $K_\mathbb{C} \simeq GL(2,\mathbb{C})$ on the full flag variety $G_\mathbb{C}/B$. (By the Matsuki duality, there exist 4 open orbits of $G = Sp(2,\mathbb{R})$ on $G_\mathbb{C}/B$. This observation will be used in the proof of Example 6.3.) By the Beilinson–Bernstein correspondence, we see that there are 4 series of discrete series representations of G. Among them, two are holomorphic

and anti-holomorphic discrete series representations, that is, π_μ^G and $(\pi_\mu^G)^*$ (the contragredient representation) with notation as in Subsection 6.1. The other two series are non-holomorphic discrete series representations. Let us parametrize them. For $\lambda := (\lambda_1, \lambda_2) \in \mathbb{Z}^2$ ($\lambda_1 > -\lambda_2 > 0$), we write W_λ for the discrete series representation of G characterized by

$Z(\mathfrak{g})$-infinitesimal character $= (\lambda_1, \lambda_2)$ (Harish-Chandra parameter),

minimal K-type $= \pi_{(\lambda_1+1, \lambda_2)}^{U(2)}$ (Blattner parameter).

Then, non-holomorphic discrete series representations are either W_λ or its contragredient representation W_λ^* for some $\lambda \in \mathbb{Z}^2$ with $\lambda_1 > -\lambda_2 > 0$. We define a θ-stable Borel subalgebra $\mathfrak{q} = \mathfrak{t}_\mathbb{C} + \mathfrak{u}$ of $\mathfrak{g}_\mathbb{C} = \mathfrak{k}_\mathbb{C} + \mathfrak{p}_\mathbb{C}$ such that

$$\Delta(\mathfrak{u} \cap \mathfrak{p}_\mathbb{C}, \mathfrak{t}) := \{2f_1, f_1 + f_2, -2f_2\}, \quad \Delta(\mathfrak{u} \cap \mathfrak{k}_\mathbb{C}, \mathfrak{t}) := \{f_1 - f_2\}.$$

Then, the Harish-Chandra module $(W_\lambda)_K$ is isomorphic to the cohomological parabolic induction $\mathcal{R}_\mathfrak{q}^1(\mathbb{C}_{(\lambda_1, \lambda_2)})$ of degree 1 as $(\mathfrak{g}_\mathbb{C}, K)$-modules with the notation and the normalization as in [86]. We set $\mu_1 := \lambda_1 + 1$ and $\mu_2 := \lambda_2$.

Example 6.2 (multiplicity of K-type of non-holomorphic discrete series W_λ). We write $m_\lambda(p, q)$ for the multiplicity of the K-type $\pi_{(p,q)}^{U(2)}$ occurring in W_λ, that is,

$$m_\lambda(p, q) := \dim \mathrm{Hom}_K(\pi_{(p,q)}^{U(2)}, W_\lambda|_K).$$

Then, $m_\lambda(p, q) \neq 0$ only if $(p, q) \in \mathbb{Z}^2$ satisfies

$$p \geq \mu_1, \ p - q \geq \mu_1 - \mu_2 \text{ and } p - q \in 2\mathbb{Z} + \mu_1 + \mu_2. \tag{6.2.1}$$

Then,

$$m_\lambda(p, q) = 1 + \min\left(\left[\frac{p - \mu_1}{2}\right], \frac{p - q - \mu_1 + \mu_2}{2}\right). \tag{6.2.2}$$

In particular, for each fixed λ, the K-multiplicity in W_λ is not uniformly bounded, namely,

$$\sup_{\tau \in \widehat{K}} \dim \mathrm{Hom}_K(\tau, W_\lambda|_K) = \sup_{(p,q) \text{ satisfies } (6.2.1)} m_\lambda(p, q) = \infty.$$

Proof. For $p, q \in \mathbb{Z}$, we write $\mathbb{C}_{(p,q)}$ for the one dimensional representation of $\mathfrak{t}_\mathbb{C}$ corresponding to the weight $pf_1 + qf_2 \in \mathfrak{t}_\mathbb{C}^*$. According to the $\mathfrak{t}_\mathbb{C}$-module isomorphism:

$$\mathfrak{u} \cap \mathfrak{p}_\mathbb{C} \simeq \mathbb{C}_{(2,0)} \oplus \mathbb{C}_{(1,1)} \oplus \mathbb{C}_{(0,-2)},$$

the symmetric algebra $S(\mathfrak{u} \cap \mathfrak{p}_\mathbb{C})$ is decomposed into irreducible representations of $\mathfrak{t}_\mathbb{C}$ as

$$S(\mathfrak{u} \cap \mathfrak{p}_\mathbb{C}) \simeq \bigoplus_{a,b,c \in \mathbb{N}} S^a(\mathbb{C}_{(2,0)}) \otimes S^b(\mathbb{C}_{(1,1)}) \otimes S^c(\mathbb{C}_{(0,-2)})$$

$$\simeq \bigoplus_{a,b,c \in \mathbb{N}} \mathbb{C}_{(2a+b, b-2c)}. \tag{6.2.3}$$

We denote by $H^j(\mathfrak{u} \cap \mathfrak{k}_\mathbb{C}, \pi)$ the jth cohomology group of the Lie algebra $\mathfrak{u} \cap \mathfrak{k}_\mathbb{C}$ with coefficients in the $\mathfrak{u} \cap \mathfrak{k}_\mathbb{C}$-module π. If π is a $\mathfrak{k}_\mathbb{C}$-module, then $H^j(\mathfrak{u} \cap \mathfrak{k}_\mathbb{C}, \pi)$ becomes naturally a $\mathfrak{t}_\mathbb{C}$-module. Then, Kostant's version of the Borel–Weil–Bott theorem (e.g. [85, Chapter 3]) shows that

$$H^j(\mathfrak{u} \cap \mathfrak{k}_\mathbb{C}, \pi^{U(2)}_{(p,q)}) = \begin{cases} \mathbb{C}_{(p,q)} & (j = 0)\,, \\ \mathbb{C}_{(q-1,p+1)} & (j = 1)\,, \\ \{0\} & (j \neq 0, 1)\,. \end{cases} \qquad (6.2.4)$$

By using the Blattner formula due to Hecht–Schmid (e.g. [85, Theorem 6.3.12]), the K-type formula of W_λ is given by

$$m_\lambda(p,q) = \dim \mathrm{Hom}_K(\pi^{U(2)}_{(p,q)}, W_\lambda|_K)$$

$$= \sum_{j=0}^{1} (-1)^j \dim \mathrm{Hom}_{\mathfrak{t}_\mathbb{C}}(H^j(\mathfrak{u} \cap \mathfrak{k}_\mathbb{C}, \pi^{U(2)}_{(p,q)}), S(\mathfrak{u} \cap \mathfrak{p}_\mathbb{C}) \otimes \mathbb{C}_{(\mu_1,\mu_2)})\,.$$

Now, comparing (6.2.3) with the above formula (6.2.4) as $\mathfrak{t}_\mathbb{C}$-modules, we see

$$m_\lambda(p,q) = \#\{(a,b,c) \in \mathbb{N}^3 : p = 2a + b + \mu_1, q = b - 2c + \mu_2\}$$
$$- \#\{(a,b,c) \in \mathbb{N}^3 : q - 1 = 2a + b + \mu_1, p + 1 = b - 2c + \mu_2\}$$
$$= \#\{(a,b,c) \in \mathbb{N}^3 : p = 2a + b + \mu_1, q = b - 2c + \mu_2\}$$
$$= 1 + \min\left(\left[\frac{p - \mu_1}{2}\right], \frac{p - q - \mu_1 + \mu_2}{2}\right)\,.$$

Thus, the formula (6.2.2) has been verified. □

6.3 Failure of finiteness of multiplicities

Multiplicities of the branching laws can be infinite in general even for reductive symmetric pairs (G, H). In this subsection, we review from [43, Example 5.5] a curious example of the branching law, in which the multiplicity of a discrete summand is non-zero and finite and that of another discrete summand is infinite. Such a phenomenon happens only when continuous spectra appear.

Example 6.3 (infinite and finite multiplicities). Let $(G_\mathbb{C}, G)$ be a reductive symmetric pair $(Sp(2,\mathbb{C}), Sp(2,\mathbb{R}))$. We note that $(G_\mathbb{C}, G)$ is locally isomorphic to the symmetric pair $(SO(5,\mathbb{C}), SO(3,2))$. We take a Cartan subgroup $H = TA$ of $G_\mathbb{C}$. We note that $T \simeq \mathbb{T}^2$ and $A \simeq \mathbb{R}^2$, and identify $\widehat{T}$ with $\mathbb{Z}^2$.

Let $\varpi \equiv \varpi^{Sp(2,\mathbb{C})}_{(a,b)}$ be the unitary principal series representation of $G_\mathbb{C}$ induced unitarily from the character χ of a Borel subgroup B containing $H = TA$ such that

$$\chi|_H \simeq \mathbb{C}_{(a,b)} \boxtimes \mathbf{1}\,.$$

We assume $a, b \geq 0$ and set

$$c(\mu_1, \mu_2; a, b) := \#\{(s, t, u) \in \mathbb{N}^3 : a = \mu_1 + 2s + t,\ b = \mu_2 + t + 2u\}.$$

Then, the discrete part of the branching law of the restriction $\varpi_{(a,b)}^{Sp(2,\mathbb{C})}|_{Sp(2,\mathbb{R})}$ is given by the following spectra:

$$\bigoplus_{\mu_1 \geq \mu_2 \geq 3} c(\mu_1, \mu_2; a, b)(\pi_{(\mu_1,\mu_2)}^{Sp(2,\mathbb{R})} \oplus \left(\pi_{(\mu_1,\mu_2)}^{Sp(2,\mathbb{R})}\right)^*) \oplus \sum_{\lambda_1 > -\lambda_2 > 0}^{\oplus} \infty(W_\lambda \oplus W_\lambda^*),\ (6.3.1)$$

with the notation as in Examples 6.1 and 6.2.

The first term of (6.3.1) is a finite sum because there are at most finitely many (μ_1, μ_2) such that $c(\mu_1, \mu_2; a, b) \neq 0$ for each fixed (a, b). For instance, the first term of (6.3.1) amounts to

$$\bigoplus_{\substack{3 \leq \mu_1 \leq a \\ \mu_1 \equiv a \bmod 2}} \pi_{(\mu_1,3)}^{Sp(2,\mathbb{R})} \oplus \bigoplus_{\substack{3 \leq \mu_1 \leq a \\ \mu_1 \equiv a \bmod 2}} \left(\pi_{(\mu_1,3)}^{Sp(2,\mathbb{R})}\right)^* \quad \text{(multiplicity-free)}$$

if $b = 3$.

The second term of (6.3.1) is nothing other than the direct sum of all non-holomorphic discrete series representations of $G = Sp(2, \mathbb{R})$ with infinite multiplicities for any a and b.

Sketch of Proof. There exist 4 open G-orbits on $G_\mathbb{C}/B$, for which the isotropy subgroups are all isomorphic to $T \simeq \mathbb{T}^2$. By the Mackey theory, the restriction $\varpi_{(a,b)}^{G_\mathbb{C}}|_G$ is unitarily equivalent to the direct sum of the regular representations realized on L^2-sections of G-equivariant line bundles $G \times_T \mathbb{C}_{(\pm a, \pm b)} \to G/T$. That is,

$$\varpi_{(a,b)}^{G_\mathbb{C}}|_G \simeq \bigoplus_{\varepsilon_1, \varepsilon_2 = \pm 1} L^2(G/T, \mathbb{C}_{(\varepsilon_1 a, \varepsilon_2 b)}).$$

Therefore, an irreducible unitary representation σ of G occurs as a discrete spectrum in $\varpi_{(a,b)}^{G_\mathbb{C}}|_G$ if and only if σ occurs as a discrete summand in $L^2(G/T, \mathbb{C}_{(\varepsilon_1 a, \varepsilon_2 b)})$ for some $\varepsilon_1, \varepsilon_2 = \pm 1$. Further, the multiplicity is given by

$$\dim \operatorname{Hom}_G(\sigma, \varpi_{(a,b)}^{G_\mathbb{C}}|_G) = \sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \dim \operatorname{Hom}_{\mathbb{T}^2}(\mathbb{C}_{(\varepsilon_1 a, \varepsilon_2 b)}, \sigma|_{\mathbb{T}^2})$$

by the Frobenius reciprocity theorem.

Since T is compact, σ must be a discrete series representation of $G = Sp(2, \mathbb{R})$ if σ occurs in $L^2(G/T, \mathbb{C}_{(\varepsilon_1 a, \varepsilon_2 b)})$ as a discrete summand. We divide the computation of multiplicities into the following two cases:

Case I. σ is a holomorphic series representation or its contragredient representation. Let $\sigma = \pi_\mu^{Sp(2,\mathbb{R})}$. Combining (6.1.3) with the weight formulae

$$S(\mathbb{C}^3)|_{\mathbb{T}^2} \simeq \bigoplus_{s,t,u\in\mathbb{N}} S^s(\mathbb{C}_{(2,0)}) \otimes S^t(\mathbb{C}_{(1,1)}) \otimes S^u(\mathbb{C}_{(0,2)}) \simeq \bigoplus_{s,t,u\in\mathbb{N}} \mathbb{C}_{(2s+t,\,t+2u)}\,,$$

$$\pi^{U(2)}_{(\mu_1,\mu_2)}|_{\mathbb{T}^2} \simeq \bigoplus_{\substack{p+q=\mu_1+\mu_2 \\ \mu_2\le p\le\mu_1}} \mathbb{C}_{(p,q)}\,,$$

we have

$$\dim \mathrm{Hom}_{\mathbb{T}^2}(\mathbb{C}_{(a,b)}, \pi^{Sp(2,\mathbb{R})}_{\mu}|_{\mathbb{T}^2}) = c(\mu_1,\mu_2;a,b)\,.$$

Case II. σ is a non-holomorphic discrete series representation. Let $\sigma = W_\lambda$. It follows from the K-type formula (6.2.2) of W_λ that we have

$$\dim \mathrm{Hom}_{\mathbb{T}^2}(\mathbb{C}_{(a,b)}, W_\lambda|_{\mathbb{T}^2}) = \sum_{p\ge q} m_\lambda(p,q) \dim \mathrm{Hom}_{\mathbb{T}^2}(\mathbb{C}_{(a,b)}, \pi^{U(2)}_{(p,q)}) = \infty\,.$$

Likewise for $\sigma = W_\lambda^*$ (the contragredient representation).

Hence, the discrete part of the branching law is given by (6.3.1). □

7 Finite Dimensional Cases — Proof of Theorems E and F

7.1 Infinite v.s. finite dimensional representations

Our method applied to infinite dimensional representations in Sections 3 and 4 also applies to **finite** dimensional representations, leading us to multiplicity-free theorems, as stated in Theorems E and F in Section 1, for the restriction with respect to symmetric pairs.

The comparison with multiplicity-free theorems in the infinite dimensional case is illustrated by the following correspondence:

a non-compact simple group G	$\leftrightarrow$	a compact simple group G_U
a unitary highest weight module	$\leftrightarrow$	a finite dimensional module
scalar type (Definition 1.3)	$\leftrightarrow$	"pan type" (Definition 7.3.3)
Theorems A and B	$\leftrightarrow$	Theorems E and F.

The main goal of this section is to give a proof of Theorems E and F by using Theorem 2.2. Geometrically, our proof is built on the fact that the H_U action on the Hermitian symmetric space is strongly visible if (G_U, H_U) is a symmetric pair (see [50]).

7.2 Representations associated to maximal parabolic subalgebras

Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra. We take a Cartan subalgebra $\mathfrak{j}$ of $\mathfrak{g}_{\mathbb{C}}$, and fix a positive system $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j})$. We denote by $\{\alpha_1, \ldots, \alpha_n\}$ the set of simple roots, and by $\{\omega_1, \ldots, \omega_n\}$ ($\subset \mathfrak{j}^*$) the set of the fundamental weights.

We denote by $\pi_\lambda^{\mathfrak{g}_\mathbb{C}}$ irreducible finite dimensional representation of $\mathfrak{g}_\mathbb{C}$ with highest weight $\lambda = \sum_{i=1}^n m_i \omega_i$ for $m_1, \ldots, m_n \in \mathbb{N}$. It is also regarded as a holomorphic representation of $G_\mathbb{C}$, a simply connected complex Lie group with Lie algebra $\mathfrak{g}_\mathbb{C}$.

We fix a simple root α_i, and define a maximal parabolic subalgebra

$$\mathfrak{p}_{i\mathbb{C}}^- := \mathfrak{l}_{i\mathbb{C}} + \mathfrak{n}_{i\mathbb{C}}^-$$

such that the nilradical $\mathfrak{n}_{i\mathbb{C}}^-$ and the Levi part $\mathfrak{l}_{i\mathbb{C}}$ ($\supset \mathfrak{j}$) are given by

$$\Delta(\mathfrak{l}_{i\mathbb{C}}, \mathfrak{j}) = \mathbb{Z}\text{-span of } \{\alpha_1, \ldots, \overset{\wedge}{\alpha_i}, \ldots, \alpha_n\} \cap \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{j}),$$
$$\Delta(\mathfrak{n}_{i\mathbb{C}}^-, \mathfrak{j}) = \Delta^-(\mathfrak{g}_\mathbb{C}, \mathfrak{j}) \setminus \Delta(\mathfrak{l}_{i\mathbb{C}}, \mathfrak{j}).$$

We shall see that irreducible finite dimensional representations realized on generalized flag varieties $G_\mathbb{C}/P_\mathbb{C}$ is multiplicity-free with respect to any symmetric pairs if $P_\mathbb{C}$ has an abelian unipotent radical.

We write $P_{i\mathbb{C}}^- = L_{i\mathbb{C}} N_{i\mathbb{C}}^-$ for the corresponding maximal parabolic subgroup of $G_\mathbb{C}$.

Let $\mathrm{Hom}(\mathfrak{p}_{i\mathbb{C}}^-, \mathbb{C})$ be the set of Lie algebra homomorphisms over $\mathbb{C}$. Since any such homomorphism vanishes on the derived ideal $[\mathfrak{p}_{i\mathbb{C}}^-, \mathfrak{p}_{i\mathbb{C}}^-]$, $\mathrm{Hom}(\mathfrak{p}_{i\mathbb{C}}^-, \mathbb{C})$ is naturally identified with

$$\mathrm{Hom}(\mathfrak{p}_{i\mathbb{C}}^-/[\mathfrak{p}_{i\mathbb{C}}^-, \mathfrak{p}_{i\mathbb{C}}^-], \mathbb{C}) \simeq \mathbb{C}\omega_i.$$

Next, let $\mathrm{Hom}(P_{i\mathbb{C}}^-, \mathbb{C}^\times)$ be the set of complex Lie group homomorphisms. Then, we can regard $\mathrm{Hom}(P_{i\mathbb{C}}^-, \mathbb{C}^\times) \subset \mathrm{Hom}(\mathfrak{p}_{i\mathbb{C}}^-, \mathbb{C})$. As its subset, $\mathrm{Hom}(P_{i\mathbb{C}}^-, \mathbb{C}^\times)$ is identified with $\mathbb{Z}\omega_i$ since $G_\mathbb{C}$ is simply connected.

For $k \in \mathbb{Z}$, we write $\mathbb{C}_{k\omega_i}$ for the corresponding character of $P_{i\mathbb{C}}^-$, and denote by

$$\mathcal{L}_{k\omega_i} := G_\mathbb{C} \times_{P_{i\mathbb{C}}^-} \mathbb{C}_{k\omega_i} \to G_\mathbb{C}/P_{i\mathbb{C}}^- \tag{7.2.1}$$

the associated holomorphic line bundle. We naturally have a representation of $G_\mathbb{C}$ on the space of holomorphic sections $\mathcal{O}(\mathcal{L}_{k\omega_i})$. Then, by the Borel–Weil theory, $\mathcal{O}(\mathcal{L}_{k\omega_i})$ is non-zero and irreducible if $k \geq 0$ and we have an isomorphism of representations of $G_\mathbb{C}$ (also of $\mathfrak{g}_\mathbb{C}$):

$$\pi_{k\omega_i}^{\mathfrak{g}_\mathbb{C}} \simeq \mathcal{O}(\mathcal{L}_{k\omega_i}). \tag{7.2.2}$$

7.3 Parabolic subalgebra with abelian nilradical

A parabolic subalgebra with abelian nilradical is automatically a maximal parabolic subalgebra. Conversely, the nilradical of a maximal parabolic subalgebra is not necessarily abelian. We recall from Richardson–Röhrle–Steinberg [75] the following equivalent characterization of such parabolic algebras:

Lemma 7.3.1. *Retain the setting of Subsection 7.2. Then, the following four conditions on the pair* $(\mathfrak{g}_{\mathbb{C}}, \alpha_i)$ *are equivalent:*

i) *The nilradical* $\mathfrak{n}_{i\mathbb{C}}^-$ *is abelian.*

ii) $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{l}_{i\mathbb{C}})$ *is a symmetric pair.*

iii) *The simple root* α_i *occurs in the highest root with coefficient one.*

iv) $(\mathfrak{g}_{\mathbb{C}}, \alpha_i)$ *is in the following list if we label the simple roots* $\alpha_1, \ldots, \alpha_n$ *in the Dynkin diagram as in Table 7.3.2.*

(7.3.1)   *Type* A_n $\alpha_1, \alpha_2, \ldots, \alpha_n$
(7.3.2)   *Type* B_n α_1
(7.3.3)   *Type* C_n α_n
(7.3.4)   *Type* D_n $\alpha_1, \alpha_{n-1}, \alpha_n$
(7.3.5)   *Type* E_6 α_1, α_6
(7.3.6)   *Type* E_7 α_7

For types G_2, F_4, E_8, *there are no maximal parabolic subalgebras with abelian nilradicals.*

Table 7.3.2.
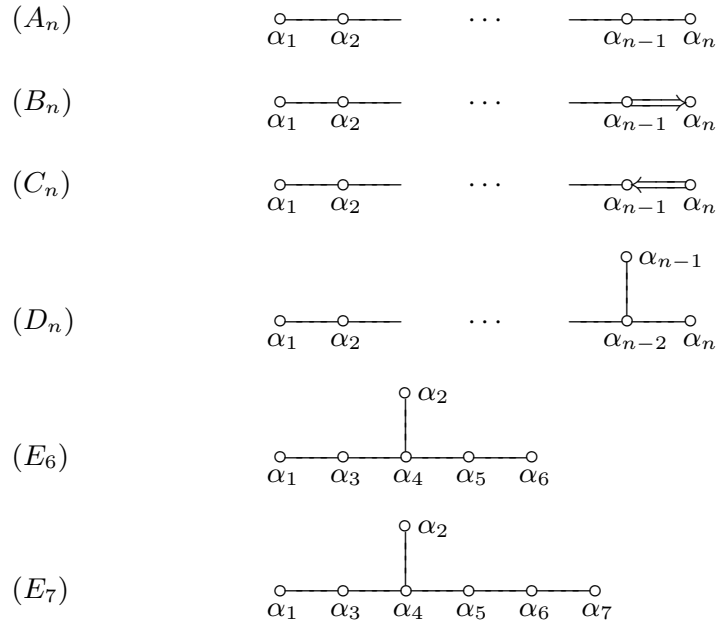
Proof. See [75] for the equivalence (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). The implication (iv) $\Rightarrow$ (ii) is straightforward. For the convenience of the reader, we present a table of the symmetric pairs $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{l}_{i\mathbb{C}})$ corresponding to the index i in (iv).

Type	$\mathfrak{g}_\mathbb{C}$	$\mathfrak{l}_{i\mathbb{C}}$	i
A_n	$\mathfrak{sl}(n+1,\mathbb{C})$	$\mathfrak{sl}(i,\mathbb{C}) + \mathfrak{sl}(n+1-i,\mathbb{C}) + \mathbb{C}$	$i = 1, 2, \ldots, n$
B_n	$\mathfrak{so}(2n+1,\mathbb{C})$	$\mathfrak{so}(2n-1,\mathbb{C}) + \mathbb{C}$	$i = 1$
C_n	$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{gl}(n,\mathbb{C})$	$i = n$
D_n	$\mathfrak{so}(2n,\mathbb{C})$	$\mathfrak{so}(2n-2,\mathbb{C}) + \mathbb{C}$	$i = 1$
	$\mathfrak{so}(2n,\mathbb{C})$	$\mathfrak{gl}(n,\mathbb{C})$	$i = n-1, n$
E_6	$\mathfrak{e}_6$	$\mathfrak{so}(10,\mathbb{C}) + \mathbb{C}$	$i = 1, 6$
E_7	$\mathfrak{e}_7$	$\mathfrak{e}_6 + \mathbb{C}$	$i = 1$

If $(\mathfrak{g}_\mathbb{C}, \mathfrak{l}_{i\mathbb{C}})$ is a symmetric pair, then $[\mathfrak{n}_{i\mathbb{C}}^-, \mathfrak{n}_{i\mathbb{C}}^-] \subset \mathfrak{n}_{i\mathbb{C}}^- \cap \mathfrak{l}_{i\mathbb{C}} = \{0\}$, whence (ii) $\Rightarrow$ (i). $\square$

Definition 7.3.3. We say the representation $\pi_{k\omega_i}^{\mathfrak{g}_\mathbb{C}}$ $(k = 0, 1, 2, \ldots)$ is of *pan type*, or a pan representation if $(\mathfrak{g}_\mathbb{C}, \alpha_i)$ satisfies one of (therefore, all of) the equivalent conditions of Lemma 7.3.1. Here, **pan** stands for a **pa**rabolic subalgebra with **a**belian **n**ilradical.

7.4 Examples of pan representations

Example 7.4. Let $\mathfrak{g}_\mathbb{C} = \mathfrak{gl}(n,\mathbb{C})$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. (This $\mathfrak{g}_\mathbb{C}$ is not a simple Lie algebra, but the above concept is defined similarly.) Then, π_λ is of pan type if and only if

$$\lambda_1 = \cdots = \lambda_i \geq \lambda_{i+1} = \cdots = \lambda_n$$

for some i $(1 \leq i \leq n-1)$. Then, $(\mathfrak{l})_{i\mathbb{C}} \simeq \mathfrak{gl}(i,\mathbb{C}) + \mathfrak{gl}(n-i,\mathbb{C})$.

In particular, the kth symmetric tensor representations $S^k(\mathbb{C}^n)$ $(k \in \mathbb{N})$ and the kth exterior representations $\Lambda^k(\mathbb{C}^n)$ $(0 \leq k \leq n)$ are examples of pan representations since their highest weights are given by $(k, 0, \ldots, 0)$ and $(\underbrace{1, \ldots, 1}_{k}, \underbrace{0, \ldots, 0}_{n-k})$, respectively.

S. Okada [68] studied branching laws for a specific class of irreducible finite dimensional representations of classical Lie algebras, which he referred to as "rectangular-shaped representations". The notion of "pan representations" is equivalent to that of rectangular-shaped representations for type (A_n), (B_n), and (C_n). For type (D_n), $\pi_{k\omega_{n-1}}, \pi_{k\omega_n}$ $(k \in \mathbb{N})$ are rectangular-shaped representations, while $\pi_{k\omega_1}$ $(k \in \mathbb{N})$ are not.

7.5 Reduction to rank condition

Suppose $(\mathfrak{g}_\mathbb{C}, \alpha_i)$ satisfies the equivalent conditions in Lemma 7.3.1. Let θ be the complex involutive automorphism of the Lie algebra $\mathfrak{g}_\mathbb{C}$ that defines the

symmetric pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{l}_{i\mathbb{C}})$. We use the same letter θ to denote the corresponding holomorphic involution of a simply connected $G_{\mathbb{C}}$. We take a maximal compact subgroup G_U of $G_{\mathbb{C}}$ such that $\theta G_U = G_U$. Then $K := G_U^{\theta} = G_U \cap L_{i\mathbb{C}}$ becomes a maximal compact subgroup of $L_{i\mathbb{C}}$.

Let τ be another complex involutive automorphism of $\mathfrak{g}_{\mathbb{C}}$, and $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ the symmetric pair defined by τ. We also use the same letter τ to denote its lift to $G_{\mathbb{C}}$. We recall from Subsection 3.7 the 'twisted' involution τ^g for $g \in G_{\mathbb{C}}$ is given by

$$\tau^g(x) = g\tau(g^{-1}xg)g^{-1} \quad (x \in G_{\mathbb{C}}).$$

Lemma 7.5. *Let (θ, τ) be as above.*
1) There exist an involutive automorphism σ of G_U and $g \in G_{\mathbb{C}}$ satisfying the following three conditions (by an abuse of notation, we write τ for τ^g):
(7.5.1) $\tau \mathfrak{g}_U = \mathfrak{g}_U$, $\sigma\theta = \theta\sigma$, $\sigma\tau = \tau\sigma$.
(7.5.2) The induced action of σ on G_U/K is anti-holomorphic.
(7.5.3) $(\mathfrak{g}_U)^{\sigma, -\tau, -\theta}$ contains a maximal abelian subspace in $(\mathfrak{g}_U)^{-\tau, -\theta}$.
2) For any $x \in G_U/K$, there exists $h \in (G_U^{\tau})_0$ such that $\sigma(x) = h \cdot x$. In particular, each $(G_U^{\tau})_0$-orbit on G_U/K is preserved by σ.

Proof. 1) See [50, Lemma 4.1] for the proof.
2) The second statement follows from the first statement and a similar argument of Lemma 3.3.

7.6 Proof of Theorem E

We are now ready to complete the proof of Theorem E in Section 1.

Let $\pi = \pi_{k\omega_i}^{\mathfrak{g}_{\mathbb{C}}}$ be a representation of pan type. As in Subsection 7.2, we consider the holomorphic line bundle $\mathcal{L}_{k\omega_i} \to G_{\mathbb{C}}/P_{i\mathbb{C}}^-$ and realize π on the space of holomorphic sections $\mathcal{O}(\mathcal{L}_{k\omega_i})$. We fix a G_U-invariant inner product on $\mathcal{O}(\mathcal{L}_{k\omega_i})$. With notation as in Subsection 7.5, we have a diffeomorphism

$$G_U/K \simeq G_{\mathbb{C}}/P_{i\mathbb{C}}^-,$$

through which the holomorphic line bundle $\mathcal{L}_{k\omega_i} \to G_{\mathbb{C}}/P_{i\mathbb{C}}^-$ is naturally identified with the G_U-equivariant holomorphic line bundle $\mathcal{L} \to D$, where we set $\mathcal{L} := G_U \times_K \mathbb{C}_{k\omega_i}$ and $D := G_U/K$ (a compact Hermitian symmetric space).

Now, applying Lemma 7.5, we take σ and set $H := (G_U^{\tau})_0$. We note that the complexification of the Lie algebra of H is equal to $\mathfrak{h}_{\mathbb{C}}$ up to a conjugation by $G_{\mathbb{C}}$. By Lemma 7.5, the condition (2.2.3) in Theorem 2.2 is satisfied. Furthermore, we see the condition (2.2.2) holds by a similar argument of Lemma 9.4. Therefore, the restriction $\pi|_{(G_U)_0^{\tau}}$ is multiplicity-free by Theorem 2.2. Hence, Theorem E holds by Weyl's unitary trick. $\square$

7.7 Proof of Theorem F

Suppose π_1 and π_2 are representations of pan type. We realize π_1 and π_2 on the space of holomorphic sections of holomorphic line bundles over compact symmetric spaces G_U/K_1 and G_U/K_2, respectively. We write θ_i for the corresponding involutive automorphisms of G_U that define K_i $(i = 1, 2)$. In light of Lemma 7.3.1 (iv), we can assume that $\theta_1\theta_2 = \theta_2\theta_1$. Then, applying Lemma 7.5 to (θ_1, θ_2) we find an involution $\sigma' \in \mathrm{Aut}(G_U)$ satisfying the following three conditions:

(7.7.1) $\sigma'\theta_i = \theta_i\sigma'$ $(i = 1, 2)$.

(7.7.2)   The induced action of σ' on G_U/K_i $(i = 1, 2)$ is anti-holomorphic.

(7.7.3) $(\mathfrak{g}_U)^{\sigma', -\theta_1, -\theta_2}$ contains a maximal abelian subspace of $(\mathfrak{g}_U)^{-\theta_1, -\theta_2}$.

We remark that the condition (7.7.2) for $i = 2$ is not included in Lemma 7.5, but follows automatically by our choice of σ.

We define three involutive automorphisms τ, θ and σ on $G_U \times G_U$ by $\tau(g_1, g_2) := (g_2, g_1)$, $\theta := (\theta_1, \theta_2)$ and $\sigma := (\sigma', \sigma')$, respectively. Then $(G_U \times G_U)^\tau = \mathrm{diag}(G_U)$. By using the identification

$$(\mathfrak{g}_U \oplus \mathfrak{g}_U)^{-\tau} = \{(X, -X) : X \in \mathfrak{g}_U\} \xrightarrow{\sim} \mathfrak{g}_U, \quad (X, -X) \mapsto X,$$

we have isomorphisms

$$(\mathfrak{g}_U \oplus \mathfrak{g}_U)^{-\tau, -\theta} \simeq (\mathfrak{g}_U)^{-\theta_1, -\theta_2},$$

$$(\mathfrak{g}_U \oplus \mathfrak{g}_U)^{\sigma, -\tau, -\theta} \simeq (\mathfrak{g}_U)^{\sigma', -\theta_1, -\theta_2}.$$

Thus, the condition (7.7.3) implies that $(\mathfrak{g}_U \oplus \mathfrak{g}_U)^{\sigma, -\tau, -\theta}$ contains a maximal abelian subspace of $(\mathfrak{g}_U \oplus \mathfrak{g}_U)^{-\tau, -\theta}$. Then, by Lemma 7.5 and by a similar argument of Lemma 3.3 again, for any $(x, y) \in G_U/K_1 \times G_U/K_2$ there exists a $g \in G_U$ such that $\sigma'(x) = g \cdot x$ and $\sigma'(y) = g \cdot y$ simultaneously. Now, Theorem F follows readily from Theorem 2.2. □

7.8 List of multiplicity-free restrictions

For the convenience of the reader, we present the list of the triple $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}, i)$ for which we can conclude from Theorem E that the irreducible finite dimensional representation $\pi^{\mathfrak{g}_\mathbb{C}}_{k\omega_i}$ of a simple Lie algebra $\mathfrak{g}_\mathbb{C}$ is multiplicity-free when restricted to $\mathfrak{h}_\mathbb{C}$ for any $k \in \mathbb{N}$ by Theorem E.

Table 7.8.1.

$\mathfrak{g}_{\mathbb{C}}$	$\mathfrak{h}_{\mathbb{C}}$	i
$\mathfrak{sl}(n+1,\mathbb{C})$	$\mathfrak{sl}(p,\mathbb{C}) + \mathfrak{sl}(n+1-p,\mathbb{C}) + \mathbb{C}$	$1, 2, \ldots, n$
$\mathfrak{sl}(n+1,\mathbb{C})$	$\mathfrak{so}(n+1,\mathbb{C})$	$1, 2, \ldots, n$
$\mathfrak{sl}(2m,\mathbb{C})$	$\mathfrak{sp}(m,\mathbb{C})$	$1, 2, \ldots, 2m-1$
$\mathfrak{so}(2n+1,\mathbb{C})$	$\mathfrak{so}(p,\mathbb{C}) + \mathfrak{so}(2n+1-p,\mathbb{C})$	1
$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{sp}(p,\mathbb{C}) + \mathfrak{sp}(n-p,\mathbb{C})$	n
$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{gl}(n,\mathbb{C})$	n
$\mathfrak{so}(2n,\mathbb{C})$	$\mathfrak{so}(p,\mathbb{C}) + \mathfrak{so}(2n-p,\mathbb{C})$	$1, n-1, n$
$\mathfrak{so}(2n,\mathbb{C})$	$\mathfrak{gl}(n,\mathbb{C})$	$1, n-1, n$
$\mathfrak{e}_6$	$\mathfrak{so}(10,\mathbb{C}) + \mathfrak{so}(2,\mathbb{C})$	$1, 6$
$\mathfrak{e}_6$	$\mathfrak{sl}(6,\mathbb{C}) + \mathfrak{sl}(2,\mathbb{C})$	$1, 6$
$\mathfrak{e}_6$	$\mathfrak{f}_4$	$1, 6$
$\mathfrak{e}_6$	$\mathfrak{sp}(4,\mathbb{C})$	$1, 6$
$\mathfrak{e}_7$	$\mathfrak{e}_6 + \mathfrak{so}(2,\mathbb{C})$	7
$\mathfrak{e}_7$	$\mathfrak{so}(12,\mathbb{C}) + \mathfrak{sl}(2,\mathbb{C})$	7
$\mathfrak{e}_7$	$\mathfrak{sl}(8,\mathbb{C})$	7

Some of the above cases were previously known to be multiplicity-free by case-by-case argument, in particular, for the case rank $\mathfrak{g}_{\mathbb{C}} = $ rank $\mathfrak{h}_{\mathbb{C}}$. Among them, the corresponding explicit branching laws have been studied by S. Okada [68] and H. Alikawa [1].

There are some few representations π that are not of pan type, but are multiplicity-free when restricted to symmetric subgroups H. Our method still works to capture such cases, but we do not go into details here (see [46, 51, 52]).

8 Generalization of the Hua–Kostant–Schmid Formula

This section discusses an explicit irreducible decomposition formula of the restriction $\pi|_H$ where the triple (π, G, H) satisfies the following two conditions:

1) π is a holomorphic discrete series representation of scalar type (Definition 1.3).

2) (G, H) is a symmetric pair defined by an involution τ of holomorphic type (Definition 1.4).

We know a priori from Theorem B (1) that the branching law is discrete and multiplicity-free. The main result of this section is Theorem 8.3, which enriches this abstract property with an explicit multiplicity-free formula. The formula for the special case $H = K$ corresponds to the Hua–Kostant–Schmid formula ([26, 32, 78]). We also present explicit formulas for the irreducible decomposition of the tensor product representation (Theorem 8.4) and of the restriction $U(p, q) \downarrow U(p-1, q)$ (Theorem 8.11).

Let us give a few comments on our proof of Theorem 8.3. Algebraically, our key machinery is Lemma 8.7 which assures that the irreducible G-decomposition is determined only by its K-structure. Geometrically, a well-known method of taking normal derivatives (e.g. S. Martens [63], Jakobsen–Vergne [31]) gives a general algorithm to obtain branching laws for highest weight modules. This algorithm yields explicit formulae by using the observation that the fiber of the normal bundle for $G^\tau/K^\tau \subset G/K$ is the tangent space of another Hermitian symmetric space $G^{\tau\theta}/K^\tau$. The key ingredient of the geometry here is the following nice properties of the two symmetric pairs (G, G^τ) and $(G, G^{\tau\theta})$:

    a) $K \cap G^\tau = K \cap G^{\tau\theta}$,

    b) $\mathfrak{p} = (\mathfrak{p} \cap \mathfrak{g}^\tau) \oplus (\mathfrak{p} \cap \mathfrak{g}^{\tau\theta})$.

Unless otherwise mentioned, we shall assume H is connected, that is, $H = G_0^\tau$ throughout this section.

8.1 Notation for highest weight modules

We set up the notation and give a parametrization of irreducible highest weight modules for both finite and infinite dimensional cases.

First, we consider finite dimensional representations. Let us take a Cartan subalgebra $\mathfrak{t}$ of a reductive Lie algebra $\mathfrak{k}$ and fix a positive system $\Delta^+(\mathfrak{k}, \mathfrak{t})$. We denote by $\pi_\mu^{\mathfrak{k}}$ the irreducible finite dimensional representation of $\mathfrak{k}$ with highest weight μ, if μ is a dominant integral weight. A $\mathfrak{k}$-module $\pi_\mu^{\mathfrak{k}}$ will be written also as π_μ^K if the action lifts to K.

Next, let G be a connected reductive Lie group, θ a Cartan involution, $K = \{g \in G : \theta g = g\}$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the corresponding Cartan decomposition and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}$ its complexification. We assume that there exists a central element Z of $\mathfrak{k}$ such that

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_+ + \mathfrak{p}_- \tag{8.1.1}$$

is the eigenspace decomposition of $\frac{1}{\sqrt{-1}}\operatorname{ad}(Z)$ with eigenvalues 0, 1, and -1, respectively. This assumption is satisfied if and only if G is locally isomorphic to a direct product of connected compact Lie groups and non-compact Lie groups of Hermitian type (if G is compact, we can simply take $Z = 0$).

We set

$$\widetilde{Z} := \frac{1}{\sqrt{-1}}Z\,. \tag{8.1.2}$$

As in Definition 1.3, we say an irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$-module V is a *highest weight module* if

$$V^{\mathfrak{p}_+} = \{v \in V : Yv = 0 \quad \text{for all } Y \in \mathfrak{p}_+\}$$

is non-zero. Then, $V^{\mathfrak{p}_+}$ is irreducible as a K-module, and the $(\mathfrak{g}_{\mathbb{C}}, K)$-module V is determined uniquely by the K-structure on $V^{\mathfrak{p}_+}$. If μ is the highest weight of $V^{\mathfrak{p}_+}$, we write V as $\pi_\mu^{\mathfrak{g}}$. That is, the irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$-module $\pi_\mu^{\mathfrak{g}}$ is characterized by the K-isomorphism:

$$(\pi_\mu^{\mathfrak{g}})^{\mathfrak{p}_+} \simeq \pi_\mu^{\mathfrak{k}}. \tag{8.1.3}$$

An irreducible unitary highest weight representation π of G will be denoted by π_μ^G if the underlying $(\mathfrak{g}_{\mathbb{C}}, K)$-module of π is isomorphic to $\pi_\mu^{\mathfrak{g}}$. Let Λ_G be the totality of μ such that $\pi_\mu^{\mathfrak{g}}$ lifts to an irreducible unitary representation of G. For simply connected G, irreducible unitary highest weight representations were classified, that is, the set Λ_G $(\subset \sqrt{-1}\mathfrak{t}^*)$ was explicitly found in [12] and [30] (see also [13]). In particular, we recall from [12] that

$$\lambda(\widetilde{Z}) \in \mathbb{R} \quad \text{for any } \lambda \in \Lambda_G$$

and

$$c_G := \sup_{\lambda \in \Lambda_G} \lambda(\widetilde{Z}) < \infty \tag{8.1.4}$$

if G is semisimple.

The highest weight module $\pi_\mu^{\mathfrak{g}}$ is the unique quotient of the generalized Verma module

$$N^{\mathfrak{g}}(\mu) := U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}}+\mathfrak{p}_+)} \pi_\mu^{\mathfrak{k}}, \tag{8.1.5}$$

where $\pi_\mu^{\mathfrak{k}}$ is regarded as a module of the maximal parabolic subalgebra $\mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_+$ by making $\mathfrak{p}_+$ act trivially. Furthermore, $\pi_\mu^{\mathfrak{g}}$ has a $Z(\mathfrak{g}_{\mathbb{C}})$-infinitesimal character $\mu + \rho_{\mathfrak{g}} \in \mathfrak{t}_{\mathbb{C}}^*$ via the Harish-Chandra isomorphism

$$\mathrm{Hom}_{\mathbb{C}\text{-algebra}}(Z(\mathfrak{g}_{\mathbb{C}}), \mathbb{C}) \simeq \mathfrak{t}_{\mathbb{C}}^*/W,$$

where $Z(\mathfrak{g}_{\mathbb{C}})$ is the center of the enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$, W is the Weyl group of the root system $\Delta(\mathfrak{g}, \mathfrak{t})$, and $\rho_{\mathfrak{g}}$ is half the sum of positive roots $\Delta^+(\mathfrak{g}, \mathfrak{t}) := \Delta^+(\mathfrak{k}, \mathfrak{t}) \cup \Delta(\mathfrak{p}_+, \mathfrak{t})$.

8.2 Strongly orthogonal roots

Let G be a non-compact simple Lie group of Hermitian type, and τ an involution of holomorphic type which commutes with the Cartan involution θ.

We take a Cartan subalgebra $\mathfrak{t}^\tau$ of the reductive Lie algebra

$$\mathfrak{k}^\tau := \{X \in \mathfrak{k} : \tau X = X\}$$

and extend it to a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$. We note that $\mathfrak{t}^\tau = \mathfrak{k}^\tau \cap \mathfrak{t}$. The pair $(\mathfrak{k}, \mathfrak{k}^\tau)$ forms a reductive symmetric pair, and $\mathfrak{t}$ plays an analogous role to the fundamental Cartan subalgebra with respect to this symmetric pair. Thus, using the same argument as in [84], we see that if $\alpha \in \Delta(\mathfrak{k}, \mathfrak{t})$ satisfies $\alpha|_{\mathfrak{t}^\tau} = 0$ then $\alpha = 0$. Thus, we can take positive systems $\Delta^+(\mathfrak{k}, \mathfrak{t})$ and $\Delta^+(\mathfrak{k}^\tau, \mathfrak{t}^\tau)$ in a compatible way such that

$$\alpha|_{\mathfrak{t}^\tau} \in \Delta^+(\mathfrak{k}^\tau, \mathfrak{t}^\tau) \quad \text{if } \alpha \in \Delta^+(\mathfrak{k}, \mathfrak{t}). \tag{8.2.1}$$

Since τ is of holomorphic type, we have $\tau Z = Z$, and therefore $\tau \mathfrak{p}_+ = \mathfrak{p}_+$. Hence, we have a direct sum decomposition $\mathfrak{p}_+ = \mathfrak{p}_+^\tau \oplus \mathfrak{p}_+^{-\tau}$, where we set

$$\mathfrak{p}_+^{\pm\tau} := \{X \in \mathfrak{p}_+ : \tau X = \pm X\}.$$

Let us consider the reductive subalgebra $\mathfrak{g}^{\tau\theta}$. Its Cartan decomposition is given by

$$\mathfrak{g}^{\tau\theta} = (\mathfrak{g}^{\tau\theta} \cap \mathfrak{g}^\theta) + (\mathfrak{g}^{\tau\theta} \cap \mathfrak{g}^{-\theta}) = \mathfrak{k}^\tau + \mathfrak{p}^{-\tau},$$

and its complexification is given by

$$\mathfrak{g}_\mathbb{C}^{\tau\theta} = \mathfrak{k}_\mathbb{C}^\tau \oplus \mathfrak{p}_+^{-\tau} \oplus \mathfrak{p}_-^{-\tau}. \tag{8.2.2}$$

The Cartan subalgebra $\mathfrak{t}^\tau$ of $\mathfrak{k}^\tau$ is also a Cartan subalgebra of $\mathfrak{g}^{\tau\theta}$.

Let $\Delta(\mathfrak{p}_+^{-\tau}, \mathfrak{t}^\tau)$ be the set of weights of $\mathfrak{p}_+^{-\tau}$ with respect to $\mathfrak{t}^\tau$. The roots α and β are said to be *strongly orthogonal* if neither $\alpha + \beta$ nor $\alpha - \beta$ is a root. We take a maximal set of strongly orthogonal roots $\{\nu_1, \nu_2, \dots, \nu_l\}$ in $\Delta(\mathfrak{p}_+^{-\tau}, \mathfrak{t}^\tau)$ such that

   i) ν_1 is the lowest root among the elements in $\Delta(\mathfrak{p}_+^{-\tau}, \mathfrak{t}^\tau)$,

   ii) ν_{j+1} is the lowest root among the elements in $\Delta(\mathfrak{p}_+^{-\tau}, \mathfrak{t}^\tau)$ that are strongly orthogonal to $\nu_1, \dots, \nu_j$.

A special case applied to $\tau = \theta$ shows $\mathfrak{k}^\tau = \mathfrak{k}$, $\mathfrak{t}^\tau = \mathfrak{t}$, $\mathfrak{p}^{-\tau} = \mathfrak{p}$, and $\Delta(\mathfrak{p}_+^{-\tau}, \mathfrak{t}^\tau) = \Delta(\mathfrak{p}_+, \mathfrak{t})$. In this case, we shall use the notation $\{\bar\nu_1, \bar\nu_2, \dots, \bar\nu_{\bar l}\}$ for a maximal set of strongly orthogonal roots in $\Delta(\mathfrak{p}_+, \mathfrak{t})$ such that

   i) $\bar\nu_1$ is the lowest root among $\Delta(\mathfrak{p}_+, \mathfrak{t})$,
   ii) $\bar\nu_{j+1}$ is the lowest root among the elements in $\Delta(\mathfrak{p}_+, \mathfrak{t})$ that are strongly orthogonal to $\bar\nu_1, \dots, \bar\nu_j$ $(1 \le j \le \bar l)$.

Then, $\bar l = \mathbb{R}$-rank $\mathfrak{g}$ by [57]. Likewise, in light of (8.2.2) for the Hermitian symmetric space $G^{\tau\theta}/G^{\tau\theta} \cap K = G^{\tau\theta}/G^{\tau,\theta}$, we have $l = \mathbb{R}$-rank $\mathfrak{g}^{\tau\theta}$. In general, $l \le \bar l$.

8.3 Branching laws for semisimple symmetric pairs

It follows from (8.1.3) that the highest weight module $\pi_\mu^\mathfrak{g}$ is of scalar type, namely, $(\pi_\mu^\mathfrak{g})^{\mathfrak{p}+}$ is one dimensional, if and only if

$$\langle \mu, \alpha \rangle = 0 \qquad \text{for any } \alpha \in \Delta(\mathfrak{k}, \mathfrak{t}). \tag{8.3.1}$$

Furthermore, the representation π_μ^G is a (relative) holomorphic discrete series representation of G if and only if

$$\langle \mu + \rho_\mathfrak{g}, \alpha \rangle < 0 \qquad \text{for any } \alpha \in \Delta(\mathfrak{p}_+, \mathfrak{t}). \tag{8.3.2}$$

We are now ready to state the branching law of holomorphic discrete series representations π_μ^G of scalar type with respect to semisimple symmetric pairs (G, H):

Theorem 8.3. *Let G be a non-compact simple Lie group of Hermitian type. Assume that $\mu \in \sqrt{-1}\,\mathfrak{t}^*$ satisfies (8.3.1) and (8.3.2). Let τ be an involutive automorphism of G of holomorphic type, $H = G_0^\tau$ (the identity component of G^τ), and $\{\nu_1, \ldots, \nu_l\}$ be the set of strongly orthogonal roots in $\Delta(\mathfrak{p}_+^{-\tau}, \mathfrak{t}^\tau)$ as in Subsection 8.2. Then, π_μ^G decomposes discretely into a multiplicity-free sum of irreducible H-modules:*

$$\pi_\mu^G|_H \simeq \sum_{\substack{a_1 \geq \cdots \geq a_l \geq 0 \\ a_1, \ldots, a_l \in \mathbb{N}}}^{\oplus} \pi_{\mu|_{\mathfrak{t}^\tau} - \sum_{j=1}^l a_j \nu_j}^H \quad \textit{(discrete Hilbert sum)}. \qquad (8.3.3)$$

The formula for the case $H = K$ (that is, $\tau = \theta$) was previously found by L.-K. Hua (implicit in the classical case), B. Kostant (unpublished) and W. Schmid [78] (see also Johnson [32] for an algebraic proof). In this case, each summand in the right side is finite dimensional.

For $\tau \neq \theta$, some special cases have been also studied by H. Jakobsen, M. Vergne, J. Xie, W. Bertram and J. Hilgert [7, 30, 31, 89]. Further, quantitative results by means of reproducing kernels are obtained in [5]. The formula (8.3.3) in the above generality was first given by the author [39].

We shall give a proof of Theorem 8.3 in Subsection 8.8.

8.4 Irreducible decomposition of tensor products

As we saw in Example 3.2.1, the pair $(G \times G, \operatorname{diag}(G))$ forms a symmetric pair. Correspondingly, the tensor product representation can be regarded as a special (and easy) case of restrictions of representations with respect to symmetric pairs. This subsection provides a decomposition formula of the tensor product of two holomorphic discrete series representations of scalar type. This is regarded as a counterpart of Theorem 8.3 for tensor product representations.

We recall from Subsection 8.2 that $\{\bar{\nu}_1, \ldots, \bar{\nu}_{\bar{l}}\}$ is a maximal set of strongly orthogonal roots in $\Delta(\mathfrak{p}_+, \mathfrak{t})$ and $\bar{l} = \mathbb{R}\text{-}\operatorname{rank}\mathfrak{g}$.

Theorem 8.4. *Let G be a non-compact simple Lie group of Hermitian type. Assume that $\mu_1, \mu_2 \in \sqrt{-1}\,\mathfrak{t}^*$ satisfy the conditions (8.3.1) and (8.3.2). Then, the tensor product representation $\pi_{\mu_1}^G \widehat{\otimes} \pi_{\mu_2}^G$ decomposes discretely into a multiplicity-free sum of irreducible G-modules:*

$$\pi_{\mu_1}^G \widehat{\otimes} \pi_{\mu_2}^G \simeq \sum_{\substack{a_1 \geq \cdots \geq a_{\bar{l}} \geq 0 \\ a_1, \ldots, a_{\bar{l}} \in \mathbb{N}}} \pi_{\mu_1 + \mu_2 - \sum_{j=1}^{\bar{l}} a_j \bar{\nu}_j}^G.$$

The proof of Theorem 8.4 will be given in Subsection 8.9.

8.5 Eigenvalues of the central element Z

Our proof of Theorems 8.3 and 8.4 depends on the algebraic lemma that the K-type formula determines the irreducible decomposition of the whole group (see Lemma 8.7). This is a very strong assertion, which fails in general for non-highest weight modules. This subsection collects some nice properties peculiar to highest weight modules that will be used in the proof of Lemma 8.7.

For a K-module V, we define a subset of $\mathbb{C}$ by

$$\mathrm{Spec}_{\widetilde{Z}}(V) := \{\text{eigenvalues of } \widetilde{Z} \text{ on } V\},$$

where we set

$$\widetilde{Z} := \frac{1}{\sqrt{-1}} Z.$$

For instance, $\mathrm{Spec}_{\widetilde{Z}}(V)$ is a singleton if V is an irreducible K-module. We also note that $\mathrm{Spec}_{\widetilde{Z}}(\mathfrak{g}_{\mathbb{C}}) = \{0, \pm 1\}$ by (8.1.1).

Lemma 8.5. *Suppose V is an irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$-module. Then,*
1) $\mathrm{Spec}_{\widetilde{Z}}(V) \subset a_0 + \mathbb{Z}$ *for some $a_0 \in \mathbb{C}$.*
2) *If $\sup \mathrm{Re}\, \mathrm{Spec}_{\widetilde{Z}}(V) < \infty$, then V is a highest weight module.*
3) *If V is a highest weight module $\pi_\lambda^{\mathfrak{g}}$, then $\mathrm{Spec}_{\widetilde{Z}}(V) \subset -\mathbb{N} + \lambda(\widetilde{Z})$ and $\sup \mathrm{Re}\, \mathrm{Spec}_{\widetilde{Z}}(V) = \mathrm{Re}\, \lambda(\widetilde{Z})$.*
4) *If V is a unitary highest weight module, then $\mathrm{Spec}_{\widetilde{Z}}(V) \subset (-\infty, c_G]$, where c_G is a constant depending on G.*
5) *If both V and F are highest weight modules of finite length, then any irreducible subquotient W of $V \otimes F$ is also a highest weight module.*

Proof. 1) For $a \in \mathbb{C}$, we write the eigenspace of $\widetilde{Z}$ as $V_a := \{v \in V : \widetilde{Z}v = av\}$. Then, it follows from the Leibniz rule that

$$\mathfrak{p}_+ V_a \subset V_{a+1}, \quad \mathfrak{k}_{\mathbb{C}} V_a \subset V_a, \quad \text{and} \quad \mathfrak{p}_- V_a \subset V_{a-1}.$$

An iteration of this argument shows that

$$\mathrm{Spec}_{\widetilde{Z}}(U(\mathfrak{g}_{\mathbb{C}})V_a) \subset a + \mathbb{Z}.$$

Now we take a_0 such that $V_{a_0} \neq \{0\}$. Since V is irreducible, we have $V = U(\mathfrak{g}_{\mathbb{C}})V_{a_0}$, and therefore $\mathrm{Spec}_{\widetilde{Z}}(V) \subset a_0 + \mathbb{Z}$.
2) Suppose $\sup \mathrm{Re}\, \mathrm{Spec}_{\widetilde{Z}}(V) < \infty$. Since $\mathrm{Re}\, \mathrm{Spec}_{\widetilde{Z}}(V)$ is discrete by (1), there exists $a \in \mathrm{Spec}_{\widetilde{Z}}(V)$ such that $\mathrm{Re}\, a$ attains its maximum. Then

$$\mathfrak{p}_+ V_a \subset V_{a+1} = \{0\}.$$

Thus, $V_a \subset V^{\mathfrak{p}_+}$. Hence, V is a highest weight module.
3) The highest weight module $\pi_\lambda^{\mathfrak{g}}$ is isomorphic to the unique irreducible quotient of the generalized Verma module $N^{\mathfrak{g}}(\lambda) = U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_+)} \pi_\lambda^{\mathfrak{k}}$. By the

Poincaré–Birkhoff–Witt theorem, $N^{\mathfrak{g}}(\lambda)$ is isomorphic to $S(\mathfrak{p}_-) \otimes \pi_\lambda^{\mathfrak{k}}$ as a $\mathfrak{k}$-module. Thus, any $\mathfrak{k}$-type $\pi_\mu^{\mathfrak{k}}$ occurring in $\pi_\lambda^{\mathfrak{g}}$ is of the form

$$\mu = \lambda + \sum_{\alpha \in \Delta(\mathfrak{p}_-, \mathfrak{t})} m_\alpha \alpha$$

for some $m_\alpha \in \mathbb{N}$. As $\alpha(\widetilde{Z}) = -1$ for any $\alpha \in \Delta(\mathfrak{p}_-, \mathfrak{t})$, we have

$$\mu(\widetilde{Z}) = \lambda(\widetilde{Z}) - \sum_{\alpha \in \Delta(\mathfrak{p}_-, \mathfrak{t})} m_\alpha \,. \qquad (8.5.1)$$

In particular, we have the following equivalence:

$$\operatorname{Re} \mu(\widetilde{Z}) = \operatorname{Re} \lambda(\widetilde{Z}) \iff \mu = \lambda \,, \qquad (8.5.2)$$

and we also have

$$\operatorname{Spec}_{\widetilde{Z}}(\pi_\lambda^{\mathfrak{g}}) \subset \{\lambda(\widetilde{Z}) - \sum_{\alpha \in \Delta(\mathfrak{p}_-, \mathfrak{t})} m_\alpha : m_\alpha \in \mathbb{N}\} = -\mathbb{N} + \lambda(\widetilde{Z}) \,. \qquad (8.5.3)$$

Furthermore, since the $\mathfrak{k}$-type $\pi_\lambda^{\mathfrak{k}}$ occurs in $\pi_\lambda^{\mathfrak{g}}$, we have $\lambda(\widetilde{Z}) \in \operatorname{Spec}_{\widetilde{Z}}(\pi_\lambda^{\mathfrak{g}})$. Here, $\sup \operatorname{Re} \operatorname{Spec}_{\widetilde{Z}}(\pi_\lambda^{\mathfrak{g}}) = \operatorname{Re} \lambda(\widetilde{Z})$.

4) This statement follows from (8.1.4) and from (3).

5) For two subsets A and B in $\mathbb{C}$, we write $A + B := \{a + b \in \mathbb{C} : a \in A, b \in B\}$. Then, $\operatorname{Spec}_{\widetilde{Z}}(V \otimes F) \subset \operatorname{Spec}_{\widetilde{Z}}(V) + \operatorname{Spec}_{\widetilde{Z}}(F)$. Therefore,

$$\sup \operatorname{Re} \operatorname{Spec}_{\widetilde{Z}}(W) \le \sup \operatorname{Re} \operatorname{Spec}_{\widetilde{Z}}(V)$$
$$\le \sup \operatorname{Re} \operatorname{Spec}_{\widetilde{Z}}(V) + \sup \operatorname{Re} \operatorname{Spec}_{\widetilde{Z}}(F) < \infty.$$

Hence, W is also a highest weight module by (2). $\quad\square$

8.6 Bottom layer map

The following lemma finds an irreducible summand ('bottom layer') from the K-type structure.

Lemma 8.6. *Let V be a $(\mathfrak{g}_{\mathbb{C}}, K)$-module. We assume that V decomposes into an algebraic direct sum of (possibly, infinitely many) irreducible highest weight modules. We set*

$$\operatorname{Supp}_{\mathfrak{k}}(V) := \{\mu \in \sqrt{-1}\mathfrak{t}^* : \operatorname{Hom}_{\mathfrak{k}}(\pi_\mu^{\mathfrak{k}}, V) \ne \{0\}\} \,.$$

If the evaluation map

$$\operatorname{Supp}_{\mathfrak{k}}(V) \to \mathbb{R}, \quad \mu \mapsto \operatorname{Re} \mu(\widetilde{Z})$$

attains its maximum at μ_0, then

$$\operatorname{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}(\pi_{\mu_0}^{\mathfrak{g}}, V) \ne \{0\} \,.$$

Proof. Take a non-zero map $q \in \operatorname{Hom}_{\mathfrak{k}}(\pi_{\mu_0}^{\mathfrak{k}}, V)$. As V is an algebraic direct sum of irreducible highest weight modules, there exists a projection $p : V \to \pi_\lambda^{\mathfrak{g}}$ for some λ such that $p \circ q \neq 0$. This means that $\pi_{\mu_0}^{\mathfrak{k}}$ occurs in $\pi_\lambda^{\mathfrak{g}}$, and therefore we have

$$\operatorname{Re}\mu_0(\widetilde{Z}) \leq \sup \operatorname{Re} \operatorname{Spec}_{\widetilde{Z}}(\pi_\lambda^{\mathfrak{g}}) = \operatorname{Re}\lambda(\widetilde{Z})\,.$$

Here, the last equality is by Lemma 8.5 (3).

Conversely, the maximality of μ_0 implies that $\operatorname{Re}\mu_0(\widetilde{Z}) \geq \operatorname{Re}\lambda(\widetilde{Z})$. Hence, $\operatorname{Re}\mu_0(\widetilde{Z}) = \operatorname{Re}\lambda(\widetilde{Z})$, and we have then $\mu_0 = \lambda$ by (8.5.2). Since $\pi_\lambda^{\mathfrak{g}}$ is an irreducible summand of V, we have $\operatorname{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}(\pi_{\mu_0}^{\mathfrak{g}}, V) \neq \{0\}$. $\square$

8.7 Determination of the $\mathfrak{g}_{\mathbb{C}}$-structure by K-types

In general, the K-type formula is not sufficient to determine the irreducible decomposition of a unitary representation even in the discretely decomposable case. However, this is the case if any irreducible summand is a highest weight module. Here is the statement that we shall use as a main machinery of the proof of Theorems 8.3 and 8.4.

Lemma 8.7. *Suppose $(\pi, \mathcal{H})$ is a K-admissible unitary representation of G, which splits discretely into a Hilbert direct sum of irreducible unitary highest weight representations of G. Let $\mathcal{H}_K$ be the space of K-finite vectors of $\mathcal{H}$. Assume that there exists a function $n_\pi : \mathfrak{t}_{\mathbb{C}}^* \to \mathbb{N}$ such that $\mathcal{H}_K$ is isomorphic to the following direct sum as $\mathfrak{k}$-modules:*

$$\mathcal{H}_K \simeq \bigoplus_\lambda n_\pi(\lambda)\pi_\lambda^{\mathfrak{g}} \quad \textit{(algebraic direct sum).} \tag{8.7.1}$$

Then, $n_\pi(\lambda) \neq 0$ only if $\lambda \in \Lambda_G$, that is, $\pi_\lambda^{\mathfrak{g}}$ lifts to an irreducible unitary representation π_λ^G of G. Furthermore, the identity (8.7.1) holds as a $(\mathfrak{g}_{\mathbb{C}}, K)$-module isomorphism, and the unitary representation π has the following decomposition into irreducible unitary representations of G:

$$\pi \simeq \sum_\lambda^\oplus n_\pi(\lambda)\pi_\lambda^G \quad \textit{(discrete Hilbert sum).} \tag{8.7.2}$$

Proof. We write an abstract irreducible decomposition of $\mathcal{H}$ as

$$\mathcal{H} \simeq \sum_{\lambda \in \Lambda_G}^\oplus m_\lambda \pi_\lambda^G \quad \text{(discrete Hilbert sum).}$$

Since $\mathcal{H}$ is K-admissible, the multiplicity $m_\lambda < \infty$ for all λ, and we have an isomorphism of $(\mathfrak{g}_{\mathbb{C}}, K)$-modules with the same multiplicity m_λ (see [43, Theorem 2.7]):

$$\mathcal{H}_K \simeq \bigoplus_{\lambda \in \Lambda_G} m_\lambda \pi_\lambda^{\mathfrak{g}} \quad \text{(algebraic direct sum).} \tag{8.7.3}$$

Let us show $n_\pi(\lambda) = m_\lambda$ for all λ. For this, we begin with an observation that

$$\mathrm{Spec}_{\widetilde{Z}}(\mathcal{H}_K) = \bigcup_{\substack{\lambda \text{ such that} \\ m_\lambda \neq 0}} \mathrm{Spec}_{\widetilde{Z}}(\pi_\lambda^{\mathfrak{g}})$$

is a subset in $\mathbb{R}$ and has an upper bound. This follows from Lemma 8.5 (4) applied to each irreducible summand in (8.7.3).

First, we consider the case where there exists $a \in \mathbb{R}$ such that

$$\lambda(\widetilde{Z}) \equiv a \bmod \mathbb{Z} \quad \text{for any } \lambda \text{ satisfying } n_\pi(\lambda) \neq 0. \tag{8.7.4}$$

Then, the set

$$\{\lambda(\widetilde{Z}) : \lambda \in \mathfrak{t}_{\mathbb{C}}^*, n_\pi(\lambda) \neq 0\} \tag{8.7.5}$$

is contained in $\mathrm{Spec}_{\widetilde{Z}}(\mathcal{H}_K)$ by (8.7.1), and is discrete by (8.7.4). Hence, it is a discrete subset of $\mathbb{R}$ with an upper bound. Thus, we can find $\mu_0 \in \mathfrak{t}_{\mathbb{C}}^*$ such that $n_\pi(\mu_0) \neq 0$ and that $\mu_0(\widetilde{Z})$ attains its maximum in (8.7.5). In turn, the evaluation map $\mathrm{Supp}_{\mathfrak{t}}(\mathcal{H}_K) \to \mathbb{R}$, $\mu \mapsto \mu(\widetilde{Z})$ attains its maximum at $\mu_0 \in \mathrm{Supp}_{\mathfrak{t}}(\mathcal{H}_K)$ by (8.7.1) and Lemma 8.5 (3). Therefore, $\mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi_{\mu_0}^{\mathfrak{g}}, \mathcal{H}_K) \neq \{0\}$ by Lemma 8.6. Thus, we have shown $m_{\mu_0} \neq 0$, that is, $\pi_{\mu_0}^G$ occurs as a subrepresentation in $\mathcal{H}$.

Next, we consider the unitary representation π' on

$$\mathcal{H}' := \sum_{\lambda \neq \mu_0}^{\oplus} m_\lambda \pi_\lambda^G \oplus (m_{\mu_0} - 1)\pi_{\mu_0}^G,$$

the orthogonal complement of a subrepresentation $\pi_{\mu_0}^G$ in $\mathcal{H}$. Then, the K-type formula (8.7.1) for $(\pi', \mathcal{H}')$ holds if we set

$$n_{\pi'}(\lambda) := \begin{cases} n_\pi(\lambda) - 1 & (\lambda = \mu_0), \\ n_\pi(\lambda) & (\lambda \neq \mu_0). \end{cases}$$

Hence, by the downward induction on $\sup \mathrm{Spec}_{\widetilde{Z}}(\mathcal{H}_K)$, we have $n_\pi(\lambda) = m_\lambda$ for all λ.

For the general case, let A be the set of complete representatives of $\{\lambda(\widetilde{Z}) \in \mathbb{C} \bmod \mathbb{Z} : n_\pi(\lambda) \neq 0\}$. For each $a \in A$, we define a subrepresentation $\mathcal{H}_a$ of $\mathcal{H}$ by

$$\mathcal{H}_a := \sum_{\lambda(\widetilde{Z}) \equiv a \bmod \mathbb{Z}}^{\oplus} m_\lambda \pi_\lambda^G \quad \text{(discrete Hilbert sum)}.$$

Then, we have an isomorphism of unitary representations of G:

$$\mathcal{H} \simeq \sum_{a \in A}^{\oplus} \mathcal{H}_a.$$

Since $\mathrm{Spec}_{\widetilde{Z}}(\pi_\lambda^{\mathfrak{g}}) \subset a + \mathbb{Z}$ if and only if $\lambda(\widetilde{Z}) \equiv a \bmod \mathbb{Z}$ by Lemma 8.5 (3), we get from (8.7.1) the following K-isomorphism

$$(\mathcal{H}_a)_K \simeq \bigoplus_{\lambda(\widetilde{Z}) \equiv a \bmod \mathbb{Z}} n_\pi(\lambda)\pi_\lambda^{\mathfrak{g}} \qquad (8.7.6)$$

for each $a \in A$. Therefore, our proof for the first step assures $n_\pi(\lambda) = m_\lambda$ for any λ such that $\lambda \equiv a \bmod \mathbb{Z}$. Since $a \in A$ is arbitrary, we obtain Lemma in the general case. $\square$

8.8 Proof of Theorem 8.3

In this section, we give a proof of Theorem 8.3. This is done by showing a more general formula in Lemma 8.8 without the scalar type assumption (8.3.1). Then, Theorem 8.3 follows readily from Lemma 8.8 because the assumption (8.3.1) makes $\dim \pi_\mu^{\mathfrak{k}} = 1$ and $\mathbb{S}_{(a_1,\ldots,a_l)}(\mu) = \{\mu - \sum_{j=1}^l a_j\nu_j\}$ (see (8.8.1) for notation).

For a discussion below, it is convenient to use the concept of a multiset. Intuitively, a multiset is a set counted with multiplicities; for example, $\{a, a, a, b, c, c\}$. More precisely, a multiset $\mathbb{S}$ consists of a set S and a function $m : S \to \{0, 1, 2, \ldots, \infty\}$. If $\mathbb{S}' = \{S, m'\}$ is another multiset on S such that $m'(x) \le m(x)$ for all $x \in S$, we say $\mathbb{S}'$ is a *submultiset* of $\mathbb{S}$ and write $\mathbb{S}' \subset \mathbb{S}$.

Suppose we are in the setting of Subsection 8.2 and recall τ is an involution of holomorphic type. For a $\Delta^+(\mathfrak{k}, \mathfrak{t})$-dominant weight μ, we introduce a multiset $\mathbb{S}(\mu)$ consisting of $\Delta^+(\mathfrak{k}^\tau, \mathfrak{t}^\tau)$-dominant weights:

$$\mathbb{S}(\mu) := \bigcup_{\substack{a_1 \ge \cdots \ge a_l \ge 0 \\ a_1,\ldots,a_l \in \mathbb{N}}} \mathbb{S}_{(a_1,\ldots,a_l)}(\mu),$$

where we define the multiset $\mathbb{S}_{(a_1,\ldots,a_l)}(\mu)$ by

$$\{\text{highest weight of irreducible } \mathfrak{k}^\tau\text{-modules occurring in} \atop \pi_{-\sum_{j=1}^l a_j\nu_j}^{\mathfrak{k}^\tau} \otimes \pi_\mu^{\mathfrak{k}}|_{\mathfrak{k}^\tau} \text{ counted with multiplicities}\}. \qquad (8.8.1)$$

Because the central element $\widetilde{Z} = \frac{1}{\sqrt{-1}}Z$ of $\mathfrak{k}_\mathbb{C}$ acts on the irreducible representation $\pi_\mu^{\mathfrak{k}}$ by the scalar $\mu(\widetilde{Z})$ and because $\nu_j(\widetilde{Z}) = 1$ for all j ($1 \le j \le l$), any element ν in $\mathbb{S}_{(a_1,\ldots,a_l)}(\mu)$ satisfies $\nu(\widetilde{Z}) = -\sum_{j=1}^l a_j + \mu(\widetilde{Z})$. Therefore, the multiplicity of each element of the multiset $\mathbb{S}(\mu)$ is finite.

Lemma 8.8. *Let τ be an involution of G of holomorphic type, and $H = G_0^\tau$. If π_μ^G is a (relative) holomorphic discrete series representation of G, then it decomposes discretely into irreducible H-modules as:*

$$\pi_\mu^G|_H \simeq \sum_{\nu \in \mathbb{S}(\mu)}^{\oplus} \pi_\nu^H \quad \text{(discrete Hilbert sum).}$$

Proof of Lemma 8.8. It follows from Fact 5.1 (1) that π_μ^G is $(H \cap K)$-admissible, and splits discretely into a Hilbert direct sum of irreducible unitary representations of H.

Applying Lemma 8.7 to $H = G_0^\tau$, we see that Lemma 8.8 is deduced from the following $\mathfrak{k}^\tau$-isomorphism:

$$\pi_\mu^{\mathfrak{g}} \simeq \bigoplus_{\nu \in \mathbb{S}(\mu)} \pi_\nu^{\mathfrak{g}^\tau} \quad \text{(algebraic direct sum)}. \tag{8.8.2}$$

The rest of the proof is devoted to showing (8.8.2).

Since π_μ^G is a holomorphic discrete series, $\pi_\mu^{\mathfrak{g}}$ is isomorphic to the generalized Verma module $N^{\mathfrak{g}}(\mu) = U(\mathfrak{g}_\mathbb{C}) \otimes_{U(\mathfrak{k}_\mathbb{C} + \mathfrak{p}_+)} \pi_\mu^{\mathfrak{k}}$ as a $\mathfrak{g}$-module, which in turn is isomorphic to the $\mathfrak{k}$-module $S(\mathfrak{p}_-) \otimes \pi_\mu^{\mathfrak{k}}$.

According to the decomposition $\mathfrak{p}_- = \mathfrak{p}_-^\tau \oplus \mathfrak{p}_-^{-\tau}$ as $\mathfrak{k}^\tau$-modules, we have then the following $\mathfrak{k}^\tau$-isomorphism:

$$\pi_\mu^{\mathfrak{g}} \simeq S(\mathfrak{p}_-) \otimes \pi_\mu^{\mathfrak{k}} \simeq S(\mathfrak{p}_-^\tau) \otimes S(\mathfrak{p}_-^{-\tau}) \otimes \pi_\mu^{\mathfrak{k}}. \tag{8.8.3}$$

Now, we consider the Hermitian symmetric space $G^{\tau\theta}/G^{\tau,\theta}$, for which the complex structure is given by the decomposition $\mathfrak{g}_\mathbb{C}^{\tau\theta} = \mathfrak{k}_\mathbb{C}^\tau \oplus \mathfrak{p}_+^{-\tau} \oplus \mathfrak{p}_-^{-\tau}$ (see (8.2.2)). Then, the Hua–Kostant–Schmid formula ([78, Behauptung c]) applied to $G^{\tau\theta}/G^{\tau,\theta}$ decomposes the symmetric algebra $S(\mathfrak{p}_-^{-\tau})$ into irreducible $\mathfrak{k}^\tau$-modules:

$$S(\mathfrak{p}_-^{-\tau}) \simeq \bigoplus_{\substack{a_1 \geq \cdots \geq a_l \geq 0 \\ a_1, \ldots, a_l \in \mathbb{N}}} \pi_{-\sum_{j=1}^l a_j \nu_j}^{\mathfrak{k}^\tau}. \tag{8.8.4}$$

It follows from the definition of $\mathbb{S}(\mu)$ that we have the following irreducible decomposition as $\mathfrak{k}^\tau$-modules:

$$S(\mathfrak{p}_-^{-\tau}) \otimes \pi_\mu^{\mathfrak{k}} \simeq \bigoplus_{\nu \in \mathbb{S}(\mu)} \pi_\nu^{\mathfrak{k}^\tau}.$$

Combining this with (8.8.3), we get a $\mathfrak{k}^\tau$-isomorphism

$$\pi_\mu^{\mathfrak{g}} \simeq \bigoplus_{\nu \in \mathbb{S}(\mu)} S(\mathfrak{p}_-^\tau) \otimes \pi_\nu^{\mathfrak{k}^\tau}.$$

Next, we consider the Verma module $N^{\mathfrak{g}^\tau}(\nu) = U(\mathfrak{g}_\mathbb{C}^\tau) \otimes_{U(\mathfrak{k}_\mathbb{C}^\tau + \mathfrak{p}_+^\tau)} \pi_\nu^{\mathfrak{k}^\tau}$ of the subalgebra $\mathfrak{g}^\tau$. Then, $\pi_\nu^{\mathfrak{g}^\tau}$ is the unique irreducible quotient of $N^{\mathfrak{g}^\tau}(\nu)$. We shall show later that $N^{\mathfrak{g}^\tau}(\nu)$ is irreducible as a $\mathfrak{g}^\tau$-module, but at this stage we denote by $\pi_\nu^{\mathfrak{g}^\tau}, \pi_{\nu'}^{\mathfrak{g}^\tau}, \pi_{\nu''}^{\mathfrak{g}^\tau}, \ldots$ the totality of irreducible subquotient modules of $N^{\mathfrak{g}^\tau}(\nu)$. (There are at most finitely many subquotients, and all of them are highest weight modules.) Then, as $\mathfrak{k}^\tau$-modules, we have the following isomorphisms:

$$S(\mathfrak{p}_-^\tau) \otimes \pi_\nu^{\mathfrak{k}^\tau} \simeq N^{\mathfrak{g}^\tau}(\nu)$$
$$\simeq \pi_\nu^{\mathfrak{g}^\tau} \oplus \pi_{\nu'}^{\mathfrak{g}^\tau} \oplus \pi_{\nu''}^{\mathfrak{g}^\tau} \oplus \cdots.$$

Therefore, we get a $\mathfrak{k}^\tau$-isomorphism:

$$\pi_\mu^\mathfrak{g} \simeq \bigoplus_{\nu \in \mathbb{S}(\mu)} (\pi_\mu^{\mathfrak{g}^\tau} \oplus \pi_{\nu'}^{\mathfrak{g}^\tau} \oplus \pi_{\nu''}^{\mathfrak{g}^\tau} \oplus \cdots).$$

Accordingly, the restriction $\pi_\mu^G|_H$ splits discretely into irreducible unitary representations of H by Lemma 8.7:

$$\pi_\mu^G|_H \simeq \sum_{\nu \in \mathbb{S}(\mu)}^{\oplus} (\pi_\nu^H \oplus \pi_{\nu'}^H \oplus \pi_{\nu''}^H \oplus \cdots).$$

Since π_μ^G is a (relative) holomorphic discrete series representation of G, all irreducible summands in the right-hand side must be (relative) holomorphic discrete series representations of H by Fact 5.1 (1). Therefore, $N^{\mathfrak{g}^\tau}(\nu)$ is irreducible, and the other subquotients $\pi_{\nu'}^{\mathfrak{g}^\tau}, \pi_{\nu''}^{\mathfrak{g}^\tau}, \ldots$ do not appear. Hence, the $\mathfrak{k}^\tau$-structures of the both sides of (8.8.2) are the same. Thus, Lemma 8.8 is proved. $\square$

8.9 Proof of Theorem 8.4

For two irreducible representations $\pi_{\mu_1}^\mathfrak{k}$ and $\pi_{\mu_2}^\mathfrak{k}$, we define a multiset $\mathbb{S}(\mu_1, \mu_2)$ consisting of $\Delta^+(\mathfrak{k}, \mathfrak{t})$-dominant weights by

$$\mathbb{S}(\mu_1, \mu_2) := \bigcup_{\substack{a_1 \geq \cdots \geq a_{\bar{l}} \geq 0 \\ a_1, \ldots, a_{\bar{l}} \in \mathbb{N}}} \mathbb{S}_{(a_1, \ldots, a_{\bar{l}})}(\mu_1, \mu_2),$$

where $\mathbb{S}_{(a_1, \ldots, a_{\bar{l}})}(\mu_1, \mu_2)$ is the multiset consisting of highest weights of irreducible $\mathfrak{k}$-modules occurring in $\pi_{-\sum_{j=1}^{\bar{l}} a_j \bar{\nu}_j}^\mathfrak{k} \otimes \pi_{\mu_1}^\mathfrak{k} \otimes \pi_{\mu_2}^\mathfrak{k}$ counted with multiplicities.

Theorem 8.4 is derived from the following more general formula:

Lemma 8.9. *The tensor product of two (relative) holomorphic discrete series representations $\pi_{\mu_1}^G$ and $\pi_{\mu_2}^G$ decomposes as follows:*

$$\pi_{\mu_1}^G \widehat{\otimes} \pi_{\mu_2}^G \simeq \sum_{\nu \in \mathbb{S}(\mu_1, \mu_2)}^{\oplus} \pi_\nu^G.$$

Proof. We define two injective maps by:

$$\text{diag} : \mathfrak{p}_+ \to \mathfrak{p}_+ \oplus \mathfrak{p}_+, \quad X \mapsto (X, X),$$
$$\text{diag}' : \mathfrak{p}_+ \to \mathfrak{p}_+ \oplus \mathfrak{p}_+, \quad X \mapsto (X, -X).$$

It then follows that we have $\mathfrak{k}$-isomorphisms:

$$S(\mathfrak{p}_-) \otimes S(\mathfrak{p}_-) \simeq S(\mathfrak{p}_- \oplus \mathfrak{p}_-)$$
$$\simeq S(\mathrm{diag}(\mathfrak{p}_-)) \otimes S(\mathrm{diag}'(\mathfrak{p}_-))$$
$$\simeq \bigoplus_{\substack{a_1 \geq \cdots \geq a_{\bar{l}} \geq 0 \\ a_1, \ldots, a_{\bar{l}} \in \mathbb{N}}} S(\mathrm{diag}(\mathfrak{p}_-)) \otimes \pi^{\mathfrak{k}}_{-\sum_{j=1}^{\bar{l}} a_j \bar{\nu}_j}.$$

This brings us the following $\mathfrak{k}$-isomorphisms:

$$\pi^{\mathfrak{g}}_{\mu_1} \otimes \pi^{\mathfrak{g}}_{\mu_2} \simeq S(\mathfrak{p}_-) \otimes \pi^{\mathfrak{k}}_{\mu_1} \otimes S(\mathfrak{p}_-) \otimes \pi^{\mathfrak{k}}_{\mu_2}$$
$$\simeq \bigoplus_{\nu \in \mathbb{S}(\mu_1, \mu_2)} S(\mathrm{diag}(\mathfrak{p}_-)) \otimes \pi^{\mathfrak{k}}_{\nu}$$
$$\simeq \bigoplus_{\nu \in \mathbb{S}(\mu_1, \mu_2)} N^{\mathfrak{g}}_{\nu}.$$

The rest of the proof goes similarly to that of Lemma 8.8.   □

8.10 Restriction $U(p, q) \downarrow U(p-1, q)$ and $SO(n, 2) \downarrow SO(n-1, 2)$

Suppose (G, H) is a reductive symmetric pair whose complexification $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is one of the following types:

$(\mathfrak{sl}(n, \mathbb{C}), \mathfrak{gl}(n-1, \mathbb{C}))$ (or $(\mathfrak{gl}(n, \mathbb{C}), \mathfrak{gl}(1, \mathbb{C}) + \mathfrak{gl}(n-1, \mathbb{C}))$),
$(\mathfrak{so}(n, \mathbb{C}), \mathfrak{so}(n-1, \mathbb{C}))$.

As is classically known (see [83]), for compact (G, H) such as $(U(n), U(1) \times U(n-1))$ or $(SO(n), SO(n-1))$, any irreducible finite dimensional representation π of G is multiplicity-free when restricted to H. For non-compact (G, H) such as $(U(p, q), U(1) \times U(p-1, q))$ or $(SO(n, 2), SO(n-1, 2))$, an analogous theorem still holds for highest weight representations π:

Theorem 8.10. *If* $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{u}(p, q), \mathfrak{u}(1) + \mathfrak{u}(p-1, q))$ *or* $(\mathfrak{so}(n, 2), \mathfrak{so}(n-1, 2))$, *then any irreducible unitary highest weight representation of G decomposes discretely into a multiplicity-free sum of irreducible unitary highest weight representations of H.*

In contrast to Theorem A, the distinguishing feature of Theorem 8.10 is that π is not necessarily of scalar type but an arbitrary unitary highest weight module. The price to pay is that the pair (G, H) is very special. We do not give the proof here that uses the vector bundle version of Theorem 2.2 (see [49]). Instead, we give an explicit decomposition formula for holomorphic discrete series π. The proof of Theorem 8.10 for the case $(G, H) = (SO_0(n, 2), SO_0(n-1, 2))$ can be also found in Jakobsen and Vergne [31, Corollary 3.1].

8.11 Branching law for $U(p, q) \downarrow U(p-1, q)$

This subsection gives an explicit branching law of a holomorphic discrete series representation π^G_{μ} of $G = U(p, q)$ when restricted to $H = U(1) \times U(p-1, q)$.

Owing to (8.3.2), such π_μ^G is parametrized by $\mu = (\mu_1, \ldots, \mu_{p+q}) \in \mathbb{Z}^{p+q}$ satisfying

$$\mu_1 \geq \cdots \geq \mu_p, \mu_{p+1} \geq \cdots \geq \mu_{p+q}, \mu_{p+q} \geq \mu_1 + p + q.$$

Here is the formula:

Theorem 8.11 (Branching law $U(p,q) \downarrow U(p-1,q)$). *Retain the above setting. Then, the branching law of π_μ^G of the restriction to the subgroup H is multiplicity-free for any μ; it is given as follows:*

$$\pi_\mu^G|_H \simeq \sum_{a=0}^{\infty}{}^{\oplus} \sum_{\substack{\mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_p \geq \mu_p \\ \lambda_{p+1} \geq \mu_{p+1} \geq \cdots \geq \lambda_{p+q} \geq \mu_{p+q} \\ \sum_{i=1}^{q}(\lambda_{p+i} - \mu_{p+i}) = a}}{}^{\oplus} \mathbb{C}_{\sum_{i=1}^{p} \mu_i - \sum_{i=1}^{p} \lambda_i - a} \boxtimes \pi_{(\lambda_2, \ldots, \lambda_p, \lambda_{p+1}, \ldots, \lambda_{p+q})}^{U(p-1,q)}.$$

$$(8.11.1)$$

Proof. For $(G, H) \equiv (G, G^\tau) = (U(p,q), U(1) \times U(p-1, q))$, we have

$$G^{\tau\theta} \simeq U(1,q) \times U(p-1),$$

$$H \cap K \ (= K^\tau = K^{\tau\theta}) \simeq U(1) \times U(p-1) \times U(q),$$

$\mathfrak{t}^\tau = \mathfrak{t}$, and

$$\Delta^+(\mathfrak{p}_+^{-\tau}, \mathfrak{t}^\tau) = \{e_1 - e_{p+i} : 1 \leq i \leq q\}$$

by using the standard basis of $\Delta(\mathfrak{g}, \mathfrak{t}) = \{\pm(e_i - e_j) : 1 \leq i < j \leq p + q\}$. Thus, $l = \mathbb{R}\text{-}\mathrm{rank}\, G^{\tau\theta} = 1$ and $\nu_1 = e_1 - e_{p+1}$. Hence, the $\mathfrak{k}^\tau$-type formula (8.8.4) amounts to

$$S(\mathfrak{p}_-^{-\tau}) \simeq \bigoplus_{a=0}^{\infty} \pi_{-a(e_1 - e_{p+1})}^{H \cap K}$$

$$\simeq \bigoplus_{a=0}^{\infty} \mathbb{C}_{-a} \boxtimes \mathbf{1} \boxtimes \pi_{(a,0,\ldots,0)}^{U(q)} \qquad (8.11.2)$$

as $H \cap K \simeq U(1) \times U(p-1) \times U(q)$ modules. Here, $\mathbf{1}$ denotes the trivial one dimensional representation of $U(p-1)$.

On the other hand, we recall a classical branching formula $U(p) \downarrow U(p-1)$:

$$\pi_{(\mu_1, \ldots, \mu_p)}^{U(p)}|_{U(1) \times U(p-1)} \simeq \bigoplus_{\mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_p \geq \mu_p} \mathbb{C}_{\sum_{i=1}^{p} \mu_i - \sum_{i=2}^{p} \lambda_i} \otimes \pi_{(\lambda_2, \ldots, \lambda_p)}^{U(p-1)},$$

whereas the classical Pieri rule says

$$\pi_{(a,0,\ldots,0)}^{U(q)} \otimes \pi_{(\mu_{p+1}, \ldots, \mu_{p+q})}^{U(q)} \simeq \bigoplus_{\substack{\lambda_{p+1} \geq \mu_{p+1} \geq \cdots \geq \lambda_{p+q} \geq \mu_{p+q} \\ \sum_{i=1}^{q}(\lambda_{p+i} - \mu_{p+i}) = a}} \pi_{(\lambda_{p+1}, \ldots, \lambda_{p+q})}^{U(q)}.$$

These two formulae together with (8.11.2) yield the following $\mathfrak{k}^\tau$-isomorphisms:

$$S(\mathfrak{p}_-^{-\tau}) \otimes \pi_\mu^{\mathfrak{k}}|_{\mathfrak{k}^\tau}$$

$$\simeq \bigoplus_{a=0}^\infty ((\mathbb{C}_{-a} \boxtimes \mathbf{1}) \otimes \pi_{(\mu_1,\ldots,\mu_p)}^{U(p)}|_{U(1)\times U(p-1)}) \boxtimes (\pi_{(a,0,\ldots,0)}^{U(q)} \otimes \pi_{(\mu_{p+1},\ldots,\mu_{p+q})}^{U(q)})$$

$$\simeq \bigoplus_{a=0}^\infty \bigoplus_{\substack{\mu_1\geq\lambda_2\geq\mu_2\geq\cdots\geq\lambda_p\geq\mu_p \\ \lambda_{p+1}\geq\mu_{p+1}\geq\cdots\geq\lambda_{p+q}\geq\mu_{p+q} \\ \sum_{i=1}^q(\lambda_{p+i}-\mu_{p+i})=a}} \mathbb{C}_{\sum_{i=1}^p \mu_i - \sum_{i=2}^p \lambda_i - a} \boxtimes \pi_{(\lambda_2,\ldots,\lambda_p)}^{U(p-1)} \boxtimes \pi_{(\lambda_{p+1},\ldots,\lambda_{p+q})}^{U(q)} .$$

In view of the $\mathfrak{k}^\tau$-isomorphisms

$$\pi_\mu^{\mathfrak{g}} \simeq S(\mathfrak{p}_-^\tau) \otimes S(\mathfrak{p}_-^{-\tau}) \otimes \pi_\mu^{\mathfrak{k}}|_{\mathfrak{k}^\tau}$$

and $N^{\mathfrak{g}^\tau}(\nu) \simeq S(\mathfrak{p}_-^\tau) \otimes \pi_\nu^{\mathfrak{k}^\tau}$, we have now shown that the $\mathfrak{k}^\tau$-structure of $\pi_\mu^{\mathfrak{g}}$ coincides with that of

$$\bigoplus_{a=0}^\infty \bigoplus_{\substack{\mu_1\geq\lambda_2\geq\mu_2\geq\cdots\geq\lambda_p\geq\mu_p \\ \lambda_{p+1}\geq\mu_{p+1}\geq\cdots\geq\lambda_{p+q}\geq\mu_{p+q} \\ \sum_{i=1}^q(\lambda_{p+i}-\mu_{p+i})=a}} N^{\mathfrak{g}^\tau}(\sum_{i=1}^p \mu_i - \sum_{i=2}^p \lambda_i - a, \lambda_2,\ldots,\lambda_{p+q}) .$$

As in the last part of the proof of Theorem 8.3, we see that any generalized Verma module occurring in the right-hand side is irreducible (and is isomorphic to the underlying $(\mathfrak{g}_\mathbb{C}^\tau, H\cap K)$-module of a holomorphic discrete series of H). Therefore, Theorem follows from Lemma 8.7. $\square$

9 Appendix: Associated Bundles on Hermitian Symmetric Spaces

In this Appendix, we explain standard operations on homogeneous vector bundles. The results are well-known and elementary, but we recall them briefly for the convenience of the reader. The main goal is Lemma 9.4 which is used to verify the condition (2.2.2) in Theorem 2.2.

9.1 Homogeneous vector bundles

Let M be a real manifold, and V a (finite dimensional) vector space over $\mathbb{C}$. Suppose that we are given an open covering $\{U_\alpha\}$ of M and transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_\mathbb{C}(V)$$

satisfying the following compatibility conditions:

$$g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} \equiv \mathrm{id} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma ; \qquad g_{\alpha\alpha} \equiv \mathrm{id} \quad \text{on } U_\alpha .$$

A complex vector bundle $\mathcal{V}$ over M with typical fiber V is constructed as the equivalence class of $\coprod_\alpha (U_\alpha \times V)$, where $(x,v) \in U_\beta \times V$ and $(y,w) \in U_\alpha \times V$ are defined to be equivalent if $y = x$ and $w = g_{\alpha\beta}(x)v$. Then, the space of sections $\Gamma(M, \mathcal{V})$ is identified with the collection

$$\{(f_\alpha) : f_\alpha \in C^\infty(U_\alpha, V),\ f_\alpha(x) = g_{\alpha\beta}(x) f_\beta(x),\ \text{for } x \in U_\alpha \cap U_\beta\}. \quad (9.1.1)$$

If M is a complex manifold and if every $g_{\alpha\beta}$ is holomorphic (or anti-holomorphic), then $\mathcal{V} \to M$ is a holomorphic (or anti-holomorphic, respectively) vector bundle.

Next, let G be a Lie group, K a closed subgroup of G, and $M := G/K$ the homogeneous manifold. Then, we can take an open covering $\{U_\alpha\}$ of M such that for each α there is a local section $\varphi_\alpha : U_\alpha \to G$ of the principal bundle $G \to G/K$. Given a representation $\chi : K \to GL_\mathbb{C}(V)$, we define the homogeneous vector bundle, $\mathcal{V} := G \times_K (\chi, V)$. Then $\mathcal{V}$ is associated with the transition functions:

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_\mathbb{C}(V), \quad g_{\alpha\beta}(x) := \chi(\varphi_\alpha(x)^{-1} \varphi_\beta(x)).$$

The section space $\Gamma(M, \mathcal{V})$ is identified with the following subspace of $C^\infty(G, V)$:

$$\{f \in C^\infty(G, V) : f(gk) = \chi^{-1}(k) f(g),\ \text{for } g \in G, k \in K\}. \quad (9.1.2)$$

9.2 Pull-back of vector bundles

Let G' be a Lie group, K' a closed subgroup of G', and $M' := G'/K'$ the homogeneous manifold. Suppose that $\sigma : G' \to G$ is a Lie group homomorphism such that $\sigma(K') \subset K$. We use the same letter σ to denote by the induced map $M' \to M$, $g'K' \mapsto \sigma(g')K$. Then the pull-back of the vector bundle $\mathcal{V} \to M$, denoted by $\sigma^* \mathcal{V} \to M'$, is associated to the representation

$$\chi \circ \sigma : K' \to GL_\mathbb{C}(V).$$

Then we have a commuting diagram of the pull-back of sections (see (9.1.2)):

$$\sigma^* : \ \Gamma(M, \mathcal{V}) \ \to \ \Gamma(M', \sigma^* \mathcal{V}), \quad (f_\alpha)_\alpha \mapsto (f_\alpha \circ \sigma)_\alpha,$$
$$\cap \qquad\qquad \cap$$
$$\sigma^* : C^\infty(G, V) \to \ C^\infty(G', V), \qquad f \quad \mapsto \quad f \circ \sigma.$$

9.3 Push-forward of vector bundles

Suppose that V and W are complex vector spaces and that $\xi : V \to W$ is an anti-linear bijective map. Then, we have an anti-holomorphic group isomorphism

$$GL_\mathbb{C}(V) \to GL_\mathbb{C}(W), \quad g \mapsto g^\xi := \xi \circ g \circ \xi^{-1}.$$

Let $\mathcal{V} \to M$ be a complex vector bundle with transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_\mathbb{C}(V)$. Then, one constructs a complex vector bundle $\xi_* \mathcal{V} \to M$ with the transition functions $g_{\alpha\beta}^\xi : U_\alpha \cap U_\beta \to GL_\mathbb{C}(W)$. We have a natural homomorphism

$$\xi_* : \Gamma(M, \mathcal{V}) \to \Gamma(M, \xi_* \mathcal{V}), \quad (f_\alpha) \mapsto (\xi \circ f_\alpha),$$

which is well-defined because the compatibility condition in (9.1.1) is satisfied as follows: If $x \in U_\alpha \cap U_\beta$ then

$$g_{\alpha\beta}^\xi(x)(\xi \circ f_\beta)(x) = (\xi \circ g_{\alpha\beta}(x) \circ \xi^{-1})(\xi \circ f_\beta)(x) = \xi \circ g_{\alpha\beta}(x)f_\beta(x) = \xi \circ f_\alpha(x).$$

If $\mathcal{V}$ is the homogeneous vector bundle $G \times_K (\chi, V)$ associated to a representation $\chi : K \to GL_\mathbb{C}(V)$, then $\xi_* \mathcal{V}$ is isomorphic to the homogeneous vector bundle $G \times_K (\chi^\xi, W)$ associated to the representation

$$\chi^\xi : K \to GL_\mathbb{C}(W), \quad k \mapsto \chi^\xi(k) := \xi \circ \chi(k) \circ \xi^{-1}.$$

9.4 A sufficient condition for the isomorphism $\xi_* \sigma^* \mathcal{V} \simeq \mathcal{V}$

We are particularly interested in the case where $G' = G$, $K' = K$, $V = W = \mathbb{C}$ and $\xi(z) := \bar{z}$ (the complex conjugate of z) in the setting of Subsections 9.2 and 9.3.

By the identification of $GL_\mathbb{C}(\mathbb{C})$ with $\mathbb{C}^\times$, we have $g^\xi = \overline{g}$ for $g \in GL_\mathbb{C}(V) \simeq \mathbb{C}^\times$. Then, χ^ξ coincides with the conjugate representation

$$\overline{\chi} : K \to GL_\mathbb{C}(W) \simeq \mathbb{C}^\times, \quad k \mapsto \overline{\chi(k)}$$

for $\chi \in \mathrm{Hom}(K, \mathbb{C}^\times)$. Thus, we have an isomorphism of G-equivariant holomorphic line bundles:

$$\xi_* \sigma^* \mathcal{V} \simeq G \times_K (\overline{\chi \circ \sigma}, \mathbb{C}) \tag{9.4.1}$$

with the following correspondence of sections:

$$\xi_* \circ \sigma^* : \Gamma(M, \mathcal{V}) \to \Gamma(M, \xi_* \sigma^* \mathcal{V}), \quad (f_\alpha) \mapsto (\overline{f_\alpha \circ \sigma}).$$

We now apply the formula (9.4.1) to the setting where $M = G/K$ is an irreducible Hermitian symmetric space.

Lemma 9.4. *Let $\chi : K \to \mathbb{C}^\times$ be a unitary character. We denote by $\mathcal{V}$ the homogeneous line bundle $G \times_K (\chi, \mathbb{C})$. Suppose σ is an involutive automorphism of G of anti-holomorphic type (see Definition 1.4). Then we have an isomorphism of G-equivariant holomorphic line bundles: $\xi_* \sigma^* \mathcal{V} \simeq \mathcal{V}$.*

Proof. In view of (9.4.1), it suffices to show $\overline{\chi \circ \sigma} = \chi$. As the character χ of K is unitary, we have $\overline{\chi(k)} = \chi(k^{-1})$ for any $k \in K$. Let Z be a generator of the center $\mathfrak{c}(\mathfrak{k})$ of $\mathfrak{k}$. Since σ is of anti-holomorphic type, we have $\sigma Z = -Z$, and then

$$\overline{\chi \circ \sigma(\exp tZ)} = \overline{\chi(\exp(-tZ))} = \chi(\exp tZ) \qquad (t \in \mathbb{R}).$$

On the other hand, if $k \in [K, K]$, then $\overline{\chi \circ \sigma(k)} = 1 = \chi(k)$ because $[K, K]$ is a connected semisimple Lie group. As $K = \exp \mathfrak{c}(\mathfrak{k}) \cdot [K, K]$, we have shown $\overline{\chi \circ \sigma} = \chi$. Hence Lemma. □

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