Rogers–Ramanujan type identities

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L. Euler (1707–1783)
C. G. J. Jacobi (1804–1851)
L. J. Rogers (1862–1933)
Throughout: Assume $|q| < 1$. 
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$$\sum_{n \geq 0} \frac{q^n}{(1 - q)(1 - q^2) \cdots (1 - q^n)} = \prod_{m=1}^{\infty} \frac{1}{1 - q^m} \quad \text{(Euler)}$$

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\sum_{n \geq 0} \frac{q^{n^2}}{(1 - q)^2(1 - q^2)^2 \cdots (1 - q^n)^2} = \prod_{m \geq 1} \frac{1}{1 - q^m} \quad \text{(Jacobi)}
\]
Precursors to the RR identities

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\[m \equiv \pm 1 \pmod{5} \quad \text{(Rogers)}\]
Rising $q$-factorial notation

$$(a)_n = (a; q)_n := (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{n-1}),$$
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Rising $q$-factorial notation

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$$(a_1, a_2, \ldots a_r; q)_\infty := (a_1)_\infty(a_2)_\infty(a_3)_\infty \cdots (a_r)_\infty$$
S. Ramanujan (1887–1920)

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Rogers–Ramanujan type identities
For $|ab| < 1$, 

$$f(a, b) := \sum_{n \in \mathbb{Z}} a^{n(n+1)/2} b^{n(n-1)/2}.$$
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Jacobi’s triple product identity

$$f(a, b) = (a, b, ab; ab)_\infty.$$
Ramanujan’s notation

\[
f(-q) := f(-q, -q^2) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n(3n-1)/2} = (q)_\infty
\]

(Euler’s pentagonal numbers thm)
Ramanujan’s notation

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(Gauss's square numbers thm)

\[ \psi(-q) := f(-q, -q^3) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n(2n-1)} = \frac{(q^2; q^2)_\infty}{(-q; q^2)_\infty} \]

(Gauss's hexagonal numbers thm)
Rogers–Ramanujan type identities

\[ \sum_{n \geq 0} \frac{q^{n^2}}{(q)_n} = \frac{f(-q^2, -q^3)}{(q)_\infty}. \]
\[ \sum_{n \geq 0} \frac{q^{n(n+1)}}{(q)_n} = \frac{f(-q, -q^4)}{(q)_\infty}. \]

Ramanujan really enjoyed identities of this type. Over 50 are recorded in the lost notebook.
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Bailey pairs, Bailey’s lemma

If \((\alpha_n(a, q), \beta_n(a, q))\) satisfies

\[
\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}},
\]

then \((\alpha_n, \beta_n)\) is called a \textit{Bailey pair with respect to} \(a\),
If \((\alpha_n(a, q), \beta_n(a, q))\) satisfies

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\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r} (aq)_{n+r}},
\]

then \((\alpha_n, \beta_n)\) is called a Bailey pair with respect to \(a\), and

\((\alpha'_n(a, q), \beta'_n(a, q))\) is also a Bailey pair, where

\[
\alpha'_r(a, q) = \frac{(\rho_1)_r (\rho_2)_r}{(aq/\rho_1)_r (aq/\rho_2)_r} \left( \frac{aq}{\rho_1 \rho_2} \right)^r \alpha_r
\]

and

\[
\beta'_n(a, q) = \sum_{j=0}^{n} \frac{(\rho_1)_j (\rho_2)_j (aq/\rho_1 \rho_2)_{n-j}}{(aq/\rho_1)_n (aq/\rho_2)_n (q)_{n-j}} \left( \frac{aq}{\rho_1 \rho_2} \right)^j \beta_j(a, q).
\]
Limiting cases of Bailey’s lemma

\[
\sum_{n \geq 0} q^{n^2} \beta_n(1, q) = \frac{1}{(q)_\infty} \sum_{r \geq 0} q^{r^2} \alpha_r(1, q) \quad \text{(PBL)}
\]

\[
\sum_{n \geq 0} q^{n^2} (-q; q^2)_n \beta_n(1, q^2) = \frac{1}{\psi(-q)} \sum_{r \geq 0} q^{r^2} \alpha_r(1, q^2) \quad \text{(HBL)}
\]

\[
\sum_{n \geq 0} q^{n(n+1)/2} (-1)_n \beta_n(1, q) = \frac{2}{\varphi(-q)} \sum_{r \geq 0} \frac{q^{r(r+1)/2}}{1 + q^r} \alpha_r(1, q) \quad \text{(SBL)}
\]
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Lucy Slater found many Bailey pairs, and used them to generate a list of 130 RR type identities.
For $d | n$, define
\[
\alpha_n^{(d,e,k)}(a, q) := \frac{(-1)^{n/d} a^{(k/d-1)n/e} q^{(k/d-1+1/2d)n^2/e-n/2e}}{(1 - a^{1/e}) (q^{d/e}; q^{d/e})_{n/d}}
\times (1 - a^{1/e} q^{2n/e}) (a^{1/e}; q^{d/e})_{n/d},
\]

Let the corresponding $\beta_n^{(d,e,k)}(a, q)$, $\tilde{\beta}_n^{(d,e,k)}(a, q)$, and $\bar{\beta}_n^{(d,e,k)}(a, q)$ be determined by the Bailey pair relation.
For $d \mid n$, define

$$
\alpha_n^{(d,e,k)}(a, q) := \frac{(-1)^{n/d} a^{(k/d-1)n/e} q^{(k/d-1+1/2d)n^2/e-n/2e}}{(1 - a^{1/e})(q^{d/e}; q^{d/e})_{n/d}} \times (1 - a^{1/e} q^{2n/e})(a^{1/e}; q^{d/e})_{n/d},
$$

and

$$
\tilde{\alpha}_n^{(d,e,k)}(a, q) := q^{n(d-n)/2de} a^{-n/de} \frac{(-a^{1/e}; q^{d/e})_{n/d}}{(-q^{d/e}; q^{d/e})_{n/d}} \alpha_n^{(d,e,k)}(a, q),
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$$

$$
\tilde{\alpha}_n^{(d,e,k)}(a, q) := q^{n(d-n)/2de} a^{-n/de} \frac{(-a^{1/e}; q^{d/e})_{n/d}}{(-q^{d/e}; q^{d/e})_{n/d}} \alpha_n^{(d,e,k)}(a, q),
$$

$$
\bar{\alpha}_n^{(d,e,k)}(a, q) := (-1)^{n/d} q^{n^2/2de} \frac{(q^{d/2e}; q^{d/e})_{n/d}}{(a^{1/e} q^{d/2e}; q^{d/e})_{n/d}} \alpha_n^{(d,e,k)}(a, q).
$$
General Bailey pairs

For $d \mid n$, define

$$\alpha_n^{(d,e,k)}(a, q) := (-1)^{n/d} a^{(k/d-1)n/e} q^{(k/d-1+1/2d)n^2/e-n/2e} \left( 1 - a^{1/e} \right) \left( q^{d/e}; q^{d/e} \right)_{n/d}$$

$$\times \left( 1 - a^{1/e} q^{2n/e} \right) \left( a^{1/e}; q^{d/e} \right)_{n/d},$$

$$\tilde{\alpha}_n^{(d,e,k)}(a, q) := q^{n(d-n)/2de} a^{-n/de} \frac{\left( -a^{1/e}; q^{d/e} \right)_{n/d}}{\left( -q^{d/e}; q^{d/e} \right)_{n/d}} \alpha_n^{(d,e,k)}(a, q),$$

$$\bar{\alpha}_n^{(d,e,k)}(a, q) := (-1)^{n/d} q^{n^2/2de} \frac{\left( q^{d/2e}; q^{d/e} \right)_{n/d}}{\left( a^{1/e} q^{d/2e}; q^{d/e} \right)_{n/d}} \alpha_n^{(d,e,k)}(a, q).$$

Let the corresponding $\beta_n^{(d,e,k)}(a, q)$, $\tilde{\beta}_n^{(d,e,k)}(a, q)$, and $\bar{\beta}_n^{(d,e,k)}(a, q)$ be determined by the Bailey pair relation.
For any positive integer triples \((d, e, k)\), upon inserting any of these \(\alpha\)'s into any of the limiting cases of Bailey’s lemma with \(a = 1\), the resulting series is summable via Jacobi’s triple product identity.
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For certain \((d, e, k)\), the resulting expression for \(\beta\) is a very well-poised \(\phi_5\), summable by a theorem of F. H. Jackson.
For any positive integer triples \((d, e, k)\), upon inserting any of these \(\alpha\)'s into any of the limiting cases of Bailey's lemma with \(a = 1\), the resulting series is summable via Jacobi's triple product identity.

For certain \((d, e, k)\), the resulting expression for \(\beta\) is a very well-poised \(6\phi_5\), summable by a theorem of F. H. Jackson.

Using only this, and an associated families of \(q\)-difference equations, one can recover the majority of Slater's list, as well as other identities.
The Bailey pair that arises from

\[ \left( \alpha_n^{(1,1,2)}(a, q), \beta_n^{(1,1,2)}(a, q) \right) \]

\[ = \left( \frac{(-1)^n a^n q^{n(3n-1)/2}(1 - aq^{2n}) (a)_n}{(1 - a)(q)_n}, \frac{1}{(q)_n} \right) \]

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\sum_{n \geq 0} \frac{q^{n^2}}{(q)_n} = \frac{f(-q^2, -q^3)}{(q)_{\infty}}
\]

upon insertion into (PBL),
The Bailey pair that arises from

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\]

gives

- \[ \sum_{n \geq 0} \frac{q^{n^2}}{(q)_n} = \frac{f(-q^2, -q^3)}{(q)_\infty} \] upon insertion into (PBL),
- \[ \sum_{n \geq 0} \frac{q^{n(n+1)}(-1)_n}{(q)_n} = \frac{\varphi(-q^2)}{\varphi(-q)} \] upon insertion into (SBL), and
The Bailey pair that arises from

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\]

yields

- \[ \sum_{n \geq 0} q^{n^2} \frac{(q)_n}{(q)_n} = \frac{f(-q^2, -q^3)}{(q)_\infty} \] upon insertion into (PBL),
- \[ \sum_{n \geq 0} q^{n(n+1)} (-1)_n \frac{(q)_n}{(q)_n} = \frac{\varphi(-q^2)}{\varphi(-q)} \] upon insertion into (SBL), and
- \[ \sum_{n \geq 0} q^{n^2} (-q; q^2)_n \frac{(q^2; q^2)_n}{(q^2; q^2)_n} = \frac{f(-q^3, -q^5)}{\psi(-q)} \] upon insertion into (HBL).
New identities arising from this framework (S.)

\[ \sum_{n,r \geq 0} \frac{q^{n^2+2nr+2r^2}(-q; q^2)_r}{(q)_{2r}(q)_n} = \frac{f(-q^{10}, -q^{10})}{(q)_\infty} \]

by insertion of \((\tilde{\alpha}_n^{(2,1,5)}(1, q), \tilde{\beta}_n^{(2,1,5)}(1, q))\) into (PBL).
New identities arising from this framework (S.)

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by insertion of \((\bar{\alpha}_n^{(2,1,5)}(1, q), \bar{\beta}_n^{(2,1,5)}(1, q))\) into (PBL).

\[ \sum_{n,r \geq 0} \frac{q^{4n^2+8nr+8r^2}(-q;q^2)_{2r}}{(q^4;q^4)_{2r}(q^4;q^4)_n} = \frac{f(q^9, q^{11})}{(q^4;q^4)_{\infty}} \]

by insertion of \((\bar{\alpha}_n^{(1,2,4)}(1, q), \bar{\beta}_n^{(1,2,4)}(1, q))\) into (PBL).
New identities arising from this framework (S.)

\[ \sum_{n,r \geq 0} \frac{q^{n^2 + 2nr + 2r^2} (-q; q^2)_r}{(q)_{2r}(q)_n} = \frac{f(-q^{10}, -q^{10})}{(q)_\infty} \]

by insertion of \((\tilde{\alpha}_n^{(2,1,5)}(1, q), \tilde{\beta}_n^{(2,1,5)}(1, q))\) into (PBL).

\[ \sum_{n,r \geq 0} \frac{q^{4n^2 + 8nr + 8r^2} (-q; q^2)_{2r}}{(q^4; q^4)_{2r}(q^4; q^4)_n} = \frac{f(q^9, q^{11})}{(q^4; q^4)_\infty} \]

by insertion of \((\bar{\alpha}_n^{(1,2,4)}(1, q), \bar{\beta}_n^{(1,2,4)}(1, q))\) into (PBL).
A family of mod 24 identities

\[
\sum_{n \geq 0} \frac{q^{n(n+2)}(-q; q^2)_n(-1; q^6)_n}{(q^2; q^2)_{2n}(-1; q^2)_n} = \frac{f(-q, -q^{11})f(-q^{10}, -q^{14})}{\psi(-q)(q^{24}; q^{24})_{\infty}}
\]

(McLaughlin.-S.)

\[
\sum_{n=0}^{\infty} \frac{q^{n^2}(-q^3; q^6)_n}{(q^2; q^2)_{2n}} = \frac{f(-q^2, -q^{10})f(-q^8, -q^{16})}{\psi(-q)(q^{24}; q^{24})_{\infty}}
\]

(Ramanujan)

\[
\sum_{n \geq 0} \frac{q^{n^2}(-q; q^2)_n(-1; q^6)_n}{(q^2; q^2)_{2n}(-1; q^2)_n} = \frac{f(-q^3, -q^9)f(-q^6, -q^{18})}{\psi(-q)(q^{24}; q^{24})_{\infty}}
\]

(M.-S.)

\[
\sum_{n \geq 0} \frac{q^{n(n+2)}(-q^3; q^6)_n}{(q^2; q^2)_{2n}(1 - q^{2n+1})} = \frac{f(-q^4, -q^8)f(-q^4, -q^{20})}{\psi(-q)(q^{24}; q^{24})_{\infty}}
\]

(M.-S.)

\[
\sum_{n \geq 0} \frac{q^{n(n+2)}(-q; q^2)_{n+1}(-q^6; q^6)_n}{(q^4; q^4)_n(q^{2n+4}; q^2)_{n+1}} = \frac{f(-q^5, -q^7)f(-q^2, -q^{22})}{\psi(-q)(q^{24}; q^{24})_{\infty}}
\]

(M.-S.)
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A *partition* $\lambda$ of $n$ is a tuple $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ of weakly decreasing positive integers (called the *parts* of $\lambda$) that sum to $n$. 

The seven partitions of 5 are 

$(5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1)$. 

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Euler’s partition theorem

The number of partitions of $n$ into odd parts equals the number of partitions of $n$ into distinct parts.
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Example:

\[ 9, 711, 531, 51111, 333, 33111, 3111111, 111111111 \]
Euler’s partition theorem

The number of partitions of $n$ into odd parts equals the number of partitions of $n$ into distinct parts.

Example:

$$9, 711, 531, 51111, 333, 33111, 3111111, 111111111$$

$$9, 81, 72, 63, 621, 54, 531, 432$$
The number of partitions of \( n \) into parts that mutually differ by at least 2 equals the number of partitions of \( n \) into parts congruent to \( \pm 1 \pmod{5} \).
The number of partitions of $n$ into parts that mutually differ by at least 2 equals the number of partitions of $n$ into parts congruent to $\pm 1 \pmod{5}$.

The number of partitions of $n$ into parts greater than 1 that mutually differ by at least 2 equals the number of partitions of $n$ into parts congruent to $\pm 2 \pmod{5}$. 
Let $k$ be a positive integer and $1 \leq i \leq k$. 

Note: The case $k = 2$ gives the standard combinatorial interpretation of the two RR identities.
B. Gordon’s combinatorial generalization of RR (1961)

Let $k$ be a positive integer and $1 \leq i \leq k$. Let $A_{k,i}(n)$ denote the number of partitions of $n$ into parts $\not\equiv 0, \pm i \pmod{2k + 1}$.
Let $k$ be a positive integer and $1 \leq i \leq k$. Let $A_{k,i}(n)$ denote the number of partitions of $n$ into parts $\not\equiv 0, \pm i \pmod{2k+1}$. Let $B_{k,i}(n)$ denote the number of partitions $\lambda$ of $n$ where

- at most $i - 1$ of the parts of $\lambda$ equal 1,
- $\lambda_j - \lambda_{j+k-1} \geq 2$ for $j = 1, 2, \ldots, l(\lambda) + 1 - k$.
Let $k$ be a positive integer and $1 \leq i \leq k$. Let $A_{k,i}(n)$ denote the number of partitions of $n$ into parts \not\equiv 0, \pm i \pmod{2k + 1}$. Let $B_{k,i}(n)$ denote the number of partitions $\lambda$ of $n$ where

1. at most $i - 1$ of the parts of $\lambda$ equal 1,
2. $\lambda_j - \lambda_{j+k-1} \geq 2$ for $j = 1, 2, \ldots, l(\lambda) + 1 - k$.

Then $A_{k,i}(n) = B_{k,i}(n)$ for all $n$. 

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Let $k$ be a positive integer and $1 \leq i \leq k$. Let $A_{k,i}(n)$ denote the number of partitions of $n$ into parts $\not\equiv 0, \pm i \pmod{2k + 1}$. Let $B_{k,i}(n)$ denote the number of partitions $\lambda$ of $n$ where

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Then $A_{k,i}(n) = B_{k,i}(n)$ for all $n$.

**Note:** The case $k = 2$ gives the standard combinatorial interpretation of the two RR identities.
G. Andrews’ analytic counterpart to Gordon’s theorem

\[ \sum_{n_{k-1} \geq n_{k-2} \geq \ldots \geq n_1 \geq 0} \frac{q^{n_1^2 + n_2^2 + \cdots + n_{k-1}^2 + n_i + n_{i+1} + \cdots + n_{k-1}}}{(q)n_1(q)n_2-n_1(q)n_3-n_2 \cdots (q)n_{k-1}-n_{k-2}} = f(-q^i, -q^{2k+1-i}) \left( \frac{1}{q} \right)_\infty. \]
Let $d \in \mathbb{N}$ and let $1 \leq i \leq k$. Let $G_{d,k,i}(n)$ denote the number of partitions $\pi$ of $n$ such that

$$m_d(\pi) \leq i - 1 \text{ and } m_{dj}(\pi) + m_{dj+d}(\pi) \leq k - 1$$

for any $j \in \mathbb{N}$. 

Combinatorial interpretations of these “$(d, e, k)$” identities (S.)
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Let \(G_{d,k,i}(n)\) denote the number of partitions \(\pi\) of \(n\) such that

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m_d(\pi) \leq i - 1 \quad \text{and} \quad m_{dj}(\pi) + m_{dj+d}(\pi) \leq k - 1
\]

for any \(j \in \mathbb{N}\).

Let \(H_{d,k,i}(n)\) denote the number of partitions of \(n\) into parts

\[
\not\equiv 0, \pm di \pmod{2d(k + 1)}.
\]
Let $d \in \mathbb{N}$ and let $1 \leq i \leq k$.
Let $G_{d,k,i}(n)$ denote the number of partitions $\pi$ of $n$ such that

$$m_d(\pi) \leq i - 1 \text{ and } m_{dj}(\pi) + m_{dj+d}(\pi) \leq k - 1$$

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Let $H_{d,k,i}(n)$ denote the number of partitions of $n$ into parts

$\not\equiv 0, \pm di \pmod{2d(k+1)}$.
Then $G_{d,k,i}(n) = H_{d,k,i}(n)$ for all integers $n$. 
Combinatorial interpretations of these “\((d, e, k)\)” identities (S.)

Let \(d \in \mathbb{N}\) and let \(1 \leq i \leq k\).
Let \(G_{d,k,i}(n)\) denote the number of partitions \(\pi\) of \(n\) such that
\[
m_d(\pi) \leq i - 1 \text{ and } m_{dj}(\pi) + m_{dj+d}(\pi) \leq k - 1
\]
for any \(j \in \mathbb{N}\).
Let \(H_{d,k,i}(n)\) denote the number of partitions of \(n\) into parts
\(\not\equiv 0, \pm di \pmod{2d(k+1)}\).
Then \(G_{d,k,i}(n) = H_{d,k,i}(n)\) for all integers \(n\).
This is a combinatorial interpretation of the identity obtained
by inserting the Bailey pair \((\alpha_n^{(d,1,k)}(1,q), \beta_n^{(d,1,k)}(1,q))\)
into (PBL) (along with associated systems of \(q\)-difference equations).
WHO
(outside the partitions and $q$-series community)
CARES?
In the 1980’s J. Lepowsky and R. Wilson showed that the principally specialized characters of standard modules for the odd levels of $A_1^{(1)}$ are given by the The Andrews–Gordon identity.
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The two Rogers–Ramanujan identities occur at level 3.

The even levels of $A_1^{(1)}$ correspond to D. Bressoud’s even modulus analog of Andrews–Gordon.
Capparelli’s identities (1988)

The Rogers–Ramanujan identities also occur at level 2 of $A_2^{(2)}$. 

Performing an analogous analysis of the level 3 modules of $A_2^{(2)}$, S. Capparelli discovered:

The number of partitions of $n$ into parts $\equiv \pm 2, \pm 3 \pmod{12}$ equals the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ of $n$ where $\lambda_i - \lambda_i + 1 \geq 2, \lambda_i - \lambda_i + 1 = 2 \Rightarrow \lambda_i \equiv 1 \pmod{3}, \lambda_i - \lambda_i + 1 = 3 \Rightarrow \lambda_i \equiv 0 \pmod{3}$.
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- $\lambda_i - \lambda_{i+1} \geq 2$,
- $\lambda_i - \lambda_{i+1} = 2 \implies \lambda_i \equiv 1 \pmod{3}$,
- $\lambda_i - \lambda_{i+1} = 3 \implies \lambda_i \equiv 0 \pmod{3}$
1 + \sum_{n,j,r \geq 0 \atop (n,j,r) \neq (0,0,0)} \frac{q^{3n^2 + \frac{9}{2}r^2 + 3j^2 + 6nj + 6nr + 6rj - \frac{5}{2}r - j} (1 + q^{2r+2j}) (1 - q^{6r+6j})}{(q^3; q^3)_n (q^3; q^3)_r (q^3; q^3)_j (q^3; q^3)_{j+1} (q^3; q^3)_{n+2r+2j}} = \frac{1}{(q^2, q^3, q^9, q^{10}; q^{12})_\infty}
1 + \sum_{n,j,r \geq 0 \atop (n,j,r) \neq (0,0,0)} \frac{q^{3n^2 + \frac{9}{2}r^2 + 3j^2 + 6nj + 6nr + 6rj - \frac{5}{2}r - j}(1 + q^{2r+2j})(1 - q^6r + 6j)}{(q^3; q^3)_n(q^3; q^3)_r(q^3; q^3)_j(-1; q^3)_j+1(q^3; q^3)_{n+2r+2j}} = \frac{1}{(q^2, q^3, q^9, q^{10}; q^{12})_\infty}

\sum_{n,j \geq 0} \frac{q^{n^2} \left( \frac{n-j+1}{3} \right)}{(q)_{2n-j}(q)_j} = \frac{1}{(q^2, q^3, q^9, q^{10}; q^{12})_\infty}.
In an analogous study of the level 4 modules of $A_2^{(2)}$, D. Nandi (2014) conjectured three partition identities.
In an analogous study of the level 4 modules of $A_2^{(2)}$, D. Nandi (2014) conjectured three partition identities. Proved by Motoki Takigiku and Shunsuke Tsuchioka (2019). One of these identities is:

The number of partitions of $n$ into parts $\equiv \pm 2, \pm 3, \pm 4 \pmod{14}$ equals the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ of $n$ where

$$\lambda_i - \lambda_i + 1 \geq 2, \quad \lambda_i - \lambda_i + 2 \geq 3, \quad \lambda_i - \lambda_i + 2 = 3, \quad 2 \nmid \lambda_i = \lambda_i + 1, \quad \lambda_i - \lambda_i + 2 \neq \lambda_i + 1, \quad \lambda_i - \lambda_i + 2 \neq 3.$$

One of these identities is:

The number of partitions of $n$ into parts $\equiv \pm 2, \pm 3, \pm 4 \pmod{14}$ equals the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ of $n$ where

- $\lambda_i - \lambda_{i+1} \geq 2$
- $\lambda_i - \lambda_{i+2} \geq 3$
- $\lambda_i - \lambda_{i+2} = 3 \implies \lambda_i \neq \lambda_{i+1}$,
- $\lambda_i - \lambda_{i+2} = 3$ and $2 \nmid \lambda_i \implies \lambda_{i+1} \neq \lambda_{i+2}$.
- $\lambda_i - \lambda_{i+2} = 4$ and $2 \nmid \lambda_i \implies \lambda_i \neq \lambda_{i+1}$.

Consider the first differences

$\Delta \lambda := (\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{l-1} - \lambda_l)$. None of the following subwords are permitted in $\Delta \lambda$:

$(3, 3, 0), (3, 2, 3, 0), (3, 2, 2, 3, 0), \ldots, (3, 2, 2, 2, \ldots, 2, 3, 0)$. 

Andrew Sills  
Rogers–Ramanujan type identities
Shashank Kanade and Matthew Russell (2014)

Related to level 3 standard modules of $D_4^{(3)}$, Kandade and Russell conjectured several partition identities, including:

Andrew Sills  
Rogers–Ramanujan type identities
Related to level 3 standard modules of $D_4^{(3)}$, Kandade and Russell conjectured several partition identities, including:

The number of partitions of $n$ into parts $\equiv \pm 1, \pm 3 \pmod{9}$ equals the number of partitions $\lambda$ of $n$ such that

- $\lambda_j - \lambda_{j+2} \geq 3$,
- $\lambda_j - \lambda_{j+1} \leq 1 \implies 3 \mid (\lambda_j + \lambda_{j+1})$. 
Kanade–Russell conjectures

Kanade and Russell have released a steady stream of $q$-series and partition identity conjectures over the past six years.
Kanade and Russell have released a steady stream of $q$-series and partition identity conjectures over the past six years. Many have been proved by

- Katherin Bringmann, Chris Jennings-Shaffer, and Karl Mahlburg;
- Kagan Kurşungöz;
- Hjalmar Rosengren;
- Kanade and Russell themselves.
WHO ELSE CARES?
Polynomial RR identities

\[ D_0(q) = D_1(q) = 1 \]

\[ D_n(q) = D_{n-1}(q) + q^{n-1}D_{n-2} \text{ if } n \geq 2 \]
Polynomial RR identities

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\[ D_n(q) = \sum_{j \geq 0} q^{j^2} \left[ \begin{array}{c} n-j \\ j \end{array} \right]_q \]  \hspace{1cm} \text{(MacMahon)}
Polynomial RR identities

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\[ = \sum_{j \in \mathbb{Z}} (-1)^j q^{j(5j+1)/2} \left[ \frac{n}{\frac{n+5j+1}{2}} \right]_q \quad \text{(Schur)} \]
Polynomial RR identities

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\[ = \sum_{k \in \mathbb{Z}} \left( q^{k(10k+1)} \tau_0(n, 5k; q) - q^{(5k+3)(2k+1)} \tau_0(n, 5k + 3; q) \right) \]  \hspace{1cm} \text{(Andrews)}
We can prove these polynomial identities via recurrences, and then the original series–infinite product identity follows via asymptotics of $q$-bi/trinomial coefficients, and the triple product identity.
$q$-binomial and $q$-trinomial coefficients

\[
\begin{bmatrix} A \\ B \end{bmatrix}_q := \frac{(q)_A}{(q)_B} \frac{(q)_A}{(q)_{A-B}} \quad \text{if } 0 \leq B \leq A; \ 0 \ o/w
\]
\[ \binom{A}{B}_q := (q)_A(q)_B^{-1}(q)_{A-B}^{-1} \text{ if } 0 \leq B \leq A; 0 \text{ o/w} \]

\[ T_0(L, A; q) := \sum_{r=0}^{L} (-1)^r \binom{L}{r}_q^2 \binom{2L - 2r}{L - A - r}_q \]
\[ \begin{bmatrix} A \\ B \end{bmatrix}_q := (q)_A(q)_B^{-1}(q)_{A-B}^{-1} \text{ if } 0 \leq B \leq A; \ 0 \text{ o/w} \]

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\[ T_1(L, A; q) := \sum_{r=0}^{L} (-q)^r \left[ \begin{array}{c} L \\ r \end{array} \right] q^2 \left[ \begin{array}{c} 2L - 2r \\ L - A - r \end{array} \right]_q \]
$q$-binomial and $q$-trinomial coefficients

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\begin{bmatrix} A \\ B \end{bmatrix}_q := (q)_A(q)_B^{-1}(q)_{A-B}^{-1} \text{ if } 0 \leq B \leq A; \ 0 \ o/w
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$$
T_0(L, A; q) := \sum_{r=0}^{L} (-1)^r \binom{L}{r} q^{2L-2r} \prod_{k=0}^{r-1} \left(L - A - r\right)_q
$$

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$$

$$
\tau_0(L, A; q) := \sum_{r=0}^{L} (-1)^r q^{Lr-L(2)} \binom{L}{r} q^{2L-2r} \prod_{k=0}^{r-1} \left(L - A - r\right)_q
$$
$q$-binomial and $q$-trinomial coefficients

$${A \choose B}_q := (q)_A(q)_B^{-1}(q)_{A-B}^{-1} \text{ if } 0 \leq B \leq A; \ 0 \ o/w$$

$$T_0(L, A; q) := \sum_{r=0}^{L} (-1)^r \left[ \begin{array}{c} L \\ r \end{array} \right] q^2 \left[ \begin{array}{c} 2L - 2r \\ L - A - r \end{array} \right]_q$$

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U(L, A; q) := T_0(L, A; q) + T_0(L, A + 1; q)
linear combinations of $q$-trinomial coefficients

$$U(L, A; q) := T_0(L, A; q) + T_0(L, A + 1; q)$$

and

$$V(L, A; q) := T_1(L - 1, A; q) + q^{L-A}T_0(L - 1, A - 1; q).$$
The Andrews Method of Finitization

\[ G(q) := \sum_{j \geq 0} \frac{q^j}{(q)_j}. \]
The Andrews Method of Finitization

\[ G(q) := \sum_{j \geq 0} \frac{q^{j^2}}{(q)_j} \cdot \]

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\[ \lim_{t \to 1^-} (1 - t) \mathcal{G}(t) = G(q) \quad \text{(by Abel's lemma)}. \]
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\[ \mathfrak{G}(t) = \sum_{n \geq 0} D_n(q) t^n. \]

\[ \lim_{n \to \infty} D_n(q) = G(q) \]
I “algorithmitized” and generalized Andrews’ heuristic, and implemented it in Maple.
I “algorithmitized” and generalized Andrews’ heuristic, and implemented it in Maple.

“Finitized” all 130 identities in Slater’s list of RR type identities.
\[
\sum_{j \geq 0} \frac{q^{(j+1)/2}(-q^2; q^2)_j}{(q)_j(q; q^2)_{j+1}} = \frac{\psi(-q^2)}{\varphi(-q)}.
\]
\[
\sum_{j \geq 0} \frac{q^{j(j+1)/2}(-q^2; q^2)_j}{(q)_j(q; q^2)_{j+1}} = \frac{\psi(-q^2)}{\varphi(-q)}.
\]

For fixed \(n\),

\[
\sum_{i,j,k \geq 0} q^{j(j+1)/2+i^2+i+k} \left[ \begin{array}{c} j \\ i \end{array} \right] q^2 \left[ \begin{array}{c} j+k \\ k \end{array} \right] q^2 \left[ \begin{array}{c} n-2i-2k \\ j \end{array} \right]_q \\
= \sum_{j \in \mathbb{Z}} (-1)^j q^{2j(2j+1)} V(n+1, 4j + 1; \sqrt{q}).
\]
\[ \sum_{j \geq 0} \frac{q^{(j+1)/2}(-q^2; q^2)_j}{(q)_j(q; q^2)_{j+1}} = \frac{\psi(-q^2)}{\varphi(-q)}. \]

For fixed \( n \),

\[ \sum_{i,j,k \geq 0} q^{(j+1)/2 + i^2 + i + k} \left[ \begin{array}{c} j \\ i \end{array} \right] q^2 \left[ \begin{array}{c} j+k \\ k \end{array} \right] q^2 \left[ \begin{array}{c} n-2i-2k \\ j \end{array} \right]_q = \sum_{j \in \mathbb{Z}} (-1)^j q^{2j(2j+1)} \vartheta(n+1, 4j+1; \sqrt{q}). \]

\( q \)-Pell numbers: \( P_0 = 1, \quad P_1 = q + 1, \quad P_2 = q^3 + q^2 + 2q + 1 \)

\[ P_n = (1 + q^n)P_{n-1} + qP_{n-2} + (q^n - q)P_{n-3}. \]
Andrew Sills

Rogers–Ramanujan type identities

\[
\sum_{j \geq 0} \frac{q^{j(j+1)}(-q^3; q^3)_j}{(-q)_j(q)_{2j+1}} = \frac{f(-q^3, -q^6)f(-q^3, -q^{15})}{(q)_\infty(q^{18}; q^{18})_\infty}
\]
\[
\sum_{j \geq 0} \frac{q^{j(j+1)}(-q^3; q^3)_j}{(-q)_j(q)_{2j+1}} = \frac{f(-q^3, -q^6)f(-q^3, -q^{15})}{(q)_\infty(q^{18}; q^{18})_\infty}
\]

For fixed \( n \),

\[
\sum_{i,j,k,l,m \geq 0} (-1)^{k+m} q^{j^2 + 2j + 3i(i+1)/2 + k+l+m} \left[ \begin{array}{c} j \\ i \end{array} \right]_q \left[ \begin{array}{c} j+k-1 \\ k \end{array} \right]_q \\
\times \left[ \begin{array}{c} j+l \\ l \end{array} \right]_q^2 \left[ \begin{array}{c} j+m-1 \\ m \end{array} \right]_q \left[ \begin{array}{c} n-3i-j-k-2l-m \\ j \end{array} \right]_q
\]

\[
= \sum_{k \in \mathbb{Z}} q^{9k(3k+1)/2} \left[ \begin{array}{c} n+1 \\ \frac{n+9k+3}{2} \end{array} \right]_q - q^3 \sum_{k \in \mathbb{Z}} q^{27k(k+1)/2} \left[ \begin{array}{c} n+1 \\ \frac{n+9k+6}{2} \end{array} \right]_q
\]
\[
\sum_{j \geq 0} \frac{q^{3j^2} (-q; q^2)_{3j}}{(q^6; q^6)_{2j}} = \frac{f(q^4, q^8)}{\psi(-q^3)}
\]
\[ \sum_{j \geq 0} \frac{q^{3j^2}(-q; q^2)_{3j}}{(q^6; q^6)_{2j}} = \frac{f(q^4, q^8)}{\psi(-q^3)} \]

For fixed \( n \),

\[ \sum_{i,j,k \geq 0} (-1)^k q^{3j^2+i^2+3k} \left[ \begin{array}{c} 3j \\ i \end{array} \right] q^{2j+k-1} \left[ \begin{array}{c} n+j-i-k \\ 2j \end{array} \right] = \sum_{j \in \mathbb{Z}} q^{6j^2+2j} (T_0(n, 2j; q^3) + T_0(n-1, 2j; q^3)) . \]
An Invitation to the ROGERS-RAMANUJAN IDENTITIES

Andrew V. Sills
THANK YOU!