(B1)

3.

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \qquad t = \cancel{to}\ \sin^2 s$$

$$1-t = \cos^2 s$$

$$dt = 2\sin s\cos s\,ds$$

$$= \int_0^{\pi/2} (\sin s)^{2(x-1)} (\cos s)^{2(y-1)}\, 2\sin s\cos s\,ds$$

$$= 2\int_0^{\pi/2} (\sin s)^{2x-1}(\cos s)^{2y-1}\,ds$$

1. 授業でやった

2. $B(x,y) = B(y,x)$ は $t = 1-s$ とする.

$$xB(x,y+1) = x\int_0^1 \underbrace{t^{x-1}}_{f'}\underbrace{(1-t)^{y}}_{g}\,dt$$

$$= x\left\{\left[\frac{t^x}{x}(1-t)^y\right]_0^1 + \int_0^1 \frac{t^x}{x}(1-t)^{y-1}\cdot y\,dt\right\}$$

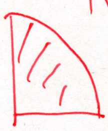
$x, y > 0$ だから

$$= y\int_0^1 t^x (1-t)^{y-1}\,dt = yB(x+1,y)$$

(B2)

1. $\iint_{D_R} e^{-x^2-y^2}\,dxdy$

$D_R.$



$x = r\cos\theta$ $\qquad 0 \le r \le R$

$y = r\sin\theta$ $\qquad 0 \le \theta \le \cancel{2\pi}\ \frac{\pi}{2}$

$$\iint e^{-r^2}\, r\,dr\,d\theta$$

$$= \int_0^{\cancel{2\pi}\ \frac{\pi}{2}} \left[-\frac{1}{2}e^{-r^2}\right]_{r=0}^{R} d\theta$$

$$= \int_0^{\cancel{2\pi}\ \frac{\pi}{2}} \left(-\frac{1}{2}e^{-R^2} + \frac{1}{2}\right) d\theta$$

$$= \frac{\pi}{4}\left(1 - e^{-R^2}\right)$$

2. Fubini より $I(R)^2 = \iint_{\square R} e^{-x^2-y^2}\,dxdy$

$D_R \subseteq \square R \subseteq D_{\sqrt{2}R}$ と $e^{-x^2-y^2} \ge 0$ より 従う

3. $R \to \infty$ とすると $\lim_{R\to\infty} I(R) = \frac{\sqrt{\pi}}{2}.$

(B3) $x = \sqrt{t}$ $\qquad dx = \frac{dt}{2\sqrt{t}}$

$\cancel{\int_{-\infty}^{\infty} e}$

$$\int_{-\infty}^{\infty} t^{-\frac{1}{2}} e^{-t}\,dt = \int_0^{\infty} e^{-x^2}\,2dx.$$

$$= 2\int_0^{\infty} e^{-x^2}dx = 2\cdot\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

(B4) $\iint_{D_R} 4e^{-x^2-y^2} x^{2p-1} y^{2q-1}\,dxdy$

$$= \int_0^{\frac{\pi}{2}} \int_0^R 4e^{-r^2} (r\cos\theta)^{2p-1} (r\sin\theta)^{2q-1}\, r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^R 4 r^{2p+2q-1} e^{-r^2} \cos^{2p-1}\theta \sin^{2q-1}\theta\, dr\,d\theta$$

$$= \underbrace{\int_0^{\frac{\pi}{2}} 2 \cos^{2p-1}\theta \sin^{2q-1}\theta\, d\theta}_{g(\theta)} \underbrace{\int_0^R 2 r^{2p+2q-1} e^{-r^2}\, dr}_{= f(R)}$$

$r^2 = s$

$2r\,dr = ds$

$$f(R) = \int_0^{R^2} s^{p+q-1} e^{-s}\, ds$$

$$\to \Gamma(p+q)$$

$$g(\theta) = \cancel{4}\, B(p,q)$$

$$\lim_{R\to\infty} \iint_{D_R} f(x,y)\,dxdy = \cancel{4}\, B(p,q)\,\Gamma(p+q)$$

~~(B2)~~ $\Gamma(p)\Gamma(q) \overset{\text{Fubini}}{=} \iint_{\square_R} 4e^{-x^2-y^2} x^{2p-1} y^{2q-1}\,dxdy$

$$\left(\int_0^\infty t^{z-1} e^{-t}\,dt = 2\int_0^\infty x^{2z-1} e^{-x^2}\,dx \text{ より} \right.$$

$t = x^2$

$dt = 2x\,dx$

(B2)と同様に

よって $\iint_{D_R} 4e^{-x^2-y^2} x^{2p-1} y^{2q-1}\,dxdy \leq \Gamma(p)\Gamma(q) \leq \iint_{D_{\sqrt{2}R}} \cdots$

で $R\to\infty$ とすれば良い。