Mono-anabelian geometry II: Mono-anabelian geometry over mixed characteristic local fields

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§1 Reconstruction Algorithms (1) – Reconstructions from G_k –

k: an MLF $\hookrightarrow$ $\overline{k}$: an algebraic closure of k

$\underline{k}$: the residue field of k whose char. $= p$

$\mathcal{O}_k$: the ring of integers of k

$\mathcal{O}_k^{\triangleright} \stackrel{\mathrm{def}}{=} \mathcal{O}_k \setminus \{0\} \quad \subseteq \quad k^{\times} \stackrel{\mathrm{def}}{=} k \setminus \{0\}$ [a multiplicative submonoid]

$\mathcal{O}_k^{\times}$: the group of invertible elements in $\mathcal{O}_k^{\triangleright}$

$G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$

P_k : the wild inertia subgp $\subseteq$ I_k: the inertia subgp $\subseteq$ G_k

$\mathrm{Frob}_{\underline{k}} \in G_k/I_k$: the frobenius

By local class field theory, we have the local reciprocity map

$$\rho_k : \; k^\times \; \hookrightarrow \; G_k^{\mathrm{ab}}$$

which induces an isom.

$$\widehat{k^\times} \; \xrightarrow{\sim} \; G_k^{\mathrm{ab}}$$

["$\wedge$" denotes the prof. compl'n]. This isom. fits into the comm. diag.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Im}(I_k \to G_k^{\mathrm{ab}}) & \longrightarrow & G_k^{\mathrm{ab}} & \longrightarrow & G_k/I_k & \longrightarrow & 1 \\
 & & \uparrow \wr & & \uparrow \wr & & \uparrow \wr & & \\
1 & \longrightarrow & \mathcal{O}_k^\times & \longrightarrow & \widehat{k^\times} & \longrightarrow & \widehat{\mathbb{Z}} & \longrightarrow & 1 \\
 & & \| & & \uparrow \cup & & \uparrow \cup & & \\
1 & \longrightarrow & \mathcal{O}_k^\times & \longrightarrow & k^\times & \xrightarrow{\mathrm{val}_k} & \mathbb{Z} & \longrightarrow & 1
\end{array}$$

where the isom. $\widehat{\mathbb{Z}} \xrightarrow{\sim} G_k/I_k$ maps 1 to $\mathrm{Frob}_{\underline{k}}$.

In particular, we have

$$G_k^{\mathrm{ab}} \overset{\sim}{\leftarrow} \widehat{k^\times} \cong \mathcal{O}_k^\times \times \widehat{\mathbb{Z}} \cong \mathbb{Z}/(p^{f_k}-1)\mathbb{Z} \times \mathbb{Z}/p^a\mathbb{Z} \times \mathbb{Z}_p^{d_k} \times \widehat{\mathbb{Z}}$$

where $f_k \overset{\mathrm{def}}{=} [\underline{k} : \mathbb{F}_p]$, $d_k \overset{\mathrm{def}}{=} [k : \mathbb{Q}_p]$, $a \geq 0$.

Thus:

- $\{p\} = \{ \, l$: prime $\mid \dim_{\mathbb{Q}_l}(G_k^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l) \geq 2 \, \}$
- $d_k = [k : \mathbb{Q}_p] = \dim_{\mathbb{Q}_p}(G_k^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_p) - 1$
- $f_k = [\underline{k} : \mathbb{F}_p] = \log_p(\sharp(G_k^{\mathrm{ab}})_{\mathrm{tor}}^{(p')} + 1)$
- $I_k = \bigcap_{K/k:\ \text{fin s.t. } d_K/f_K = d_k/f_k} G_K \quad [\subseteq G_k]$
- P_k: the unique pro-p Sylow subgroup of I_k
- $\{ \, \mathrm{Frob}_{\underline{k}} \in G_k/I_k \, \} = \{ \, \gamma \in G_k/I_k \mid \gamma$ acts on I_k/P_k by $p^{f_k} \, \}$

Theorem (cf. [AbsAnab], Prop 1.2.1)

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for constructing the subgroups and the submonoid of G^{ab} corresp'g to

$$\rho_k(\mathcal{O}_k^{\times}) \subseteq \rho_k(\mathcal{O}_k^{\triangleright}) \subseteq \rho_k(k^{\times}) \quad (\subseteq G_k^{\mathrm{ab}})$$

Indeed, by using the above facts, we can reconstruct:

(1) $p(G)$: the [unique] prime l s.t. $\dim_{\mathbb{Q}_l}(G^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l) \geq 2$

(2) $d(G) \overset{\mathrm{def}}{=} \dim_{\mathbb{Q}_{p(G)}}(G^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_{p(G)}) - 1$

(3) $f(G) \overset{\mathrm{def}}{=} \log_{p(G)}(\sharp(G^{\mathrm{ab}})_{\mathrm{tor}}^{(p(G)')} + 1)$

(4) $I(G) \overset{\mathrm{def}}{=} \bigcap_{H \subseteq G\text{: open s.t. } d(G)/f(G) = d(H)/f(H)} H \quad [\subseteq G]$

(5) $P(G)$: the unique pro-$p(G)$ Sylow subgroup of $I(G)$

(6) $\mathrm{Frob}(G) \in G/I(G)$: the [unique] element $G/I(G)$ which acts on
$$I(G)/P(G) \ \text{ by } \ p(G)^{f(G)}$$

(7) $k^\times(G) \stackrel{\mathrm{def}}{=} G^{\mathrm{ab}} \times_{G/I(G)} \mathrm{Frob}(G)^{\mathbb{Z}} \quad [\subseteq G^{\mathrm{ab}}]$

(8) $\mathcal{O}^{\triangleright}(G) \stackrel{\mathrm{def}}{=} G^{\mathrm{ab}} \times_{G/I(G)} \mathrm{Frob}(G)^{\mathbb{N}} \quad [\subseteq k^\times(G)]$

(9) $\mathcal{O}^\times(G) \stackrel{\mathrm{def}}{=} \mathrm{Im}(I(G) \to G^{\mathrm{ab}}) \quad [\subseteq \mathcal{O}^{\triangleright}(G)]$

In the following, we write

$$\begin{aligned} \overline{\mathcal{O}}^\times(G) \stackrel{\mathrm{def}}{=} \varinjlim \mathcal{O}^\times(H) \quad &\subseteq \quad \overline{\mathcal{O}}^{\triangleright}(G) \stackrel{\mathrm{def}}{=} \varinjlim \mathcal{O}^{\triangleright}(H) \\ &\subseteq \quad \overline{k}^\times(G) \stackrel{\mathrm{def}}{=} \varinjlim k^\times(H) \end{aligned}$$

where $\varinjlim$ is taken over the direct. set $\{H \subseteq G : \text{open}\}$; the transition morphism $\square(H_2) \to \square(H_1)$ $[H_1 \subseteq H_2]$ is defined to be the morphism induced by the transfer $H_2^{\mathrm{ab}} \to H_1^{\mathrm{ab}}$.

§2 Reconstruction Algorithms (2) – Cyclotomic Rigidity via LCFT –

$\mathcal{O}_{\overline{k}}$: the ring of integers of $\overline{k}$

$\mathcal{O}^{\rhd}_{\overline{k}} \stackrel{\text{def}}{=} \mathcal{O}_{\overline{k}} \setminus \{0\} \quad \subseteq \quad \overline{k}^{\times} \stackrel{\text{def}}{=} \overline{k} \setminus \{0\}$ [a multiplicative submonoid]

$\mathcal{O}^{\times}_{\overline{k}}$: the group of invertible elements in $\mathcal{O}^{\rhd}_{\overline{k}} \supseteq \boldsymbol{\mu}_{\overline{k}} \stackrel{\text{def}}{=} (\mathcal{O}^{\times}_{\overline{k}})_{\mathrm{tor}}$

$\Lambda(\overline{k}) \stackrel{\text{def}}{=} \varprojlim_n (\boldsymbol{\mu}_{\overline{k}}[n])$

In the notation of §1, the reciprocity map ρ_k induces

$$\boldsymbol{\mu}_{\overline{k}} \stackrel{\sim}{\to} \boldsymbol{\mu}_{\overline{k}}(G_k) \stackrel{\text{def}}{=} (\overline{\mathcal{O}}^{\times}(G_k))_{\mathrm{tor}}$$

hence, this isom. induces the cyclotomic rigidity isomorphism

$$\Lambda(\overline{k}) \stackrel{\sim}{\to} \Lambda(G_k) \stackrel{\text{def}}{=} \varprojlim_{n} (\boldsymbol{\mu}_{\overline{k}}(G_k)[n])$$

Theorem (cf. [AbsTopIII], Rem 3.2.1; Prop 3.3, (i); [Alien], Ex 2.12.1)

Let $(G \curvearrowright M) \cong (G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}})$ (resp. $(G_k \curvearrowright \mathcal{O}^{\times}_{\overline{k}})$; $(G_k \curvearrowright \overline{k}^{\times})$)

$\Gamma \stackrel{\text{def}}{=} \{1\}$ (resp. $\widehat{\mathbb{Z}}^{\times}$; $\{\pm 1\}$)

$\Lambda(M) \stackrel{\text{def}}{=} \varprojlim_n (M_{\text{tor}}[n])$

$\Lambda(G) \stackrel{\text{def}}{=} \varprojlim_n (\overline{\mathcal{O}}^{\times}(G)_{\text{tor}}[n])$

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$$(G \curvearrowright M) \quad \longmapsto \quad \text{the } \Gamma\text{-orbit of } \Lambda(M) \stackrel{\sim}{\to} \Lambda(G)$$

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$$(G \curvearrowright M) \quad \longmapsto \quad \text{the } \Gamma\text{-orbit of } \Lambda(M) \overset{\sim}{\to} \Lambda(G)$$

where $\Lambda(M) \overset{\sim}{\to} \Lambda(G)$ corresp's to the cyc. rig. isom. $\Lambda(\overline{k}) \overset{\sim}{\to} \Lambda(G_k)$.

In the following, we give a proof of this thm in the case where

$$(G \curvearrowright M) \cong (G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}).$$

To verify this thm, it suffices to show:

Proposition (2.1)

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$$(G \curvearrowright M) \quad \longmapsto \quad \text{a natural isom. } H^2(G, \Lambda(M)) \overset{\sim}{\to} \widehat{\mathbb{Z}}$$

Indeed, by applying this prop. to $(G \curvearrowright \overline{\mathcal{O}}^{\triangleright}(G))$ [cf. §1], we have

$$H^2(G, \Lambda(G)) \overset{\sim}{\to} \widehat{\mathbb{Z}}.$$

Then $\boxed{\text{the cyc. rig. isom. } \Lambda(M) \overset{\sim}{\to} \Lambda(G)}$ may be characterized as the unique isom. $\Lambda(M) \overset{\sim}{\to} \Lambda(G)$ which is compatible with

$$H^2(G, \Lambda(M)) \overset{\sim}{\to} \widehat{\mathbb{Z}} \quad \text{and} \quad H^2(G, \Lambda(G)) \overset{\sim}{\to} \widehat{\mathbb{Z}}.$$

Proof of Proposition (2.1)

First, we note that $G \curvearrowright (M_{\mathrm{tor}} \hookrightarrow M^{\mathrm{gp}})$ induces

$$\boxed{H^2(G, M_{\mathrm{tor}}) \stackrel{\sim}{\to} H^2(G, M^{\mathrm{gp}})} \qquad (1)$$

[cf. the fact that $H^i(G, M^{\mathrm{gp}}/M_{\mathrm{tor}}) \cong H^i(G_k, \overline{k}^{\times}/\boldsymbol{\mu}_{\overline{k}}) = 0$ for $i > 0$].

Recall: One can reconstruct $I(G) \subseteq G$ corresp'g to $I_k \subseteq G_k$ [cf. §1].

Write

- $G^{\mathrm{unr}} \stackrel{\mathrm{def}}{=} G/I(G)$
- $M^{\mathrm{unr}} \stackrel{\mathrm{def}}{=} \{m \in M \mid g \cdot m = m \text{ for all } g \in I(G)\} \subseteq M$

Then, by considering the Leray-Serre spectral sequence assoc. to

$$1 \to I(G) \to G \to G^{\mathrm{unr}} \to 1$$

we obtain

$$\boxed{H^2(G^{\mathrm{unr}}, (M^{\mathrm{unr}})^{\mathrm{gp}}) \ \stackrel{\sim}{\to}\ H^2(G, M^{\mathrm{gp}})} \qquad (2)$$

[cf. the fact that $H^i(I(G), M^{\mathrm{gp}}) \cong H^i(I_k, \overline{k}^{\times}) = 0$ for $i = 1,\ 2$].

Next, note that $G^{\mathrm{unr}} \curvearrowright (\ (M^{\mathrm{unr}})^{\mathrm{gp}} \twoheadrightarrow (M^{\mathrm{unr}})^{\mathrm{gp}}/(M^{\mathrm{unr}})^{\times}\ )$ induces

$$\boxed{H^2(G^{\mathrm{unr}}, (M^{\mathrm{unr}})^{\mathrm{gp}}) \ \stackrel{\sim}{\to}\ H^2(G^{\mathrm{unr}}, (M^{\mathrm{unr}})^{\mathrm{gp}}/(M^{\mathrm{unr}})^{\times})} \qquad (3)$$

[cf. the fact that $H^i(G^{\mathrm{unr}}, (M^{\mathrm{unr}})^{\times}) \cong H^i(G_k^{\mathrm{unr}}, \mathcal{O}_{k^{\mathrm{unr}}}^{\times}) = 0$ for $i > 0$].

On the other hand, by applying the following isom's

- $G^{\mathrm{unr}} \stackrel{\sim}{\to} \widehat{\mathbb{Z}}$ which maps $\mathrm{Frob}(G)$ to 1 [cf. §1]
- $(M^{\mathrm{unr}})^{\mathrm{gp}}/(M^{\mathrm{unr}})^{\times} \stackrel{\sim}{\to} \mathbb{Z}$ which is obtained by considering the generator $\in (M^{\mathrm{unr}})/(M^{\mathrm{unr}})^{\times} \cong \mathcal{O}_{k^{\mathrm{unr}}}^{\triangleright}/\mathcal{O}_{k^{\mathrm{unr}}}^{\times} \cong \mathbb{N}$

we obtain

$$\boxed{H^2(G^{\mathrm{unr}}, (M^{\mathrm{unr}})^{\mathrm{gp}}/(M^{\mathrm{unr}})^{\times}) \ \overset{\sim}{\to}\ H^2(\widehat{\mathbb{Z}}, \mathbb{Z})} \qquad (4)$$

Finally, we note that $\widehat{\mathbb{Z}} \curvearrowright (1 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 1)$ induces

$$\boxed{H^2(\widehat{\mathbb{Z}}, \mathbb{Z}) \ \overset{\sim}{\to}\ H^1(\widehat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) \ = \ \mathrm{Hom}(\widehat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) \ \overset{\sim}{\to}\ \mathbb{Q}/\mathbb{Z}} \qquad (5)$$

where the last isom. maps ϕ to $\phi(1)$. Then, by (1), (2), (3), (4), (5), we obtain a natural isom.

$$\boxed{H^2(G, M_{\mathrm{tor}}) \ \overset{\sim}{\to}\ \mathbb{Q}/\mathbb{Z}}$$

Thus, by applying the functor $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, -)$ to this isom., we obtain

$$\boxed{H^2(G, \Lambda(M)) \ \overset{\sim}{\to}\ \widehat{\mathbb{Z}}}$$

Corollary (cf. [AbsTopIII], Prop's 3.2, (iv); 3.3, (ii))

Let $(G \curvearrowright M) \cong (G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}})$ (resp. $(G_k \curvearrowright \mathcal{O}^{\times}_{\overline{k}})$; $(G_k \curvearrowright \overline{k}^{\times})$)

$\Gamma \overset{\mathrm{def}}{=} \{1\}$ (resp. $\widehat{\mathbb{Z}}^{\times}$; $\{\pm 1\}$)

Then we have an isom.

$$\mathrm{Aut}(G \curvearrowright M) \overset{\sim}{\to} \Gamma \times \mathrm{Aut}(G)$$

In the following, we give a proof of this cor. in the case where

$$(G \curvearrowright M) \cong (G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}).$$

Let us verify that the hom. induced by "forgetting"

$$\mathrm{Aut}(G \curvearrowright M) \to \mathrm{Aut}(G)$$

is bijective.

- The surjectivity follows from the existence of a func'l algorithm

$$G \longmapsto (G \curvearrowright \overline{\mathcal{O}}^{\triangleright}(G))$$

for constructing a pair $(G \curvearrowright \overline{\mathcal{O}}^{\triangleright}(G)) \cong (G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}})$ [cf. §1].

- To verify the injectivity, let us recall Kummer theory.

$$G \curvearrowright (1 \to M_{\mathrm{tor}}[n] \to M^{\mathrm{gp}} \stackrel{\times n}{\to} M^{\mathrm{gp}} \to 1)$$

$$\stackrel{H^*(G,-)}{\Longrightarrow} \quad M^G \hookrightarrow (M^{\mathrm{gp}})^G = H^0(G, M^{\mathrm{gp}}) \to H^1(G, M_{\mathrm{tor}}[n])$$

$$\stackrel{\varprojlim_n}{\Longrightarrow} \quad M^G \hookrightarrow H^1(M, \Lambda(M))$$

$$\Longrightarrow \quad M \hookrightarrow {}_{\infty}H^1(G, \Lambda(M)) \stackrel{\mathrm{def}}{=} \varinjlim_{J \subseteq G:\ \text{open}} H^1(J, \Lambda(M))$$

In particular, by the previous thm., we obtain a natural embedding

$$M \hookrightarrow {}_{\infty}H^1(G, \Lambda(M)) \overset{\text{CRI}}{\overset{\sim}{\to}} {}_{\infty}H^1(G, \Lambda(G)).$$

Now we verify the injectivity. Let $\alpha \in \mathrm{Aut}(G \curvearrowright M)$ s.t. $\alpha|_G = \mathrm{id}_G$.

Then by the comm. diag.

$$\begin{tikzcd} M \arrow[r, "\subseteq"] \arrow[d, "\alpha|_M"', "\wr"] & {}_{\infty}H^1(G, \Lambda(G)) \arrow[d, "\mathrm{id}\ \text{[induced by } \alpha|_G\text{]}"] \\ M \arrow[r, "\subseteq"] & {}_{\infty}H^1(G, \Lambda(G)) \end{tikzcd}$$

we conclude that $\alpha|_M = \mathrm{id}|_M$.

[This argument is a typical example of a "mono-anabelian transport".]

§3 Reconstruction Algorithms (3) – Reconstructions from Π_X –

k: an MLF $\hookrightarrow$ $\overline{k}$: an alg. clos. of k $\hookleftarrow$ $\overline{k}_{\mathrm{NF}}$: the alg. clos. of $\mathbb{Q}$ in $\overline{k}$

X: a hyperbolic curve of strictly Belyi type over k [i.e., that is defined over an NF and isogenous to a hyperbolic curve of genus 0]

Then $(X \times_k \overline{k} \to X \to \mathrm{Spec}(k))$ induces an exact seq. of prof. gps

$$1 \longrightarrow \Delta_X \longrightarrow \Pi_X \longrightarrow G_k \longrightarrow 1$$

where $\Delta_X \stackrel{\mathrm{def}}{=} \Pi_{X \times_k \overline{k}}$. Now let us recall the following well-known fact:

Fact

In general, we can not reconstruct the field k from G_k. Indeed, there exists a pair of MLF's (k_1, k_2) s.t. $G_{k_1} \cong G_{k_2}$ but $k_1 \not\cong k_2$.

On the other hand, we can prove the following:

Theorem (cf. [AbsTopIII], Cor 1.10)

Let Π be a top. gp $\cong \Pi_X$. Then $^{\exists}$functorial group-theoretic algorithms

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Let Π be a top. gp $\cong \Pi_X$. Then $^{\exists}$functorial group-theoretic algorithms

$$\Pi \quad \longmapsto \quad k(\Pi),\ K_X(\Pi)$$

Theorem (cf. [AbsTopIII], Cor 1.10)

Let Π be a top. gp $\cong \Pi_X$. Then $^\exists$functorial group-theoretic algorithms

$$\Pi \quad \longmapsto \quad k(\Pi),\ K_X(\Pi)$$

for constructing fields corresp'g to k and the function field K_X.

In the following, we give a sketch of a pf of this thm. First, note that:

Proposition (3.1)

Let Π be a top. gp $\cong \Pi_X$. Then $^\exists$functorial group-theoretic algorithm

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$$\Pi \quad \longmapsto \quad \Delta(\Pi)$$

for constructing the subgroup $\Delta(\Pi) \subseteq \Pi$ corresp'g to $\Delta_X \subseteq \Pi_X$.

Write $G(\Pi) \stackrel{\mathrm{def}}{=} \Pi/\Delta(\Pi)$. ($\Longrightarrow\ 1 \to \Delta(\Pi) \to \Pi \to G(\Pi) \to 1$)

In the following, for simplicity, we assume that

the genus of the compactification X^{cpt} of X is > 1.

Write $\Delta_X^{\mathrm{cpt}} \stackrel{\mathrm{def}}{=} \Pi_{X^{\mathrm{cpt}} \times_k \overline{k}}$, $\Lambda_X \stackrel{\mathrm{def}}{=} \mathrm{Hom}(H^2(\Delta_X^{\mathrm{cpt}}, \widehat{\mathbb{Z}}), \widehat{\mathbb{Z}})$.

- Reconstruction of k First, reconstruct $k^{\times}$. By Kum. theory, we have

$$\alpha : k^{\times} \hookrightarrow H^1(G_k, \Lambda(\overline{k})) \xrightarrow{\sim} H^1(G_k, \Lambda_X)$$

where $\xrightarrow{\sim}$ is ind. by a nat. isom. $\Lambda(\overline{k}) \xrightarrow{\sim} \Lambda_X$ arising from sch. theory.

Proposition (3.2)

Let Π be a top. gp $\cong \Pi_X$. Then $^{\exists}$functorial group-theoretic algorithm

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$$\Pi \quad \longmapsto \quad (\ \Delta(\Pi) \twoheadrightarrow \Delta^{\mathrm{cpt}}(\Pi)\ )$$

In the following, for simplicity, we assume that

$$\text{the genus of the compactification } X^{\mathrm{cpt}} \text{ of } X \text{ is } > 1.$$

Write $\Delta_X^{\mathrm{cpt}} \stackrel{\mathrm{def}}{=} \Pi_{X^{\mathrm{cpt}} \times_k \overline{k}}$, $\Lambda_X \stackrel{\mathrm{def}}{=} \mathrm{Hom}(H^2(\Delta_X^{\mathrm{cpt}}, \widehat{\mathbb{Z}}), \widehat{\mathbb{Z}})$.

- Reconstruction of k First, reconstruct $k^\times$. By Kum. theory, we have

$$\alpha : k^\times \hookrightarrow H^1(G_k, \Lambda(\overline{k})) \xrightarrow{\sim} H^1(G_k, \Lambda_X)$$

where $\xrightarrow{\sim}$ is ind. by a nat. isom. $\Lambda(\overline{k}) \xrightarrow{\sim} \Lambda_X$ arising from sch. theory.

Proposition (3.2)

Let Π be a top. gp $\cong \Pi_X$. Then $^\exists$functorial group-theoretic algorithm

$$\Pi \longmapsto (\Delta(\Pi) \twoheadrightarrow \Delta^{\mathrm{cpt}}(\Pi))$$

for constructing a quotient corresp'g to the natural surj. $\Delta_X \twoheadrightarrow \Delta_X^{\mathrm{cpt}}$.

Write $\Lambda(\Pi) \stackrel{\text{def}}{=} \mathrm{Hom}(H^2(\Delta^{\mathrm{cpt}}(\Pi), \widehat{\mathbb{Z}}), \widehat{\mathbb{Z}})$. Thus, we can reconstruct

a "container" $H^1(G(\Pi), \Lambda(\Pi))$ $[\cong H^1(G_k, \Lambda_X)]$.

We want to reconstruct the subgp $\subseteq H^1(G(\Pi), \Lambda(\Pi))$ corresp'g to

$$[\ k^{\times} \stackrel{\sim}{\to}\ ]\ \ \alpha(k^{\times}) \ \subseteq\ H^1(G_k, \Lambda_X).$$

Now, by applying a result in the prev. talk [cf. [AbsTopIII], Thm 1.9], we can reconstruct a field $\overline{k}_{\mathrm{NF}}(\Pi)$ corresp'g to $\overline{k}_{\mathrm{NF}}$, and the diag.

$$H^1(G(\Pi), \Lambda(\Pi))$$

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$$\begin{array}{ccc} H^1(G(\Pi), \Lambda(\Pi)) & \xrightarrow{\subset} & (\text{a [larger] "container" recon. from } \Pi) \\ & & \uparrow \cup \\ & & \overline{k}_{\mathrm{NF}}^\times(\Pi) \end{array}$$

where $\overline{k}_{\mathrm{NF}}^\times(\Pi) \stackrel{\text{def}}{=} \overline{k}_{\mathrm{NF}}(\Pi) \setminus \{0\}$.

Write $k_{\mathrm{NF}}^{\times}(\Pi) \stackrel{\mathrm{def}}{=} H^1(G(\Pi), \Lambda(\Pi)) \cap \overline{k}_{\mathrm{NF}}^{\times}(\Pi)$. In particular,

$$k_{\mathrm{NF}}(\Pi) \stackrel{\mathrm{def}}{=} k_{\mathrm{NF}}^{\times}(\Pi) \cup \{0\} \ [\subseteq \overline{k}_{\mathrm{NF}}^{\times}(\Pi) \cup \{0\} = \overline{k}_{\mathrm{NF}}(\Pi)\ ]$$

admits a str. of field [relative to the additive str. of $\overline{k}_{\mathrm{NF}}(\Pi)$].

Now let us reconstruct a valuation on the field $k_{\mathrm{NF}}(\Pi)$ as follows:

First, note that, in the notations of §1, §2, we can reconstruct a surj.

$$\boxed{H^1(G(\Pi), \Lambda(G(\Pi))) \xrightarrow{\sim} G(\Pi)^{\mathrm{ab}} \twoheadrightarrow G(\Pi)^{\mathrm{unr}} \xrightarrow{\sim} \widehat{\mathbb{Z}}}$$

where

- 1st $\xrightarrow{\sim}$ is det. by the nat. isom. $H^2(G(\Pi), \Lambda(G(\Pi))) \xrightarrow{\sim} \widehat{\mathbb{Z}}$ [cf. Prop (2.1)] and the cup-product [in group cohomology]
- $G(\Pi)^{\mathrm{ab}} \twoheadrightarrow G(\Pi)^{\mathrm{unr}} \xrightarrow{\sim} \widehat{\mathbb{Z}}$ corresp'g to $G_k^{\mathrm{ab}} \twoheadrightarrow G_k^{\mathrm{unr}} \xrightarrow{\sim} \widehat{\mathbb{Z}}$ [cf. §1]

Thus, by choosing a non-canonical ism $\Lambda(\Pi) \xrightarrow{\sim} \Lambda(G(\Pi))$, we obtain

$$\boxed{\widehat{\mathrm{val}}(\Pi): \ H^1(G(\Pi), \Lambda(\Pi)) \twoheadrightarrow \widehat{\mathbb{Z}}}$$

well-def'd up to multiplication by $\widehat{\mathbb{Z}}^\times$. [Here, let us recall the exact seq.

$$1 \longrightarrow \mathcal{O}_k^\times \longrightarrow \widehat{k^\times} \xrightarrow{\widehat{\mathrm{val}}_k} \widehat{\mathbb{Z}} \longrightarrow 1$$

[cf. §1].] Hence, the subring $\subseteq k_{\mathrm{NF}}(\Pi)$ generated by

$$\mathrm{Ker}(\widehat{\mathrm{val}}(\Pi)) \cap \overline{k}_{\mathrm{NF}}^\times(\Pi) \ [\subseteq k_{\mathrm{NF}}(\Pi)]$$

determines a valuation on $k_{\mathrm{NF}}(\Pi)$.

Therefore, one may construct a subgroup

$$k^\times(\Pi) \subseteq H^1(G(\Pi), \Lambda(\Pi)) \ [\text{ corresp'g to } \alpha(k^\times) \subseteq H^1(G_k, \Lambda_X) \ ]$$

as the completion of $k_{\mathrm{NF}}^\times(\Pi)$ w.r.t. the valuation on $k_{\mathrm{NF}}(\Pi)$.

Moreover, one may construct the additive str. on

$$k(\Pi) \stackrel{\mathrm{def}}{=} k^{\times}(\Pi) \cup \{0\}$$

as the unique cont. ext. of the add. str. on $(k^{\times}(\Pi) \cap \overline{k}^{\times}_{\mathrm{NF}}(\Pi)) \cup \{0\}$.

- Reconstruction of K_X (Rough Sketch)

Let S be a fin. set of closed pts of X; $U_S \stackrel{\mathrm{def}}{=} X \setminus S$.

Then we have the following injection

$$\Gamma(U_S, \mathcal{O}^{\times}_{U_S}) \hookrightarrow H^1(\Pi_{U_S}, \Lambda(\overline{k})) \stackrel{\sim}{\to} H^1(\Pi_{U_S}, \Lambda_X) \stackrel{\sim}{\to} H^1(\Pi^{\text{c-ab}}_{U_S}, \Lambda_X)$$

- $\hookrightarrow$ is induced by considering the Kum. ext seq. on the ét site of U_S.
- 1st $\stackrel{\sim}{\to}$ is ind. by the nat. ism $\Lambda(\overline{k}) \stackrel{\sim}{\to} \Lambda_X$ arising from sch. theory.
- 2nd $\stackrel{\sim}{\to}$ is ind. by the "max'l abelian cuspidalization" $\Pi_{U_S} \twoheadrightarrow \Pi^{\text{c-ab}}_{U_S}$.

To reconstruct K_X, we first reconstruct

$$\text{a ``container''} \quad H^1(\Pi_{U_S}^{\text{c-ab}}, \Lambda_X)$$

and then reconstruct the image in the container of $\Gamma(U_S, \mathcal{O}_{U_S}^{\times})$.

[Note: By applying the Belyi cuspidalizations and the fact "k: an MLF",

one can recon. the decomposition groups in Π_X of arbitrary closed pts.]

Finally, by taking $\varinjlim_S$, we can reconstruct the image of an injection

$$K_X^{\times} \hookrightarrow \varinjlim_{S} H^1(\Pi_{U_S}^{\text{c-ab}}, \Lambda_X)$$

Moreover, by applying Uchida's Lemma [cf. [AbsTopIII], Prop 1.3],

we can reconstruct the additive structure on K_X.

§4 Appendix – Terminology –

Definition

- a(n) (arithmetic) holomorphic structure

  $\overset{\text{def}}{\Leftrightarrow}$ a (structure which determines a) ring structure

- a(n) (arithmetic) mono-analytic structure

  $\overset{\text{def}}{\Leftrightarrow}$ an "underlying" ("non-holomorphic") structure of a hol. str.

Example

- an arith. holomorphic str. — $\mathbb{Q}_p$ (resp. $\pi_1^{\text{ét}}(\mathbb{P}^1_{\mathbb{Q}_p} \setminus \{0, 1, \infty\})$)
- an arith. mono-analytic str. — $(\mathbb{Q}_p)_+$, $\mathbb{Q}_p^\times$ (resp. $G_{\mathbb{Q}_p}$)

[cf. a [classical] hol. str. on $\mathbb{C}$ v.s. a [classical] real-an. str. on $\mathbb{R}^{\oplus 2}$]