Fusion rules of Lattice cosets with an application to Feigin-Semikhatov duality

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This talk is based on a joint work with Thomas Creutzig, Naoki Genra, and Ryo Sato.

Main topic of this talk is to understand the representation theory of the (simple) principal affine W-superalgebra

$$W_k(\mathfrak{sl}_{n|1}, f_{\mathrm{prin}})$$

$$\left(k = -(n-1) + \frac{n-1}{n+r}, \quad r \in \mathbb{Z}_{>0} \ s.t.\ \gcd(n+r, n-1) = 1\right)$$

Example 0.1 ($n = 2$)

$$W_{-1+\frac{1}{r+2}}(\mathfrak{sl}_{2|1}, f_{\mathrm{prin}}) \simeq L_{\frac{3r}{r+2}}(\mathfrak{ns}_2)$$

is the unitary series of $\mathcal{N} = 2$ superconformal algebra with central charge $\frac{3r}{r+2}$. It is generated by the fields

$$J(z) \text{ (Heisenberg field)}, \quad L(z) \text{ (Virasoro field)}$$

$$G^{\pm}(z) \text{ (odd primary fields w/ conformal weights } \frac{3}{2} \text{ with spin } \pm 1).$$

The basic problems of the representation theory of vertex operator superalgebras V are

- Rationality (semisimplicity) of the module category V-mod.
- Classification of the simple objects.
- Description of the fusion products.

These problems for $L_{\frac{3r}{r+2}}(\mathfrak{ns}2)$ are solved by Adamović ('99).

Example 0.2 ($n = 2, r = 1$)

$$W_{-1+\frac{1}{3}}(\mathfrak{sl}_{2|1}, f_{\mathrm{prin}}) \simeq L_1(\mathfrak{ns}_2)\ (\simeq V_{\sqrt{3}\mathbb{Z}}).$$

- $\mathrm{Irr}(L_1(\mathfrak{ns}_2)) = \{M(0,0), M(1/6,1/3), M(2/3,2/3)\}$
-

⊠	M(0,0)	M(1/6,1/3)	M(2/3,2/3)
M(0,0)	M(0,0)	M(1/6,1/3)	M(2/3,2/3)
M(1/6,1/3)	M(1/6,1/3)	M(2/3,2/3)	M(0,0)
M(2/3,2/3)	M(2/3,2/3)	M(0,0)	M(1/6,1/3)

- $K_{\mathbb{C}}(L_1(\mathfrak{ns}_2)) \simeq \mathbb{C}[\mathbb{Z}/3\mathbb{Z}]$.

Fusion rules

Fact 1 (Huang-Lepowsky-Zhang)

Let V be a Möbius vertex (super)algebra such that C_2-cofinite and of CFT type. Then the category of "nice" V-modules has a natural structure of braided tensor category.

Example 0.3

Let V be a finite dimensional commutative $\mathbb{C}$-algebra. Then V-mod is just the category of finite dimensional V-modules and the fusion product is just the tensor product

$$\boxtimes = \otimes_V.$$

Rationality 1

Theorem 2 (Creutzig-Linshaw/Creutzig-Genra-N)

$$W_k(\mathfrak{sl}_{n|1}, f_{\mathrm{prin}})$$

$$\left(k = -(n-1) + \frac{n-1}{n+r}, \quad r \in \mathbb{Z}_{>0} \ s.t. \ \mathrm{gcd}(n+r, n-1) = 1\right)$$

is rational, C_2-cofinite and of CFT type. In particular, the category of its ordinary modules has a structure of braided tensor category and is semisimple with finitely many inequivalent simple objects.

The assertion is a consequence of Kazama-Suzuki coset construction

$$W_k(\mathfrak{sl}_{n|1}, f_{\mathrm{prin}}) \simeq Com(\pi_{\mathrm{diag}}, W_\ell(\mathfrak{sl}_n, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}}),$$

$$\left((k+n-1)(\ell+n) = 1, \quad i.e., \ell = -n + \frac{n+r}{n-1}.\right).$$

and the fact that $W_\ell(\mathfrak{sl}_n, f_{\mathrm{sub}})$ is C_2-cofinite by Arakawa (exceptional) and rational by Arakawa-van Ekeren.

Rationality 2

The rationality and C_2-cofiniteness of $W_k(\mathfrak{sl}_{n|1}, f_{\mathrm{prin}})$ can also be deduced from the following description of its Heisenberg coset:

Theorem 3 (Creutzig-Linshaw)

$$Com(\pi, W_k(\mathfrak{sl}_{n|1}, f_{\mathrm{prin}})) \simeq W_{-r+\frac{r+n}{r+1}}(\mathfrak{sl}_r, f_{\mathrm{prin}}) \simeq Com(\pi, W_\ell(\mathfrak{sl}_n, f_{\mathrm{sub}}))$$

$$\left(k = -(n-1) + \frac{n-1}{n+r}, \quad \ell = -n + \frac{n+r}{n-1}.\right)$$

Then we have

- $W_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, f_{\mathrm{sub}}) \simeq \bigoplus_{p\in\mathbb{Z}/r\mathbb{Z}} \mathbb{L}(n\Lambda_p, 0) \otimes V_{\frac{np}{\sqrt{nr}}+\sqrt{nr}\mathbb{Z}}$
- $W_{-(n-1)+\frac{n-1}{n+r}}(\mathfrak{sl}_{n|1}, f_{\mathrm{prin}}) \simeq \bigoplus_{p\in\mathbb{Z}/r\mathbb{Z}} \mathbb{L}(n\Lambda_p, 0) \otimes V_{\frac{(n+r)p}{\sqrt{r(n+r)}}+\sqrt{r(n+r)}\mathbb{Z}}.$

In the above, we have used

- $W_{-r+\frac{r+n}{r+1}}(\mathfrak{sl}_r, f_{\mathrm{prin}})$ is the $(r+n, r+1)$-minimal series representation. In particular, it is rational and C_2-cofinite with

$$\mathrm{Irr}(W_{-r+\frac{r+n}{r+1}}(\mathfrak{sl}_r, f_{\mathrm{prin}})) = \{\mathbb{L}(\lambda, 0) \mid \lambda \in P_+^n(\mathfrak{sl}_r)\}.$$

- The lattice vertex superalgebras $V_{\sqrt{nr}\mathbb{Z}}$ and $V_{\sqrt{r(n+r)}\mathbb{Z}}$ are rational and C_2-cofinite with

$$\mathrm{Irr}(V_{\sqrt{nr}\mathbb{Z}}) = \{V_{\frac{p}{\sqrt{nr}}+\sqrt{nr}\mathbb{Z}}\}_{p\in\mathbb{Z}/nr\mathbb{Z}}$$
$$\mathrm{Irr}(V_{\sqrt{r(n+r)}\mathbb{Z}}) = \{V_{\frac{p}{\sqrt{r(n+r)}}+\sqrt{nr}\mathbb{Z}}\}_{p\in\mathbb{Z}/r(n+r)\mathbb{Z}}$$

- The isomorphisms are as vertex superalgerbas. In the RHS, the vertex superalgebra structure is given by the simple current extension of the vertex (super)algebras

$$W_{-r+\frac{r+n}{r+1}}(\mathfrak{sl}_r, f_{\mathrm{prin}}) \otimes V_{\sqrt{nr}\mathbb{Z}}, \quad W_{-r+\frac{r+n}{r+1}}(\mathfrak{sl}_r, f_{\mathrm{prin}}) \otimes V_{\sqrt{r(n+r)}\mathbb{Z}}.$$

Fusion rules of Simple current extension 1

The representation theory of simple current extensions are studied by many people, H.Li, Dong-Li-Mason, M.Miyamoto, H.Yamauchi, S.Carnahan, Creutzig-Kanade-MacRae to name a few. Let us consider the following setting:

$$\mathcal{E} \simeq \bigoplus_{\lambda \in N/L} \mathcal{C}_\lambda \otimes V_{\lambda+L} \quad (|N/L| < \infty).$$

Then the rationality and C_2-cofiniteness of $\mathcal{C}$ and $\mathcal{E}$ are equivalent by a result of Carnahan and Miyamoto and we assume them. Then we use the induction functor

$$\begin{array}{ccc} C\text{-mod} \otimes V_L\text{-mod} & \xrightarrow{\mathcal{E}\boxtimes} & \mathrm{Rep}(\mathcal{E}) \\ \uparrow & & \uparrow \\ (C\text{-mod} \otimes V_L\text{-mod})^{loc} & \xrightarrow{\mathcal{E}\boxtimes} & \mathrm{Rep}^{loc}(\mathcal{E}) = \mathcal{E}\text{-mod} \end{array}$$

Fusion rules of Simple current extension 2

Here Rep($\mathcal{E}$) is the category of $\mathcal{E}$-module objects in C-mod $\otimes$ V_L-mod:

$$(M, \mu_M : \mathcal{E} \boxtimes M \to M)$$

and "loc" means taking the full subcategory consisting of monodromy free objects i.e., M such that

$$\mathcal{M}_{\mathcal{E},M} = \mathrm{id}_{\mathcal{E}\boxtimes M}.$$

The set of simple currents $\{C_\lambda\}_{\lambda \in N/L}$ gives the block decomposition of C-mod by monodromy:

$$C\text{-mod} = \bigoplus_{\phi \in (N/L)^\vee} C\text{-mod}_\phi, \quad C\text{-mod}_\phi = \{M \mid \mathcal{M}_{C_\lambda,M} = \phi(\lambda)\mathrm{id}_{C_\lambda \boxtimes M}\}$$

Then

$$(C\text{-mod} \otimes V_L\text{-mod})^{loc} = \bigoplus_{\lambda \in (N/L)^\vee} C\text{-mod}_\phi \otimes V_L\text{-mod}_{\phi^{-1}}.$$

Fusion rules of Simple current extension 3

Note that $\mathcal{E} \boxtimes (M \otimes N) \simeq \mathcal{E} \boxtimes ((C_\lambda \otimes V_{\lambda+L}) \boxtimes (M \otimes N))$.

Proposition 0.4 (Creutzig-Genra-Sato-N)

We have the following isomorphism of fusion algebras:

$$K_{\mathbb{C}}(\mathcal{E}) \simeq \left(K_{\mathbb{C}}(\mathcal{C}) \bigotimes_{\mathbb{C}[N/L]} \mathbb{C}[L'/L] \right)^{N/L}.$$

Here

$$(M \boxtimes C_\lambda) \otimes V_{\mu+L} = M \otimes V_{-\lambda+\mu+L}, \quad L' = \{a \in \mathbb{Q} \otimes_{\mathbb{Z}} L \mid (a|L) \subset \mathbb{Z}\}.$$

Theorem 4 (Creutzig-Genra-Sato-N)

(1) $\mathrm{Irr}(W_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, f_{\mathrm{sub}})) = \left\{ M(\lambda, a) \mid \begin{smallmatrix} (\lambda, a) \in P^n_+(\mathfrak{sl}_r) \times (\mathbb{Z}/nr\mathbb{Z})/\sim \\ \lambda \equiv a \in \mathbb{Z}/r\mathbb{Z} \end{smallmatrix} \right\}$

$$M(\lambda, a) \boxtimes M(\mu, b) \simeq \bigoplus_{\nu \in P^n_+(\mathfrak{sl}_r)} N(L_n(\mathfrak{sl}_r))^{\nu}_{\lambda\mu} M(\nu, a+b).$$

(2) $\mathrm{Irr}(W_{-(n-1)+\frac{n-1}{n+r}}(\mathfrak{sl}_{n|1}, f_{\mathrm{prin}})) = \left\{ M(\lambda, a) \mid \begin{smallmatrix} (\lambda, a) \in P^n_+(\mathfrak{sl}_r) \times (\mathbb{Z}/r(n+r)\mathbb{Z})/\sim \\ \lambda \equiv a \in \mathbb{Z}/r\mathbb{Z} \end{smallmatrix} \right\}$

$$M(\lambda, a) \boxtimes M(\mu, b) \simeq \bigoplus_{\nu \in P^n_+(\mathfrak{sl}_r)} N(L_n(\mathfrak{sl}_r))^{\nu}_{\lambda\mu} M(\nu, a+b).$$

Remark 0.5

Arakawa-van Ekeren proved when n is even,
$\mathrm{Irr}(W_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, f_{\mathrm{sub}})) = \{\mathbb{L}(\lambda) \mid \lambda \in P^r_+(\mathfrak{sl}_n)\}$.

$$\mathbb{L}(\lambda) \boxtimes \mathbb{L}(\mu) \simeq \bigoplus_{\nu \in P^r_+(\mathfrak{sl}_n)} N(L_r(\mathfrak{sl}_n))^{\nu}_{\lambda\mu} \mathbb{L}(\nu).$$

Fusion rules of Lattice cosets

Here we reconstruct the fusion rules of the lattice coset $\mathcal{C} = Com(V_L, \mathcal{E})$ from the simple current extension

$$\mathcal{E} \simeq \bigoplus_{\lambda \in N/L} \mathcal{C}_\lambda \otimes V_{\lambda+L} \quad (|N/L| < \infty).$$

Theorem 5 (Creutzig-Genra-Sato-N)

We have the following isomorphism of fusion algebras:

$$K_{\mathbb{C}}(\mathcal{C}) \simeq \left(K_{\mathbb{C}}(\mathcal{E}) \bigotimes_{\mathbb{C}[N'/L]} \mathbb{C}[L'/L] \right)^{N'/L}.$$

Here

$$(M \boxtimes C_\lambda) \otimes V_{\mu+L} = M \otimes V_{\lambda+\mu+L}, \quad N' = \{a \in \mathbb{Q} \otimes_{\mathbb{Z}} N \mid (a|N) \subset \mathbb{Z}\}.$$

The proof is obtained in the following way:

- The induction functor $\mathcal{E} \boxtimes ? \colon \mathcal{C}\text{-mod} \to \mathrm{Rep}(\mathcal{E})$ is an embedding.
- The tensor category $\mathrm{Rep}(\mathcal{E})$ is semisimple and every simple object can be expressed as

$$M \boxtimes_{\mathcal{E}} (\mathcal{E} \boxtimes V_{\lambda+L}), \quad M \in Ob(\mathcal{E}\text{-mod}),\ V_{\lambda+L} \in Ob(V_L\text{-mod}).$$

- The ambiguity of the above expression is

$$(M \boxtimes_{\mathcal{E}} (\mathcal{E} \boxtimes V_{\lambda+L})) \boxtimes_{\mathcal{E}} (\mathcal{E} \boxtimes V_{\mu+L}) \simeq M \boxtimes_{\mathcal{E}} (\mathcal{E} \boxtimes V_{\lambda+\mu+L}), \quad \lambda \in N'/L.$$

The above theorem can be used to deduce the fusion rules of $W_{-(n-1)+\frac{n-1}{n+r}}(\mathfrak{sl}_{n|1}, f_{\mathrm{prin}})$ from those of $W_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, f_{\mathrm{sub}})$ via Kazama-Suzuki coset construction:

$$W_{-n+\frac{n+r}{n-1}}(\mathfrak{sl}_n, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}} \simeq \bigoplus_{\substack{p\in\mathbb{Z}/r\mathbb{Z} \\ t\in\mathbb{Z}/(n+r)\mathbb{Z}}} \mathbb{L}(n\Lambda_r, 0) \otimes V_{\frac{(n+r)p+rt}{\sqrt{r(n+r)}}+\sqrt{r(n+r)}\mathbb{Z}} \otimes V_{\frac{-nt}{\sqrt{n(n+r)}}+\sqrt{n(n+r)}\mathbb{Z}}$$

Example 0.6 ($\mathcal{N}=2$ superconformal algebra)

$$K_{\mathbb{C}}(L_{\frac{3r}{r+2}}(\mathfrak{ns}_2)) \simeq \left(K_{\mathbb{C}}(L_r(\mathfrak{sl}_2)) \bigotimes_{\mathbb{C}[\mathbb{Z}/2\mathbb{Z}]} \mathbb{C}[\mathbb{Z}/2(k+2)\mathbb{Z}] \right)^{\mathbb{Z}/2\mathbb{Z}}.$$

For $r=1$,

$$RHS = \left(\mathbb{C}[\mathbb{Z}/2\mathbb{Z}] \bigotimes_{\mathbb{C}[\mathbb{Z}/2\mathbb{Z}]} \mathbb{C}[\mathbb{Z}/6\mathbb{Z}] \right)^{\mathbb{Z}/2\mathbb{Z}} \simeq \mathbb{C}[\mathbb{Z}/3\mathbb{Z}].$$