59. The Restriction of $A_q(\lambda)$ to Reductive Subgroups

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1. Discrete decomposability with respect to symmetric pairs. Let $G$ be a real reductive linear Lie group and $\hat{G}$ the unitary dual of $G$. Suppose $G'$ is a reductive subgroup of $G$. The representation $\pi \in \hat{G}$ is called $G'$-admissible if the restriction $\pi_{|G'}$ splits into a discrete sum of irreducible representations of $G'$ with finite multiplicity. It may well happen that the restriction $\pi_{|G'}$ contains continuous spectrum (even worse, with infinite multiplicity) which is sometimes difficult to analyse. Thus, the notion of admissibility is emphasized here to single out a very nice pair $(\pi, G')$ for the study of the restriction $\pi_{|G'}$. Here are famous examples where $\pi \in \hat{G}$ is $G'$-admissible.

(1.1)(a) If $G'$ is a maximal compact subgroup of $G$, then any $\pi \in \hat{G}$ is $G'$-admissible (Harish-Chandra). An explicit decomposition formula is known as a generalized Blattner formula if $\pi = A_q(\lambda)$ (attached to elliptic orbits in the sense of orbit method; see [2], [9] Theorem 6.3.12).

(1.1)(b) A restriction formula of a holomorphic discrete series $G'$ is found with respect to some reductive subgroups $G'$ (eg. [7], [4]). Also the restriction of the Segal-Shale-Weil representation $\pi$ with respect to dual reductive pair with one factor compact is intensively studied (Howe's correspondence).

We remark that $G'$ is compact in the case (1.1)(a), while $\pi \in \hat{G}$ is a highest weight module in (1.1)(b). On the other hand, in some special settings, explicit restriction formulas have been found where $\pi \in \hat{G}$ does not belong to unitary highest weight modules but is $G'$-admissible for noncompact $G' \subset G$, such as $(G, G') \simeq (SO(4,2), SO(4,1))$ and $\pi$ is non-holomorphic discrete series ([5] Example 3.4.2), $(G, G') = (SO(4,3), G_2(R))$ and $\pi$ is in some family of derived functor modules (Kobayashi-Uzawa, 1989 at Math. Soc. Japan), and a recent work of Howe and Tan [3]. See also an explicit formula of the discrete part of $\pi_{G'}$ for $(G, G') \simeq (SO(3,2), SO(2,2))$ and $\pi$ non-holomorphic discrete series in [1] in the non-admissible case. In this section we find a more general but still good framework to study the restriction $\pi_{|G'}$.

Let $\theta$ be a Cartan involution of $G$. Write $g_0$ for the Lie algebra of $G$, $g = g_0 \otimes C$ for its complexification, $K = G^\theta$ for the fixed point group of $\theta$, and $g_0 = t_0 + p_0$ for the corresponding Cartan decomposition. Take a fundamental Cartan subalgebra $\mathfrak{h}_0 = \mathfrak{g}_0$. Then $\mathfrak{t}_0 := \mathfrak{h}_0 \cap \mathfrak{t}_0$ is a Cartan subalgebra of $\mathfrak{t}_0$. A $\theta$-stable parabolic subalgebra $q = q(\lambda) = t(\lambda) + u(\lambda) \subset g$ and a Levi part $L(\lambda) \subset G$ are given by an elliptic element $\lambda \in \sqrt{-1} (t_0^*)_*$ (see [9] Definition 5.2.1). Let $R_0^j \equiv (R_0^j)^{j \in N}$ be the Zuckerman’s derived functor from the category of metaplectic $(l, (L \cap K)^\sim)$-modules to that of $(g, K)$-modules. In this paper, we follow the normalization in [10] Definition
6.20 and some terminologies such as weakly fair in [11] Definition 2.5.

Let \( \sigma \) be an involutive automorphism of \( G \). If \( G' \) is an open subgroup of the fixed points of \( \sigma \), \( (G, G') \) is called a reductive symmetric pair. Choose a Cartan involution \( \theta \) of \( G \) so that \( \theta \mathfrak{g} = \mathfrak{g} \). Then \( K' := K \cap G' \) is a maximal compact subgroup of \( G' \). We write \( \mathfrak{t}_{\theta \mathfrak{g}} := \{ X \in \mathfrak{t}_{\mathfrak{g}} : \sigma(X) = \pm X \} \). Fix a \( \sigma \)-stable Cartan subalgebra \( \mathfrak{t}_0^{\mathfrak{c}} \) of \( \mathfrak{t}_0 \) such that \( \mathfrak{t}_0^{\mathfrak{c}} := \mathfrak{t}_0^{\mathfrak{c}} \cap \mathfrak{t}_{\theta \mathfrak{g}} \) is a maximal abelian subspace in \( \mathfrak{t}_{\theta \mathfrak{g}} \). Choose a positive system \( \mathfrak{c}^{+}(\mathfrak{t}, \mathfrak{t}^{\mathfrak{c}}) \) of the restricted root system \( \mathfrak{c}^{+}(\mathfrak{t}, \mathfrak{t}^{\mathfrak{c}}) \) and a positive system \( \mathfrak{c}^{+}(\mathfrak{t}, \mathfrak{t}^{\mathfrak{c}}) \) which is compatible with \( \mathfrak{c}^{+}(\mathfrak{t}, \mathfrak{t}^{\mathfrak{c}}) \). Let \( q = q(\mu) = t + u \) be a \( \theta \)-stable parabolic subalgebra of \( g \) given by an element \( \mu \in \sqrt{-1}(\mathfrak{t}_0^{\mathfrak{c}})^* \), which we can assume to be dominant with respect to \( \Delta^+(\mathfrak{t}, \mathfrak{t}^{\mathfrak{c}}) \) without loss of generality. Define a closed cone in \( \sqrt{-1}(\mathfrak{t}_0^{\mathfrak{c}})^* \) by

\[
R_+ (u \cap p) := \{ \sum_{\beta \in \mathfrak{d}(u \cap p)^*} n_{\beta} \beta : n_{\beta} \geq 0 \}.
\]

**Theorem 1.2.** In the setting as above, if \( R_+ (u \cap p) \cap \sqrt{-1}(\mathfrak{t}_0^{\mathfrak{c}})^* \neq \{0\} \), then \( R_{\mathfrak{c}}(C_{\mathfrak{c}}) \) is \( K' \)-admissible for any metaplectic unitary character \( C_{\mathfrak{c}} \) of \( L \) in the weakly fair range. In particular, \( R_{\mathfrak{c}}(C_{\mathfrak{c}}) \) is \( G' \)-admissible.

**Remark 1.3.** In Proposition 4.1.3 in [6], we have established a different type of admissibility in the case where \( \mathfrak{t} \) has a direct sum decomposition \( \mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{t}_{\mathfrak{g}} \), \( \mathfrak{g} \supseteq K_1 \) and \( q = q(\mu) \) such that \( \mu \mathfrak{t}_1 \cap \mathfrak{t}_{\mathfrak{g}} = 0 \).

2. Discrete series for homogeneous spaces of reductive type. Let \( G \) be a Lie group and \( G' \) its closed subgroup. Then \( G' \) naturally acts on \( X = G/H \) from the left. Given \( x \in G/H \), we write the isotropy subgroup \( H' \equiv G'_x := \{ g \in G' : g \cdot x = x \} \) and put \( X' = G'/H' \). As a representation theoretic counterpart of an embedding \( f : X' \hookrightarrow X \) we consider the restriction of representations of \( G \) with respect to \( G' \) which arises as the pullback of function spaces \( f^* : \Gamma(X) \rightarrow \Gamma(X') \).

If \( H \) is a reductive algebraic subgroup of a real reductive linear Lie group \( G \), we say the homogeneous space \( G/H \) of reductive type. An irreducible unitary representation \( \pi \in \hat{G} \) is called discrete series for \( L^2(G/H) \) if \( \pi \) can be realized as a closed invariant subspace of \( L^2(G/H) \). The totality of discrete series for \( L^2(G/H) \) is denoted by \( \text{Disc}(G/H) \subset \hat{G} \). We also write \( \text{Disc}(G/H) \) for the multiset of \( \text{Disc}(G/H) \) counted with multiplicity occurring in \( L^2(G/H) \). Analogous notation is used for \( L^2 \)-sections of \( G \)-equivariant vector bundles over \( G/H \) associated to a unitary representation of \( H \). On the other hand, given \( (\pi, V) \in \hat{G} \), we write \( \text{Disc}(\pi_{|H}) \subset \hat{G} \) for the set of irreducible discrete summands of the restriction \( \pi_{|H} \), and \( \text{Disc}(\pi_{|H}) \) for the corresponding multiset counted with multiplicity.

**Theorem 2.1.** Suppose \( G \) is a real reductive linear group and \( G' \), \( H \) are reductive subgroups stable under \( \theta \) simultaneously. Let \( H' := H \cap G' \). Assume there exists a minimal parabolic subgroup \( P' \) of \( G' \) such that

\[
(2.1)(a) \quad \dim H + \dim G' = \dim G + \dim (H \cap G'),
(2.1)(b) \quad \dim H' + \dim P' = \dim G' + \dim (H' \cap P').
\]

Then we have a bijection between multisets \( \cup_{\pi \in \text{Disc}(G/H)} \text{Disc}(\pi_{|G'}) \cong \text{Disc}(G'/H') \).
In particular, \( \text{Disc}(G'/H') = \emptyset \) if and only if either \( \text{Disc}(G/H) = \emptyset \) or \( \pi_{|G'} \).
is decomposed into only continuous spectrum for any $\pi \in \text{Disc}(G/H)$. Moreover, if discrete series for $G'/H'$ is multiplicity free, then the discrete part of the restriction of $\pi_{G'}$ is multiplicity free for all $\pi \in \text{Disc}(G/H) \subset \hat{G}$.

An abundant theory on the harmonic analysis on $G/H$ has been developed in these fifteen years when $G/H$ is a semisimple symmetric space, while very little has been studied when it is non-symmetric. We note that if one knows $\text{Disc}(G/H)$ and the restriction formula $\pi_{G'}$ for $\pi \in \text{Disc}(G/H)$, then Theorem (2.1) gives a construction and exhaustion of discrete series for $G'/H'$. More weakly, only a combination of Theorem (1.2) and Theorem (2.1) gives new results on the existence of discrete series of some non-symmetric spherical homogeneous spaces such as

**Corollary 2.2**

1) $\text{Disc}(SU(2p - 1,2q)/Sp(p - 1, q)) \neq \emptyset$ for any $p, q$.  
2) $\text{Disc}(SO(2p - 1,2q)/U(p - 1, q)) \neq \emptyset$ if and only if $pq \in 2\mathbb{Z}$.  
3) $\text{Disc}(SO(4, 3)/G_2(R)) \neq \emptyset$, $\text{Disc}(G_2(R)/SL(3, R)) \neq \emptyset$.

Now, relax the assumption (2.1)(a). In the setting at the beginning §2, we say $f : G'/H' \subset G/H$ regular if there exists a submanifold $I$ of $G/H$ such that $G'_y = H'$ for any $y \in I$ and that $\varphi : G'/H' \times I \rightarrow G/H$, $(g, y) \mapsto g \cdot y$ is an open embedding.

**Example 2.3** (group manifolds). If $H' = H = \{e\}$, then $G' \subset G$ is regular. We can take $I$ to be a local section of the principal bundle $G \rightarrow G/G'$.

**Example 2.4** (semisimple orbits in symmetric spaces). Let $\sigma, \tau$ be commutative involutive automorphisms of $G$, $(G, G')$ and $(G, H)$ the corresponding symmetric pairs. Fix a maximally abelian semisimple subspace $\alpha$ in $\{X \in g_0 : \sigma(X) = \tau(X) = -X\}$ and define $M' := \{g \in G' \cap H : \text{Ad}(g)X = X \text{ for } X \in \alpha\}$. Then $G'/M' \subset G/H$ is regular. The regular semisimple orbit in $G$ under the adjoint action of $G$ is a typical example.

**Theorem 2.5.** In the setting of Theorem (2.1), suppose $\varphi_j : G' \times H'_j \times I_j \rightarrow G/H (j \in J)$ define regular orbits such that the disjoint union of $\varphi_j(G'/H'_j \times I_j)$ is open dense in $G/H$. Then we have $\bigcup_{\pi \in \text{Disc}(G/H)} \text{Disc}(\pi_{G'}) \subset \bigcup_{j} \text{Disc}(G'/H'_j)$. In particular, if $\text{Disc}(G'/H'_j) = \emptyset$ ($j \in J$), then either $\text{Disc}(G/H) = \emptyset$ or $\text{Disc}(\pi_{G'}) = \emptyset$ for any $\pi \in \text{Disc}(G/H)$. Moreover, if $\pi \in \text{Disc}(G/H)$ is $K'$-admissible, then $\text{Disc}(\pi_{G'}) \subset \bigcap_{j} \text{Disc}(G'/H'_j)$.

Here is a very special case corresponding to Example (3.2):

**Corollary 2.6.** Suppose $\pi = \hat{A}_q \in \hat{G}$ is a (Harish-Chandra’s) discrete series for $G$. If $\pi$ is $G'$-admissible, then $\pi_{G'}$ is decomposed into discrete series for $G'$. In particular, if rank$G' > \text{rank}K'$ and rank$G = \text{rank}K$, then $\pi_{G'}$ is decomposed into only continuous spectrum.

**Remark 2.7.** In general, if $\pi \in \text{Disc}(G)$, then $\pi_{G'}$ is supported on tempered representations of $G'$ by Mackey-Anh’s reciprocity theorem.

3. Examples of decomposition formulas. In the framework of §1, §2 we present some explicit branching formulas joint with B.Ørsted.

Let $G = SO(p, q) \supset K = SO(p) \times SO(q)$ ($p \geq 1, q \geq 0$). We take a (standard) basis $\{f_i\}$ of $\sqrt{-1}T(t_0)^*$ as in [6] §2.5 and define $\theta$-stable parabolic subalgebras by $q_+ := q(f_i) = i + u_+$, $q_- := q(-f_i) = i + u_-(p \geq 2)$. 
Then \(L(f_{\lambda}) = L(-f_{\lambda}) = T \times SO_{0}(\phi - 2, q)\). Put \(Q := \frac{1}{2}(\phi + q) - 2\). For 
\(\lambda \in Z + Q\), we write \(C_{i, j}\) for the metaplectic representation of \(L\) corresponding to \(\lambda_{i, j} \in \sqrt{-1}(t_{0})^{*}\). If \(\lambda \in Z + Q\) and \(\lambda \geq 0\) (moreover if \(\lambda \geq \frac{1}{2} \phi - 1\) when \(q = 0\)), we define \((g, K)\)-modules by 
\[U_{+}(\lambda) \equiv U_{+}^{SO_{0}(\phi, q)}(\lambda) := (B_{\phi}^{0})^{p, q - 2}(C_{i, j})^{*},\]
\[U_{-}(\lambda) \equiv U_{-}^{SO_{0}(\phi, q)}(\lambda) := (B_{\phi}^{0})^{p, q - 2}(C_{i, j}).\]
Then \(U_{\pm}(\lambda)\) are non-zero irreducible \((g, K)\)-modules and \(U_{\pm}(\lambda) \in \text{Disc}(SO_{0}(\phi, q) / SO_{0}(\phi - 1, q))\) if \(\lambda > 0\).

Next, let \(G' = U(\phi, q) \supset K' = U(\phi) \times U(q)\). We represent the root system of \(t'\) as \(\Delta(t', t'^{\vee}) = \{ (e_{i} - e_{j}) : 1 \leq i < j \leq \phi \text{ or } \phi + 1 \leq i < j \leq \phi + q \}\). We define \(\theta\)-stable parabolic subalgebras of \(g'\) by
\(1\) For \(\phi \geq 1, q \geq 1, q_{0} := q(2e_{1} + e_{\phi + 1})\) and \(q_{-} := q(-2e_{\phi} - e_{\phi + q})\).
\(2\) For \(\phi \geq 2, q \geq 0, q_{0} := q(e_{1} - e_{\phi})\).

For \(\lambda \in N_{+}, l \in Z\) such that \(l \equiv \lambda + \phi + q + 1 \mod 2\), we define \((g', K')\)-modules by:
\[V_{+}(\lambda, l) := (\mathcal{R}_{\phi}^{0})^{p, q - 2}(C_{\phi + q - 2})\text{ if } l > \lambda > 0, pq \geq 1,\]
\[V_{0}(\lambda, l) := (\mathcal{R}_{\phi}^{0})^{2p, q - 4}(C_{2h_{i} + e_{\phi} - \frac{1}{2}e_{r_{\phi}}})\text{ if } \lambda \equiv \frac{l}{2}, p \geq 2,\]
\[V_{-}(\lambda, l) := (\mathcal{R}_{\phi}^{0})^{p, q - 2}(C_{-h_{i} + e_{\phi} + \frac{1}{2}e_{r_{\phi}}})\text{ if } -l > \lambda > 0, pq \geq 1.\]
If \(q \geq 1\), then we have (cf. [6] Theorem 2):
\(3.1\)(a) \(V_{+}(\lambda, l), V_{0}(\lambda, l), V_{-}(\lambda, l)\) are non-zero and irreducible \((g', K')\)-modules with 
\(Z(g')\)-infinitesimal character \((\frac{\lambda + l}{2}, \frac{-\lambda + l}{2}, q', Q' - 1, \ldots, -Q')\) in the Harish-Chandra parametrization, where \(Q' := \frac{p + q - 3}{2}\).
\(3.1\)(b) \(V_{+}(\lambda, l) \cong V_{-}(\lambda, -l)^{\vee} (l > \lambda > 0), V_{0}(\lambda, l) \cong V_{0}(\lambda, -l)^{\vee} (\lambda \leq |l|).\)

\(3.1\)(c) \(\text{Disc}(U(p, q) / U(1) \times U(p - 1, q)) ; \chi_{l}) (q \geq 1, l \in Z)\) are given by,
\[
\begin{align*}
\{ V_{+}(\lambda, l) : |l| > \lambda > 0 \} \cup \{ V_{0}(\lambda, l) : \lambda \geq |l| \} & (p \geq 2, l \neq 0, \epsilon = \text{sgn} l), \\
\{ V_{0}(\lambda, l) : \lambda > 0 \} & (p \geq 2, l = 0), \\
\{ V_{+}(\lambda, l) : |l| - q \geq \lambda > 0 \} & (p = 1, l > q, \epsilon = \text{sgn} l), \\
\{ V_{0}(\lambda, l) : |l| \leq q \} & (p = 1, l \leq q).
\end{align*}
\]
Here, \(\chi_{l}\) is a character of \(U(1)\) and \(\lambda\) runs over \(\lambda \in Z^{2} + l + \phi + q + 1\) (resp. \(\lambda \in 2Z + l + \phi + q\)).

Third, let \(G'' = Sp(p, q) \supset K'' = Sp(p) \times Sp(q)\), and represent the root system of \(t''\) as \(\Delta(t'', t'^{\vee}) = \{ (h_{i} - h_{j}) , \pm 2h_{i} : 1 \leq i < j \leq p \text{ or } p + 1 \leq i \leq j \leq p + q, 1 \leq l \leq p + q \}\). We define
\(1\) For \(p \geq 1, q \geq 1, q_{0} := q(2h_{1} + h_{p+1}), L_{+}'' \cong T^{2} \times Sp(p - 1, q - 1).\)
\(2\) For \(p \geq 2, q \geq 0, q_{0} := q(2h_{1} + h_{p}), L_{0}'' \cong T^{2} \times Sp(p - 2, q).\)

For \(\lambda \in N_{+}, j \in N\) such that \(j \equiv \lambda + 1 \mod 2\) we define \((g'', K'')\)-modules by:
\[W_{+}(\lambda, j) := W_{+}^{Sp(p, q)}(\lambda, j) := (R_{\phi}^{p, q - 2}(C_{2h_{1} - 1 + h_{p+1}}^{1} + 1, -l + 1, \lambda)) \text{ if } j + 1 > \lambda, pq \geq 1,\]
\[W_{0}(\lambda, j) := W_{0}^{Sp(p, q)}(\lambda, j) := (R_{\phi}^{p, q - 2}(C_{-h_{1} + 1 + h_{p+1}}^{1} + 1, -1, \lambda)) \text{ if } \lambda \geq j + 1, p \geq 2.\]
If \(q \geq 1\), then we have (cf. [6] Theorem 1):
(3.2)(a) \( W_\ast(\lambda, j), W_0(\lambda, j) \) are non-zero and irreducible \((q^\ast, K^\ast)\)-modules with \( Z(q^\ast) \)-infinitesimal character \( (\lambda + j + 1, -\lambda + j + 1, \ldots, 1) \) in the Harish-Chandra parametrization, where \( Q^\ast := p + q - 2 \).

(3.2)(b) \( \text{Disc}(Sp(p, q) / Sp(p - 1, q); \sigma_j) (q \geq 1 \text{ and } j \in \mathbb{N}) \) are given by,
\[
\begin{align*}
\{W_\ast(\lambda, j) : \lambda > j\} \cup \{W_0(\lambda, j) : j > \lambda > 0\} & \quad (p \geq 2), \\
\{W_+(\lambda, j) : j - 2q + 1 \geq \lambda > 0\} & \quad (p = 1, j \geq 2q), \\
\{W_0(\lambda, j) : j > 2q + 1 \geq \lambda \geq 0\} & \quad (p = 1, j < 2q).
\end{align*}
\]
Here, \( \sigma_j \) is the irreducible \( j \) dimensional representation of \( Sp(1) \). In (3.2)(b), \( \lambda \) runs over \( \lambda \in \mathbb{Z} + j + 1 \) and the multiplicity of discrete series is uniformly \( j + 1 \) or 0.

We write \( \mathcal{H}_k^Z(R^p) \) for spherical harmonics on \( S^{p-1} \) of degree \( k (k \in \mathbb{N}) \), which is isomorphic to \( U_+^{SO(p)}(k + \frac{1}{2} p - 1) \) if \( p \geq 3 \) or \( (p, k) = (2, 0) \), to \( U_+^{SO(p)}(k) \oplus U_-^{SO(p)}(k) \) if \( p = 2 \) and \( k \geq 1 \). If \( p = 1 \), we put \( \mathcal{H}_0(R^1) := \mathbb{C} \) for \( k = 0, 1 \) and \( := 0 \) for \( k \geq 2 \). Next, we write spherical harmonics of degree \( (\alpha, \beta) (\alpha, \beta \in \mathbb{N}) \) as \( \mathcal{H}_k^{a,\beta}(R^p) = V_0^{U(p)}(\alpha + \beta + p - 1, \alpha - \beta) \subset \mathcal{H}_k^{a,\beta}(R^p) \) for \( p \geq 2 \). In the case \( p = 1 \), it is non-zero only if \( \alpha \beta = 0 \). Finally, we write \( F_{sp}(x, y) \) \( x \geq y \geq 0 \) for the irreducible representation of \( Sp(p) \) with an extremal weight \( xf_1 + yf_2 \). In the case \( p = 1 \), it is non-zero only if \( y = 0 \).

Theorem 3.3 \((SO_\ast(p, q) \downarrow SO_\ast(p, s) \times SO(q - s)) \). Let \( p \geq 2, s \geq 1, q - s \geq 1, \lambda \in \mathbb{Z} + \frac{1}{2} (p + q), \lambda > 0 \).
\[
U_+^{SO_\ast(p, q)}(\lambda)_{SO_\ast(p, s) \times SO(q - s)} \cong \bigoplus_{a, k \in \mathbb{N}} U_+^{SO_\ast(p, s)}(\lambda + \frac{1}{2} q - s + a + 2k) \otimes \mathcal{H}_a^Z(R^{s-1}).
\]

Theorem 3.4 \((U(p, q) \downarrow U(p, s) \times U(q - s)) \). Let \( s \geq 1, q - s \geq 1, \lambda \in \mathbb{N}_+, l \in \mathbb{Z} + \lambda + p + q + 1 \). For convenience, we define an irreducible representation of \( U(p, s) \times U(q - s) \) by
\[
V_\delta(\alpha, \beta, k; \lambda, l) := V_\delta^{U(p, q)}(\lambda + q - s + \alpha + \beta + 2k, l - \alpha - \beta) \otimes \mathcal{H}_a^{a,\beta}(C^{s-1}).
\]
1(i) Suppose \( p \geq 2, l > \lambda + q - s \).
\[
V_\delta^{U(p, q)}(\lambda, l)_{U(p, s) \times U(q - s)} \cong \bigoplus_{a, k \in \mathbb{N}} V_\delta(\alpha, \beta, k; \lambda, l) \otimes \mathcal{H}_a^{a,\beta}(C^{s-1}).
\]
(ii) Suppose \( p \geq 2. \) We put \( \delta = + \) if \( \lambda + q - s \geq l > \lambda \) and \( \delta = 0 \) if \( \lambda + 1 \geq \lambda \) in the left side.
\[
V_\delta^{U(p, q)}(\lambda, l)_{U(p, s) \times U(q - s)} \cong \bigoplus_{a, k \in \mathbb{N}} V_\delta(\alpha, \beta, k; \lambda, l).
\]

Use the duality (3.1)(b) if \( - \lambda > l \geq - \lambda - q + s \) or \( - \lambda - q + s > l \).
2) Suppose \( p = 1, l \geq \lambda + q \). (Use the duality (3.1)(b) if \( - l \geq \lambda + q \).)
\[
V_\delta^{U(1, q)}(\lambda, l)_{U(1, s) \times U(q - s)} \cong \bigoplus_{a, k \in \mathbb{N}} V_\delta(\alpha, \beta, k; \lambda, l).
\]
Theorem 3.5 \((\text{Sp}(p, q) \downarrow \text{Sp}(p, s) \times \text{Sp}(q, s))\). Let \(s \geq 1, q - s \geq 1, \lambda \in \mathbb{N}_+, j \in \mathbb{N}, j \in 2\mathbb{Z} + \lambda + 1\). For convenience, we define an irreducible representation of \(\text{Sp}(p, s) \times \text{Sp}(q, s)\) by \(W_\delta(y, v, k, t; \lambda, j) := W_\delta^{\text{Sp}(p, s)}(\lambda + 2q - 2s + 2y + 2k + t + v, j + v - t) \otimes F^{\text{Sp}(q, s)}(y + t + v, y)\).

1(i) Suppose \(p \geq 2, j + 1 > \lambda + 2q - 2s\). Then, \(W_+^{\text{Sp}(p, s)}(\lambda, j)_{\text{Sp}(p, s) \times \text{Sp}(q, s)} \simeq \bigoplus_{y, v, k, t \in \mathbb{N} \atop 0 \leq i \leq j} W_+^{\text{Sp}(p, s)}(y, v, k, t; \lambda, j) \oplus \bigoplus_{y, v, k, t \in \mathbb{N} \atop 0 \leq i \leq j} W_0(y, v, k, t; \lambda, j)\).

(ii) Suppose \(p \geq 2\). We put \(\delta = +\) if \(\lambda + 2q - 2s \geq j + 1 > \lambda\) and \(\delta = 0\) if \(p \geq 2, \lambda \geq j + 1\) in the left side.

\[ W_\delta^{\text{Sp}(p, q)}(\lambda, j)_{\text{Sp}(p, s) \times \text{Sp}(q, s)} \simeq \bigoplus_{y, v, k, t \in \mathbb{N} \atop 0 \leq i \leq j} W_\delta(y, v, k, t; \lambda, j). \]

2) Suppose \(p = 1, j \geq \lambda + 2q - 1\).

\[ W_+^{\text{Sp}(p, q)}(\lambda, j)_{\text{Sp}(p, s) \times \text{Sp}(q, s)} \simeq \bigoplus_{y, v, k, t \in \mathbb{N} \atop 0 \leq i \leq j} W_+(y, v, k, t; \lambda, j). \]

A detailed proof is to appear elsewhere.

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