(A6) $D = \{(x, y) \in \mathbb{R}^2 \mid (x + \frac{1}{2}y)^2 + (\frac{\sqrt{3}}{2}y)^2 \leq a^2\}$ より.

$A = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$ とすると $A : \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} X \\ Y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$ は1対1写像で

D は $D' = \{(X, Y) \in \mathbb{R}^2 \mid X^2 + Y^2 \leq a^2\}$ にうつる。これで変数変換して

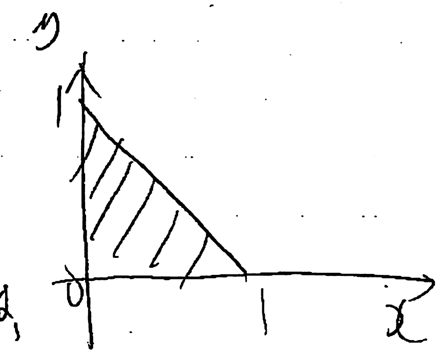
$$\iint_D \sqrt{a^2 - x^2 - xy - y^2}\, dx\, dy$$

$$= \iint_{D'} \sqrt{a^2 - (X^2 + Y^2)}\ \frac{1}{\det A}\, dX\, dY$$

$$= \frac{2}{\sqrt{3}} \times \int_{r=0}^{r=a} \int_{\theta=0}^{\theta=2\pi} \sqrt{a^2 - r^2} \cdot r\, d\theta\, dr$$

$$= \frac{4\pi}{\sqrt{3}} \cdot \frac{a^3}{3}$$

(A-1)



$x+y=u$ とおくと,

(x,y) の (x,u) に関する Jacobi 行列は,

$$\begin{pmatrix} \partial_x x & \partial_u x \\ \partial_x y & \partial_u y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

$$\therefore \iint_D \frac{dx\,dy}{1+(x+y)^4} = \int_0^1 du \int_0^u dx \, \frac{1}{1+u^4}$$

$$= \int_0^1 du \, \frac{u}{1+u^4} = \int_0^1 du \, \frac{\frac{1}{2}\frac{d}{du}(u^2)}{1+u^4}$$

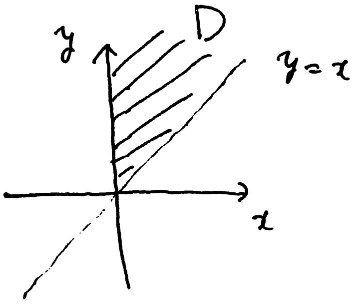
$$= \int_0^1 du^2 \cdot \frac{1}{2(1+u^4)}$$

$$= \frac{1}{2} \cdot \int_0^{\frac{\pi}{4}} d\theta \qquad (u^2 = \tan\theta)$$

$$= \frac{\pi}{8}$$

(A8) $D = \{ (x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq y \}$

$$I = \iint_D \frac{dx\,dy}{1 + (x^2 + y^2)^2}$$



$$8I = \iint_{\mathbb{R}^2} \frac{dx\,dy}{1 + (x^2+y^2)^2}$$

$$= \int_0^\infty \int_0^{2\pi} \frac{r\,dr\,d\theta}{1 + (r^2\cos^2\theta + r^2\sin^2\theta)^2}$$

$$= 2\pi \int_0^\infty \frac{r}{1+r^4}\,dr$$

$t = r^2$

$$= 2\pi \int_0^\infty \frac{1}{2} \cdot \frac{1}{1+t^2}\,dt$$

$$= \pi \cdot \left[\arctan t \right]_0^\infty = \frac{\pi^2}{2}$$

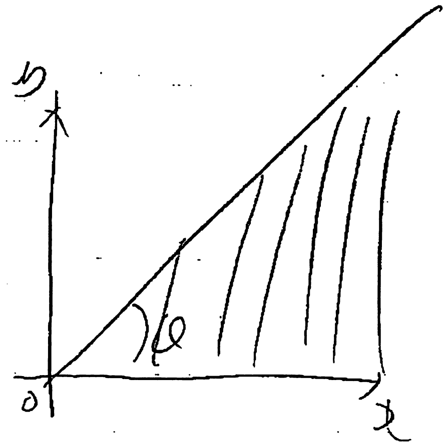
$$I = \frac{\pi^2}{16}$$

(A8)

次のような近似増加列を考える。

$0 \le x^2 + y^2 \le n^2$

$0 \le \theta \le \frac{\pi}{4}$



$$\iint_D \frac{1}{1+(x^2+y^2)^2}\,dx\,dy$$

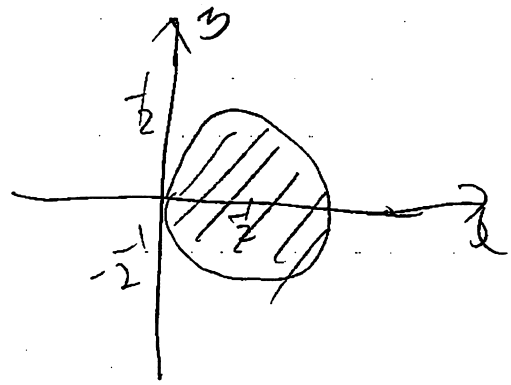
$$= \lim_{n\to\infty} \int_0^n dr \cdot r \int_0^{\frac{\pi}{4}} d\theta \, \frac{1}{1+r^4}$$

$$= \frac{\pi}{4} \cdot \frac{1}{2} \cdot \lim_{n\to\infty} \int_0^n dr^2 \, \frac{1}{1+(r^2)^2}$$

$$= \frac{\pi}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi^2}{16}$$

(A9)



$\therefore x = \frac{1}{2} + r\cos\theta,$

$y = r\sin\theta$ とおくと

- $$\iint_D x\,dx\,dy = \int_0^{\frac{1}{2}} dr \int_0^{2\pi} d\theta \cdot r \cdot \left(\frac{1}{2} + r\cos\theta\right)$$

$$= \frac{\pi}{8}$$

-

(A9) $D = \{x^2+y^2 \leq x\}$ $\quad x^2+y^2-x \leq 0$

$\Leftrightarrow (x-\frac{1}{2})^2+y^2 \leq (\frac{1}{2})^2$

$D' = \{X^2+Y^2 \leq (\frac{1}{2})^2\}$

$\begin{cases} X = x-\frac{1}{2} \\ Y = y \end{cases}$ $\quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} X \\ Y \end{pmatrix}$

Jacobi行列は $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\iint_D x \, dx\, dy$$

$$= \iint_{D'} (X+\tfrac{1}{2}) \, dX dY$$

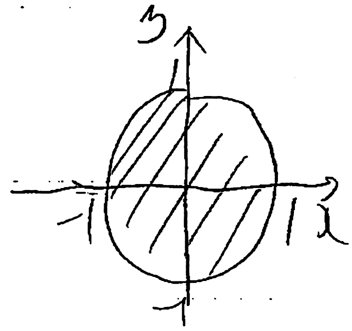
$$= \int_0^{2\pi}\int_{r=0}^{r=\frac{1}{2}} (r\cos\theta+\tfrac{1}{2}) r \, dr d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^3}{3}\cos\theta + \frac{r^2}{4}\right]_0^{\frac{1}{2}} d\theta$$

$$= \int_0^{2\pi} \frac{1}{24}\cos\theta + \frac{1}{16} \, d\theta$$

$$= \frac{1}{24}[\sin\theta]_0^{2\pi} + \frac{1}{16} \times 2\pi = \frac{\pi}{8}$$

(B-1) $x = r\cos\theta,\ y = r\sin\theta$ とおくと、



$$\iint_A \frac{dx\,dy}{(x^2+y^2)^{\frac{\alpha}{2}}}$$

$$= 2\pi \cdot \int_0^1 dr \cdot r^{1-\alpha}$$

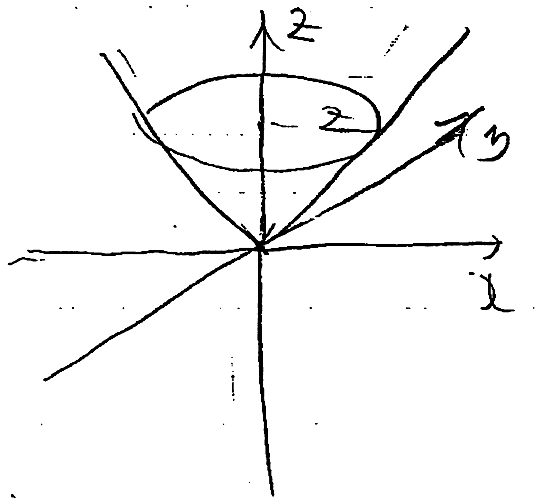
$$= 2\pi \cdot \begin{cases} \frac{1}{2-\alpha}\left[r^{2-\alpha}\right]_0^1 & (\text{if } r \neq 2) \\ \left[\ln r\right]_0^1 & (\text{if } r = 2) \end{cases}$$

よって $0<\alpha<2$ のとき収束

$2 \leq \alpha$ のとき発散。

(B4) $x = z\cos\theta$, $y = z\sin\theta$ とおくと

$$\iiint_V \frac{1}{(1+x^2+y^2)z^{\frac{3}{2}}}\,dx\,dy\,dz$$



$$= \lim_{\substack{A\to\infty\\B\to 0}} \int_B^A dz \int_0^z dr \cdot r \int_0^{2\pi} d\theta \cdot \frac{1}{(1+r^2)\cdot z^{\frac{3}{2}}}$$

$$= \lim_{\substack{A\to\infty\\B\to 0}} \int_B^A dz \cdot \pi \cdot \int_0^z d(r^2)\, \frac{1}{(1+r^2)\cdot z^{\frac{3}{2}}}$$

$$= \lim_{\substack{A\to\infty\\B\to 0}} \int_B^A dz \cdot \pi \cdot \frac{1}{z^{\frac{3}{2}}} \cdot \log(1+z^2)$$

$$= \lim_{\substack{A\to\infty\\B\to 0}} -\frac{1}{2}\left[\pi \frac{1}{z^{\frac{1}{2}}} \log(z^2+1)\right]_B^A + \pi \int_B^A dz \cdot \frac{z^{\frac{1}{2}}}{z^2+1}$$

$$= 0 + \lim_{\substack{A\to\infty\\B\to 0}} \pi \int_{B^2}^{A^2} dk \cdot \frac{2k^2}{k^4+1} \quad (\because k = \sqrt{z}\,)$$

$$\left(\because \lim_{\substack{z\to\infty\\0}} \frac{1}{z^{\frac{1}{2}}} \log(z^2+1) = -\frac{1}{2} \lim_{\substack{z\to\infty\\0}} \frac{z^{\frac{3}{2}}}{z^2+1} = 0 \right)$$

$\frac{\sin\theta}{\cos^2\theta \cdot \cos\theta}$

$\frac{\sin^4\theta}{\cos^4\theta} + 1$

$\frac{\cos\theta}{\sin^4\theta - \cos^4\theta}$

$k^2 = e^{(1-2\sin\theta\cos\theta)}$

$d k$

∘ $\frac{t^2}{1+t^4}\,dt$ の計算

$$+t^4 = (1+t^2)^2 - 2t^2$$

$$= (1+\sqrt{2}t+t^2)(1-\sqrt{2}t+t^2)$$

$$\frac{t^2}{+t^4} = \frac{t}{2\sqrt{2}}\left(\frac{1}{1-\sqrt{2}t+t^2} - \frac{1}{1+\sqrt{2}t+t^2}\right)$$

· $$\frac{t}{1-\sqrt{2}t+t^2} = \frac{t}{\frac{1}{2}+\left(t-\frac{\sqrt{2}}{2}\right)^2} = \frac{t-\frac{\sqrt{2}}{2}}{\left(t-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}} + \frac{\frac{\sqrt{2}}{2}}{\left(t-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}$$

· $$\frac{t}{1+\sqrt{2}t+t^2} = \frac{t}{\frac{1}{2}+\left(t+\frac{\sqrt{2}}{2}\right)^2} = \frac{t+\frac{\sqrt{2}}{2}}{\left(t+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}} - \frac{\frac{\sqrt{2}}{2}}{\left(t+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}$$

$$\frac{t^2}{1+t^4}\,dt$$

$$\underbrace{\int_\varepsilon^R\left(\frac{t-\frac{\sqrt{2}}{2}}{\left(t-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}} - \frac{t+\frac{\sqrt{2}}{2}}{\left(t+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}\right)dt}_{(*)}$$

$$+\underbrace{\frac{1}{4}\int_\varepsilon^R \frac{1}{\left(t-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}\,dt}_{(**)} + \underbrace{\frac{1}{4}\int_\varepsilon^R \frac{1}{\left(t+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}\,dt}_{(***)}$$

$$\left[\frac{1}{2}\log\left(\left(t-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}\right) - \frac{1}{2}\log\left(\left(t+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}\right)\right]_\varepsilon^R$$

$$\left[\frac{1}{2}\log\frac{\left(t-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}{\left(t+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}\right]_\varepsilon^R \xrightarrow[\substack{\varepsilon\to 0\\ R\to\infty}]{} 0$$

(B4つづき (2))

$$(**) = \frac{1}{4}\int_{\varepsilon}^{R} \frac{1}{\frac{1}{2}} \cdot \frac{1}{\left(\sqrt{2}\left(t-\frac{\sqrt{2}}{2}\right)\right)^2+1}\,dt$$

$$= \frac{1}{2}\int_{\sqrt{2}\left(\varepsilon-\frac{\sqrt{2}}{2}\right)}^{\sqrt{2}\left(R-\frac{\sqrt{2}}{2}\right)} \frac{1}{s^2+1}\,\frac{1}{\sqrt{2}}\,ds$$

$s = \sqrt{2}\left(t-\frac{\sqrt{2}}{2}\right)$

$$= \frac{1}{2\sqrt{2}}\Big[\arctan s\Big]_{\sqrt{2}\left(\varepsilon-\frac{\sqrt{2}}{2}\right)}^{\sqrt{2}\left(R-\frac{\sqrt{2}}{2}\right)}$$

$$(***) = \underset{\text{上と同様に}}{\cdots} = \frac{1}{2\sqrt{2}}\Big[\arctan s\Big]_{\sqrt{2}\left(\varepsilon+\frac{\sqrt{2}}{2}\right)}^{\sqrt{2}\left(R+\frac{\sqrt{2}}{2}\right)}$$

$$\int_0^{\infty} \frac{t^2}{1+t^4}\,dt = \frac{1}{2\sqrt{2}}\left(\frac{\pi}{2} - \arctan(-1) + \frac{\pi}{2} - \arctan(1)\right)$$

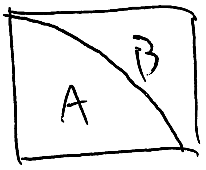
$$= \frac{\pi}{2\sqrt{2}}$$

$$\iiint_V \frac{1}{(1+x^2+y^2)z^{\frac{3}{2}}}\,dx\,dy\,dz = 2\pi\int_0^{\infty}\frac{t^2}{1+t^4}\,dt$$

$$= 2\pi\cdot\frac{\pi}{2\sqrt{2}} = \frac{\pi^2}{\sqrt{2}}$$

B5 $D = A \cup B$

f は D 上 一様連続より
($|f|$) 有界.



$\int_A f\,dxdy$ が絶対収束しないことをいう

$A = \bigcup_n A_n$ $\quad A_n =$

とする

$|f|$ は A_n 上 一様連続より

$$\int_{A_n} |f|\,dxdy = \int_{\frac{1}{n}}^{1}\int_0^{\frac{\pi}{2}} \frac{r^2|\cos^2\theta - \sin^2\theta|}{r^4}\, r\,dr\,d\theta$$

$$= \int_{\frac{1}{n}}^{1} \frac{dr}{r}\left(\int_0^{\frac{\pi}{2}} |\sin 2\theta|\,d\theta\right)$$

$\#_6$

$\int_{\frac{1}{n}}^{1} \frac{dr}{r}$ は $n\to\infty$ で ~~絶対~~収束しない.

2. $K_n^{(2)}$ 上 f は一様連続. また $\frac{d}{dx}\left(\frac{x}{x^2+y^2}\right) = f(x,y)$

$K_n^{(1)}$ $\quad \frac{d}{dy}\left(\frac{-y}{x^2+y^2}\right) = f(x,y)$ より $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\mathrm{Arctan}\frac{x}{a}$

$$\iint_{K_n^{(1)}} f\,dxdy = \int_0^1 dx \left[\frac{-y}{x^2+y^2}\right]_{\frac{1}{n}}^{1}$$

$$= \int_0^1 dx\left(\frac{-1}{x^2+1} + \frac{+\frac{1}{n}}{x^2+(\frac{1}{n})^2}\right)$$

$$= \left[-\mathrm{Arctan}\,x\right]_0^1 + \left[\mathrm{Arctan}\,nx\right]_0^1$$

$$= -\frac{\pi}{4} + \mathrm{Arctan}\,n.$$

$n\to\infty$ すると $\frac{\pi}{4}$

$-\frac{1}{5}$ $\iint_{K_n^{(2)}} f\,dx\,dy$

$= \int_{\frac{1}{n}}^{1} dx \left[\frac{-y}{x^2+y^2} \right]_0^1 = \int_{\frac{1}{n}}^{1} dx \, \frac{-1}{x^2+1}$

$= \left[-\mathrm{Arctan}\, x \right]_{\frac{1}{n}}^{1}$

$= -\frac{\pi}{4} + \mathrm{Arctan}\frac{1}{n}$

$n \to \infty$ すると。 $-\frac{\pi}{4}$

(3) $\int_0^1 dy \int_0^1 dx\, f(x,y)$ ($K_n^{(1)}$ に相当)

$= \int_0^1 dy \left[\frac{x}{x^2+y^2} \right]_0^1 = \int_0^1 \frac{1}{1+y^2} dy = \left[\mathrm{Arctan}\, y \right]_0^1$

$= \frac{\pi}{4}$

$\int_0^1 dx \int_0^1 dy\, f(x,y)$ ($K_n^{(2)}$ に相当)

$= \int_0^1 dx \left[\frac{-y}{x^2+y^2} \right]_0^1 = -\int_0^1 \frac{1}{1+x^2} dx = -\frac{\pi}{4}$