The Restriction of $A_q (\lambda)$ to Reductive Subgroups II

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§ 1. Introduction. In this paper we continue the investigation of the restriction of irreducible unitary representations of real reductive groups, with emphasis on the discrete decomposability. We recall that a representation $\pi$ of a reductive Lie group $G$ on a Hilbert space $V$ is $G$-admissible if $(\pi, V)$ is decomposed into a discrete Hilbert direct sum with finite multiplicities of irreducible representations of $G$. The same terminology is used for a $(g, K)$-module on a pre-Hilbert space, if its completion is $G$-admissible.

Let $H$ be a reductive subgroup of a real reductive Lie group $G$, and $(\pi, V)$ an irreducible unitary representation of $G$. The restriction $(\pi|_H, V)$ is decomposed uniquely into irreducible unitary representations of $H$, which may involve a continuous spectrum if $H$ is noncompact. In [5],[6], we have posed a problem to single out the triplet $(G, H, \pi)$ such that the restriction of $(\pi|_H, V)$ is $H$-admissible, together with some application to harmonic analysis on homogeneous spaces. The purpose of this paper is to give a new insight of such a triplet $(G, H, \pi)$ from view points of algebraic analysis. In particular, we will give a sufficient condition on the triplet $(G, H, \pi)$ for the $H$-admissible restriction as a generalization of [5],[6] to arbitrary $H$, and also present an obstruction for the $H$-admissible restriction.

§ 2. A sufficient condition for discrete decomposability. Let $K$ be a compact Lie group. We write $\mathfrak{t}_0$ for the Lie algebra of $K$, and $\mathfrak{t}$ for its complexification. Analogous notation is used for other groups. Take a Cartan subalgebra $\mathfrak{t}_0^c$ of $\mathfrak{t}_0$. The weight lattice $L$ in $\sqrt{-1} (\mathfrak{t}_0^c)^*$ is the additive subgroup of $\sqrt{-1} (\mathfrak{t}_0^c)^*$ consisting of differentials of the weights of finite dimensional representations of $K$. Let $C \subset \sqrt{-1} (\mathfrak{t}_0^c)^*$ be a dominant Weyl chamber. We write $K_0$ for the identity component of $K$, and $K_0^*$ for the unitary dual of $K_0$. The Cartan-Weyl theory of finite dimensional representations establishes a bijection:

$L \cap C \sim K_0, \lambda \mapsto F(K_0, \lambda)$.

Suppose $X$ is a $K$-module (possibly, of infinite dimension) which carries an algebraic action of $K$. The $K_0$-multiplicity function of $X$ is given by

$m := m_X : L \cap C \rightarrow N \cup \infty,
\quad m(\lambda) := \dim \text{Hom}_{K_0} (F(K_0, \lambda), X)$.

The asymptotic $K$-support $T(X) \subset C$ was introduced in [3] as follows:

$S(X) := \{ \lambda \in L \cap C : m_X (\lambda) \neq 0 \},
T(X) := \{ \lambda \in C : V \cap S(X) \text{ is not relatively compact for any open cone } V \text{ containing } \lambda \}.$

Hereafter we assume a growth condition on $m_X$; there are constants $A, R > 0$ such that

(2.1) $m_X (\lambda) \leq A \exp (R |\lambda|)$ for any $\lambda \in L \cap C$.

This condition assures that the character of the representation $X$ is a hyperfunction on $K$, whose singularity spectrum we can estimate in terms of $T(X)$.

Suppose $H$ is a closed subgroup of $K$. Let $\text{pr}_{K-H} : \mathfrak{t}^k \rightarrow \mathfrak{h}^k$ be the projection dual to the inclusion of Lie algebras $\mathfrak{h} \subset \mathfrak{t}$, and $\mathfrak{h}^k := \text{Ker} (\text{pr}_{K-H} : \mathfrak{t}^k \rightarrow \mathfrak{h}^k)$. We set

(2.2) $C (\mathfrak{h}) := C \cap \text{Ad}^0 (K) \mathfrak{h}^k \subset \sqrt{-1} (\mathfrak{t}_0^c)^*$.

Note that $C (\mathfrak{t} (0)) = \{ 0 \}$ and $C (0) = C$.

Theorem 2.3. Let $X$ be a $K$-module satisfying (2.1). If a closed subgroup $H$ of $K$ satisfies $T(X) \cap C (\mathfrak{h}) = \{ 0 \}$, then the restriction $X|_H$ is $H$-admissible.

Now, let us apply Theorem (2.3) to some standard $(q, K)$-modules. Suppose that $G$ is a real reductive linear Lie group and that $K$ is a maximal compact subgroup of $G$. A dominant element $a \in \sqrt{-1} \mathfrak{t}_0^c$ defines a $\theta$-stable parabolic subalgebra $q = l + u$, where $l, u$ are the sum of eigenspaces of $\text{ad} (a)$ with 0, positive eigenvalues, respectively. Let $L$ be the centralizer of $a$ in $G$. Zuckerman introduced the cohomological parabolic induction $\mathcal{P}_q^l = (\mathcal{P}_q^l)^j (j \in \mathbb{N})$, which is a covariant functor from the category of metaplectic $(l, (L \cap K)^-)$-modules to that of $(q, K)$-modules, as a generalization of the Borel-Weil-Bott con-
strucrion of finite dimensional representations of compact groups. In particular, we write $A_q(\lambda) \equiv (R_q^b)^\prime (C)$ for a metaplectic unitary character $C$ in the good range of parameter (see [8] Definition 2.5), where $S \equiv \dim_{C}(u \cap tf)$. Then $A_q(\lambda)$ is an irreducible unitarizable $(g, K)$-module (see [7] Theorem 6.8), and we write $A_q(\lambda) \in G'$ for its completion.

The $K$-module structure of the alternating sum $\sum (-1)^j (R_q^b)^\prime (W)_{ikj}$ is known as a generalized Blattner formula (see [7] Theorem 6.34). Its proof also gives an upper estimate of each term $(R_q^b)^\prime (W)_{ikj}$ which leads us to:

**Theorem 2.4.** If $W$ is a finite dimensional metaplectic $(L \cap K)^-$-module, then the restriction $(R_q^b)^\prime (W)_{ikj}$ satisfies (2.1) and 

$$T((R_q^b)^\prime (W)_{ikj}) \subset R^+ \langle u \cap p \rangle \cap \mathcal{C} \quad (j \in J).$$

Here, we recall $a = t + u$ (Levi decomposition) and $g = tf + pf$ (Cartan decomposition), and we define a closed cone by 

$$R^+ \langle u \cap p \rangle := \{ \sum_{\beta \in \Phi_{u \cap p}} n_{\beta} \beta : n_{\beta} \geq 0 \} \subset \mathcal{A}(t_f^*).$$

**Corollary 2.5.** In the setting of Theorem (2.3), if 

$$\mathcal{C}(\gamma) \cap R^+ \langle u \cap p \rangle = (0),$$

then the restriction $(R_q^b)^\prime (W)_{ij}$ is $H$-admissible for any finite dimensional metaplectic $(L \cap K)^-$-module $W$ and for any $j \in J$. In particular, $A_q(\lambda) \in H$ is decomposed discretely into irreducible unitary representations of $H$.

As a special case of Corollary (2.5), we obtain a new and unified proof of some of the main results in [5],[6], where we imposed some assumptions on a subgroup $H$.

First, suppose that $H \subset K$ is a symmetric pair defined by an involution $\sigma \in \text{Aut}(K)$. Take a maximal abelian subspace $a_0$ in $\{ Y \in t_0 : \sigma(Y) = -Y \}$ and extend it to a Cartan subalgebra $t_0^a \subset t_0$. We take a dominant Weyl chamber $C$ so that $\mathcal{A}(t_0^a) \cap C$ is a dominant Weyl chamber for the restricted root system $\Sigma(t_0, \alpha)$.

**Corollary 2.6** (cf. [5] Theorem 1.2; [6] Theorem 3.2). Retain the above setting. If 

$$\mathcal{A}(t_0^a) \cap C \cap R^+ \langle u \cap p \rangle = \{0\},$$

then the restriction $\text{End}(W)_{ij}$ of $W$ is $H$-admissible for any finite dimensional metaplectic $(L \cap K)^-$-module $W$.

Next, suppose $K$ is (locally) isomorphic to a direct product $K_1 \times K_2$. We note that the Cartan subalgebra $t_0^2$ is also decomposed into a direct sum $t_0^2 = (t_0^1)^0 + (t_0^2)^0$.

**Corollary 2.7** (cf. [6] Corollary 4.4; [4] Proposition 4.1.3). In the setting as above, if a $\theta$-stable parabolic subalgebra $\gamma$ is given by $a \equiv \sqrt{-1}(t_0^1)^0$ then the restriction $\text{End}(W)_{ij}$ of $W$ is $K_i$-admissible for any finite dimensional metaplectic $(L \cap (K_1)^-)$-module $W$.

We note that Corollaries (2.6), (2.7) are deduced from Corollary (2.5) by using $\mathcal{C}(\gamma) = \mathcal{A}(t_0^2) \cap \mathcal{C}$, $\mathcal{C}(t_0^1) = \mathcal{A}(t_0^2) \cap \mathcal{C}$, respectively.

**Remark 2.8.** The above corollaries (2.4), (2.5), (2.6) are valid if we replace $H$ by any reductive subgroup $H'$ containing $H$, because of Corollary (1.3) in [6].

§ 3. A necessary condition for discrete decomposability. In § 2, we have given a sufficient condition that the restriction of a $(g, K)$-module $X$ has an $H$-admissible restriction with respect to a subgroup $H$. Conversely, we will find a necessary condition in terms of associated varieties of $g$-modules in this section.

We recall that the associated variety of a $(g, K)$-module $X$ of finite length is defined by 

$$\mathcal{V}(X) \equiv \mathcal{V}_{\gamma}(X) = \text{Supp}_{r(\gamma)}(\text{Gr}(X)) \subset g^*,$$

as the support in $g^*$ of the associated graded module $\text{Gr}(X)$ over the symmetric algebra $S(g)$, with respect to a good filtration (see [1]). It is known that $\mathcal{V}(X)$ is a subset of the nilpotent cone $\mathcal{N}^* \equiv \mathcal{N}^*(g) \subset g^*$. Let $H$ be a closed subgroup that is reductive in $G$. We fix a Cartan involution $\theta$ of $G$ which makes $H$ stable so that $H \cap K$ is a maximal compact subgroup of $H$. Write the projection $\text{pr}_{g-H} : g^* \rightarrow h^*$ as before.

**Theorem 3.1.** Suppose $X$ is a $(g, K)$-module of finite length. Assume that the restriction $X_{\gamma}$ is $H$-admissible. Let $Y$ be any $(g, H \cap K)$-module occurring as a direct summand of $X$. Then we have 

$$\text{pr}_{g-H}(\mathcal{V}_{\gamma}(X)) \subset \mathcal{V}_{\gamma}(Y).$$

This theorem gives rise to an obstruction for the admissibility of the restriction of a unitary representation.

**Corollary 3.2.** Suppose $X$ is a $(g, K)$-module of finite length. Assume that the restriction $X_{\gamma}$ is $H$-admissible. Then 

$$\text{pr}_{g-H}(\mathcal{V}_{\gamma}(X)) \subset \mathcal{N}^*(\gamma).$$

Applying Corollary (3.2) to $X = A_q(\lambda)$, we have:
Corollary 3.3. Let us identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via the Killing form. Assume a 0-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{t} + \mathfrak{u}$ of $\mathfrak{g}$ satisfies
\[ \text{pr}_{\mathfrak{g} - \mathfrak{h}}(\text{Ad}(K_G)(\mathfrak{u} \cap \mathfrak{p})) \subset N^*(\mathfrak{h}). \]
Then the restriction of $A_{\mathfrak{q}}(\lambda) \in \hat{G}$ to $H$ is not $H$-admissible.

Remark 3.4. If $H = K$, then the assumption of Theorem (3.1) is always satisfied. In this special case, Theorem (3.1) implies a well-known result $\text{pr}_{\mathfrak{g} - \mathfrak{k}}(\mathcal{V}_{\mathfrak{g}}(X)) = \{0\}$ (see [9] Corollary 5.13) because the associated variety of a finite dimensional representation is zero. In a general case where $H$ is non-compact, $\text{pr}_{\mathfrak{g} - \mathfrak{h}}(\mathcal{V}_{\mathfrak{g}}(X))$ is not necessarily $\{0\}$.

Finally, we mention a useful information about $\hat{H}$ occurring as direct summands of the restriction $X_{\mathfrak{h}H}$ as an elementary application of associated varieties. This helps us to understand a strange phenomenon about the direct summands occurring in the restriction of $A_{\mathfrak{q}}(\lambda)_{\mathfrak{h}H}$, which was pointed out in [6] Introduction.

Theorem 3.5. Suppose $X$ is an irreducible $(\mathfrak{g}, K)$-module. Assume that $X$ is $H$-admissible. Let $Y_1, Y_2$ be any irreducible $(\mathfrak{h}, H \cap K)$-module occurring as a direct summand of the restriction $X$ to $(\mathfrak{h}, H \cap K)$. Then we have
\[ \mathcal{V}_{\mathfrak{h}}(Y_1) = \mathcal{V}_{\mathfrak{h}}(Y_2). \]

References


