

Étale theta functions and mono-theta environments I

Arata Minamide

RIMS, Kyoto University

July 19, 2016

Introduction

Summary of This Talk

- (i) By using **theta functions**, we can define **mono-theta environments**.
- (ii) There exist [various] **group-theoretic reconstruction algorithms** for reconstructing [various objects related to] theta functions.

(i) What is a mono-theta environment? — a triple

$$(\Pi, \mathcal{D}_\Pi, s_\Pi^\Theta)$$

where Π is a topological group, $\mathcal{D}_\Pi \subseteq \text{Out}(\Pi)$, \dots . It may be regarded as a “bridge” between tempered fund. gps and tempered frobenioids.

Moreover, a mono-theta env. satisfies important **rigidities** [cf. next talk].

(ii) What is a group-theoretic reconstruction of a function?

— For instance, consider the following well-known example:

k : a p -adic local field (i.e., $[k : \mathbb{Q}_p] < \infty$) $\hookrightarrow \bar{k}$: an alg. closure of k

$$G_k \stackrel{\text{def}}{=} \text{Gal}(\bar{k}/k)$$

$$\mu_N(\bar{k}) \stackrel{\text{def}}{=} \{a \in \bar{k}^\times \mid a^N = 1\} \quad (N: \text{ a positive integer})$$

$$\Lambda(\bar{k}) \stackrel{\text{def}}{=} \varprojlim_N \mu_N(\bar{k})$$

U : an affine hyperbolic curve/ k $\hookrightarrow X$: the smooth compactification

Π_U (resp. Π_X): the étale π_1 of U (resp. X)

$$\Delta_X \stackrel{\text{def}}{=} \text{Ker}(\Pi_X \rightarrow G_k)$$

In the following, we assume that the genus of X is > 1 .

By considering the Kummer exact sequence on the étale site of U , and passing to \varprojlim_n , we obtain a natural injection

$$\begin{aligned} \Gamma(U, \mathcal{O}_U^\times) &\hookrightarrow H^1(\Pi_U, \Lambda(\bar{k})) \\ f &\longmapsto \eta_f. \end{aligned}$$

We refer to η_f as the **Kummer class** associated to f .

Moreover, for a closed point x of U , write

$$D_x \subseteq \Pi_U$$

for the decomposition group of x [well-defined up to Π_U -conj.].

Note: D_x maps, via $\Pi_U \twoheadrightarrow G_k$, isom. onto $G_{k(x)} \subseteq G_k$

[where $k(x)$ is the residue field of U at x , and $G_{k(x)} \stackrel{\text{def}}{=} \text{Gal}(\bar{k}/k(x))$].

Then, the image of η_f via the composite

$$H^1(\Pi_U, \Lambda(\bar{k})) \xrightarrow{\text{res.}} H^1(D_x, \Lambda(\bar{k})) \xrightarrow{\sim} H^1(G_{k(x)}, \Lambda(\bar{k})) \xrightarrow{\sim} (k(x)^\times)^\wedge$$

[“ \wedge ” denotes the prof. completion] coincides with the image in $(k(x)^\times)^\wedge$ of the **value** of f at x . (**Galois evaluation**)

Point: Using a natural isom. $\Lambda(\bar{k}) \xrightarrow{\sim} \Lambda(X) \stackrel{\text{def}}{=} \text{Hom}(H^2(\Delta_X, \widehat{\mathbb{Z}}), \widehat{\mathbb{Z}})$, we may regard

$$\eta_f \in H^1(\Pi_U, \Lambda(X)).$$

Then the “**container**” $H^1(\Pi_U, \Lambda(X))$ may be reconstructed by π_1 -like objects Π_U, Δ_X [which are often reconstructed by **anabelian geometry!**].

\implies We want to reconstruct [a subset containing] $\eta_f \in H^1(\Pi_U, \Lambda(X))$.

Moreover, if we know $D_x \subseteq \Pi_U$, then we can reconstruct the value $f(x)$.

Tempered Coverings of a Once-punctured Elliptic Curve

p, l : distinct prime numbers ($\neq 2$)

k : a p -adic local field $\supseteq \mathcal{O}_k$: the ring of integers of k

X : an elliptic curve/ k which has split multiplicative reduction/ \mathcal{O}_k

$q \in \mathcal{O}_k \setminus \{0\}$: the “ q -parameter” of X

$X^{\log} \stackrel{\text{def}}{=} (X, \{o\} \subseteq X)$: the smooth log curve/ k associated to X

In particular, $\{\pm 1\} \curvearrowright X^{\log}$ induces $X^{\log} \rightarrow C^{\log} \stackrel{\text{def}}{=} X^{\log} // \{\pm 1\}$.

In the following, we assume that

- $\sqrt{-1} \in k$
- $X[2l](\bar{k}) = X[2l](k)$
- C^{\log} is a k -core.

Now we have the following diagram of log tempered coverings:

$$\begin{array}{ccccc}
 \underline{\underline{\ddot{Y}}}^{\log} & \xrightarrow{\mu_l} & \ddot{Y}^{\log} & & \\
 \downarrow \mu_2 & & \downarrow \mu_2 & & \\
 \underline{\underline{Y}}^{\log} & \xrightarrow{\mu_l} & Y^{\log} & & \\
 \downarrow l \cdot \underline{\mathbb{Z}} & & \downarrow l \cdot \underline{\mathbb{Z}} & & \\
 \underline{\underline{X}}^{\log} & \xrightarrow{\mu_l} & \underline{X}^{\log} & \xrightarrow{\underline{\mathbb{F}}_l} & X^{\log} \\
 & & \downarrow \{\pm 1\} & & \downarrow \{\pm 1\} \\
 & & \underline{C}^{\log} & \xrightarrow{\quad} & C^{\log}
 \end{array}$$

Diag.1

- The three squares are **cartesian**.
- The composite

$$\begin{array}{ccc}
 & Y^{\log} & \\
 & \downarrow & \\
 \underline{X}^{\log} & \longrightarrow & X^{\log}
 \end{array}$$

is the covering det. by the [graph-theoretic] **universal covering** of the dual graph of the special fiber of X^{\log} . Write

$$\underline{\mathbb{Z}} \stackrel{\text{def}}{=} \text{Gal}(Y^{\log}/X^{\log}) [\cong \mathbb{Z}].$$

The covering $\underline{X}^{\log} \longrightarrow X^{\log}$ corresponds to $l \cdot \underline{\mathbb{Z}} \subseteq \underline{\mathbb{Z}}$. Write

$$\underline{\mathbb{F}}_l \stackrel{\text{def}}{=} \text{Gal}(\underline{X}^{\log}/X^{\log}) [\cong \mathbb{F}_l].$$

For a curve $(-)$ over k , write

$\text{Csp}((-)$): the set of cusps of $(-)$

$\text{Ver}((-)$): the set of irreducible components of the special fiber of $(-)$

In the following, fix $\exists \in \text{Csp}(\underline{X}^{\log})$, and refer to the **zero cusp** of \underline{X}^{\log} .

$\implies \underline{X}$ admits a str. of **elliptic curve** whose origin is the zero cusp.

In particular, $\{\pm 1\} \curvearrowright \underline{X}^{\log}$ induces $\underline{X}^{\log} \rightarrow \underline{C}^{\log} \stackrel{\text{def}}{=} \underline{X}^{\log} // \{\pm 1\}$.

- $\underline{\underline{X}}^{\log} \rightarrow \underline{X}^{\log}$: a Galois finite log étale covering of $\text{deg} = l$
totally ramified at $\forall \in \text{Csp}(\underline{X}^{\log})$
- $\dot{Y}^{\log} \rightarrow Y^{\log}$: the double cover. determined by “ $u = \ddot{u}^2$ ”

Evaluation Points on Tempered Coverings

$0_{\underline{X}} \in \text{Ver}(\underline{X}^{\log})$: the irreducible comp. which contains the zero cusp

In the following, fix a lift. $\exists \in \text{Ver}(Y^{\log})$ of $0_{\underline{X}} \in \text{Ver}(\underline{X}^{\log})$, and write

$$0_Y \in \text{Ver}(Y^{\log}).$$

Note: Since $\text{Ver}(Y^{\log})$ is a \mathbb{Z} -torsor, we obtain a **labeling**

$$\mathbb{Z} \xrightarrow{\sim} \text{Ver}(Y^{\log}) \left[\xrightarrow{\sim} \text{Ver}(\ddot{Y}^{\log}) \xrightarrow{\sim} \text{Ver}(\underline{\ddot{Y}}^{\log}) \right].$$

$\mu_- \in \underline{X}(k)$: the pt of **order 2** whose closure intersects $0_{\underline{X}} \in \text{Ver}(\underline{X}^{\log})$

$\mu_-^Y \in Y(k)$: a $\exists!$ lift. of μ_- whose closure intersects $0_Y \in \text{Ver}(Y^{\log})$

For $j \in \mathbb{Z}$, write

$\xi_j^Y \in Y(k)$: the image of μ_-^Y by the action of $j \in \mathbb{Z} = \text{Gal}(Y^{\log}/X^{\log})$.

Definition

an **evaluation point** of \ddot{Y}^{\log} labeled by $j \in \mathbb{Z}$

$\stackrel{\text{def}}{\Leftrightarrow}$ a lifting $\in \ddot{Y}(k)$ of $\xi_j^Y \in Y(k)$

an **evaluation point** of $\underline{\underline{\ddot{Y}}}^{\log}$ labeled by $j \in \mathbb{Z}$

$\stackrel{\text{def}}{\Leftrightarrow}$ a lifting $\in \underline{\underline{\ddot{Y}}}(k)$ of an ev. pt $\in \ddot{Y}(k)$ labeled by $j \in \mathbb{Z}$

Theta Function $\ddot{\Theta}$

$0_{\ddot{Y}} \in \text{Ver}(\ddot{Y}^{\log})$: the irreducible comp. lying over $0_Y \in \text{Ver}(Y^{\log})$

Then the function

$$\ddot{\Theta}(\ddot{u}) = q^{-\frac{1}{8}} \cdot \sum_{n \in \mathbb{Z}} (-1)^n \cdot q^{\frac{1}{2}(n+\frac{1}{2})^2} \cdot \ddot{u}^{2n+1}$$

on $0_{\ddot{Y}} \in \text{Ver}(\ddot{Y}^{\log})$ extends uniquely to a meromorphic function $\ddot{\Theta}$ on the stable model of \ddot{Y} , and satisfies the following properties:

(i) (the divisor of zeroes of $\ddot{\Theta}$) = $\sum_{c \in \text{Csp}(\ddot{Y}^{\log})} [\overline{\{c\}}]$

(the divisor of poles of $\ddot{\Theta}$) = $\sum_{j \in \underline{\mathbb{Z}} \cong \text{Ver}(\ddot{Y}^{\log})} \frac{j^2 \cdot \text{ord}_k(q)}{2} \cdot [j]$

(ii) $\xi_j \in \ddot{Y}(k)$: an evaluation point of \ddot{Y} labeled by $j \in \underline{\mathbb{Z}} [\cong \mathbb{Z}]$

Then the following equality holds:

$$\ddot{\Theta}(\xi_j)^{-1} = \pm \ddot{\Theta}(\xi_0)^{-1} \cdot q^{\frac{j^2}{2}}$$

Note: If we write

$$\ddot{\Theta}_{\text{st}} \stackrel{\text{def}}{=} \ddot{\Theta}(\xi_0)^{-1} \cdot \ddot{\Theta},$$

then we have

$$\ddot{\Theta}_{\text{st}}(\xi_j)^{-1} = \pm q^{\frac{j^2}{2}}.$$

We refer to $\ddot{\Theta}_{\text{st}}$ as the **standard type** of $\ddot{\Theta}$.

Étale Theta Functions

$\Pi_{(-)}^{\text{tp}}$: the log tempered π_1 of $(-)^{\text{log}}$

$$\Delta_{(-)}^{\text{tp}} \stackrel{\text{def}}{=} \text{Ker}(\Pi_{(-)}^{\text{tp}} \rightarrow G_k)$$

$$\Pi_{(-)} \stackrel{\text{def}}{=} (\Pi_{(-)}^{\text{tp}})^{\wedge} \quad [“\wedge” \text{ denotes the profinite comp.}]$$

$$\Delta_{(-)} \stackrel{\text{def}}{=} (\Delta_{(-)}^{\text{tp}})^{\wedge}$$

$$\Delta_{\Theta} \stackrel{\text{def}}{=} [\Delta_X, \Delta_X] / [\Delta_X, [\Delta_X, \Delta_X]]$$

Note: We have a natural isom.

$$\Lambda(\bar{k}) \xrightarrow{\sim} \Delta_{\Theta}$$

arising from scheme theory.

Let

$$\ddot{\eta}^\Theta \in H^1(\Pi_{\underline{Y}}^{\text{tp}}, \Lambda(\bar{k})) \xrightarrow{\sim} H^1(\Pi_{\underline{Y}}^{\text{tp}}, \Delta_\Theta)$$

be the Kum. class assoc. to an $\alpha \in \mathcal{O}_k^\times \cdot \ddot{\Theta}$ [cf. the property (i) of $\ddot{\Theta}$].

We refer to $\ddot{\eta}^\Theta$ as an **étale theta function**.

Then we have a cohom. class

$$\underline{\underline{\ddot{\eta}}}^\Theta \in H^1(\Pi_{\underline{\underline{Y}}}^{\text{tp}}, l \cdot \Delta_\Theta)$$

— an l -th root of the **étale theta function** — i.e., the Kummer class assoc. to an $\alpha \in \mathcal{O}_k^\times \cdot \ddot{\Theta}^{\frac{1}{l}}$ s.t. the image of $\underline{\underline{\ddot{\eta}}}^\Theta$ via

$$H^1(\Pi_{\underline{\underline{Y}}}^{\text{tp}}, l \cdot \Delta_\Theta) \longrightarrow H^1(\Pi_{\underline{\underline{Y}}}^{\text{tp}}, \Delta_\Theta)$$

[induced by $l \cdot \Delta_\Theta \hookrightarrow \Delta_\Theta$] coincides with $\ddot{\eta}^\Theta|_{\Pi_{\underline{\underline{Y}}}} \in H^1(\Pi_{\underline{\underline{Y}}}^{\text{tp}}, \Delta_\Theta)$.

Construction of a (Mod N) Mono-theta Environment

For $N \geq 1$, write $\mu_N \stackrel{\text{def}}{=} \mu_N(\bar{k})$.

$\Pi \rightarrow G_k$: a topological group with an augmentation

$$\Pi[\mu_N] \stackrel{\text{def}}{=} \mu_N \rtimes \Pi$$

In particular, we have a **tautological section**

$$\Pi \rightarrow \Pi[\mu_N]$$

of the natural surj. $\Pi[\mu_N] \rightarrow \Pi$.

By Kummer theory, we obtain

$$k^\times \rightarrow k^\times / (k^\times)^N \hookrightarrow H^1(\underline{\underline{\Pi}}_Y^{\text{tp}}, \mu_N) \rightarrow \text{Out}(\underline{\underline{\Pi}}_Y^{\text{tp}}[\mu_N]).$$

$$\underline{\underline{\mathcal{D}}}_Y \stackrel{\text{def}}{=} \langle \text{Im}(k^\times), \text{Gal}(\underline{\underline{Y}}^{\text{log}} / \underline{\underline{X}}^{\text{log}}) \cong l \cdot \underline{\underline{\mathbb{Z}}} \rangle \subseteq \text{Out}(\underline{\underline{\Pi}}_Y^{\text{tp}}[\mu_N])$$

Let

$$\underline{\underline{\eta}}^{\Theta, l \cdot \underline{\mathbb{Z}} \times \underline{\mu}_2} \subseteq H^1(\Pi_{\underline{\underline{Y}}}^{\text{tp}}, l \cdot \Delta_{\Theta})$$

be the orbit of $\underline{\underline{\eta}}^{\Theta}$ by $\text{Gal}(\underline{\underline{Y}}/\underline{\underline{X}}) \cong (l \cdot \underline{\mathbb{Z}}) \times \underline{\mu}_2$.

Then

$$(\text{each } \in \underline{\underline{\eta}}^{\Theta, l \cdot \underline{\mathbb{Z}} \times \underline{\mu}_2}) \bmod N \in H^1(\Pi_{\underline{\underline{Y}}}^{\text{tp}}, l \cdot \Delta_{\Theta} \otimes \mathbb{Z}/N\mathbb{Z}) \xrightarrow{\sim} H^1(\Pi_{\underline{\underline{Y}}}^{\text{tp}}, \underline{\mu}_N)$$

[where “ $\xrightarrow{\sim}$ ” is induced by the natural isom. $l \cdot \Delta_{\Theta} \otimes \mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} \underline{\mu}_N$]
may be represented by the difference

$$(\exists! \text{section}) - (\text{taut. section})$$

$$\text{of } \Pi_{\underline{\underline{Y}}}^{\text{tp}}[\underline{\mu}_N] \rightarrow \Pi_{\underline{\underline{Y}}}^{\text{tp}}.$$

Definition

We refer to the composite

$$s_{\underline{Y}}^{\Theta} : \Pi_{\underline{Y}}^{\text{tp}} \xrightarrow{\text{section}} \Pi_{\underline{Y}}^{\text{tp}}[\mu_N] \hookrightarrow \Pi_{\underline{Y}}^{\text{tp}}[\mu_N]$$

as a (mod N) **theta section**.

Definition

We refer to the triple

$$\left(\Pi_{\underline{Y}}^{\text{tp}}[\mu_N], \mathcal{D}_{\underline{Y}}, \{ \gamma \cdot \text{Im}(s_{\underline{Y}}^{\Theta}) \cdot \gamma^{-1} \}_{\gamma \in \mu_N} \right)$$

as a (mod N) **model mono-theta environment**. We refer to an isomorph

$\mathbb{M}^{\Theta} = (\Pi, \mathcal{D}_{\Pi}, s_{\Pi}^{\Theta})$ of a (mod N) model mono-theta environment as a

(mod N) **mono-theta environment**.

Various Reconstructions

Theorem ($\Pi \Rightarrow \pi_1$ of various coverings — cf. [EtTh], Prop 2.4)

Π : a topological group $\cong \Pi_{\underline{X}}^{\text{tp}}$

Then \exists functorial group-theoretic algorithm

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$$\Pi \longmapsto \left(\begin{array}{ccccc} \underline{\underline{\Pi}}_{\underline{\underline{Y}}} & \longrightarrow & \underline{\underline{\Pi}}_{\underline{\underline{Y}}} & & \\ \downarrow & & \downarrow & & \\ \underline{\underline{\Pi}}_{\underline{\underline{Y}}} & \longrightarrow & \underline{\underline{\Pi}}_{\underline{\underline{Y}}} & & \\ \downarrow & & \downarrow & & \\ \underline{\underline{\Pi}} & \longrightarrow & \underline{\underline{\Pi}}_{\underline{\underline{X}}} & \longrightarrow & \underline{\underline{\Pi}}_X \\ & & \downarrow & & \downarrow \\ & & \underline{\underline{\Pi}}_{\underline{\underline{C}}} & \longrightarrow & \underline{\underline{\Pi}}_C \end{array} \right)$$

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for constructing a diag. of open inj. of top. gps corresp. to **Diag.1.**

$$\begin{array}{ccccc}
 \underline{\underline{\ddot{Y}}}^{\log} & \longrightarrow & \ddot{Y}^{\log} & & \\
 \downarrow & & \downarrow & & \\
 \underline{\underline{Y}}^{\log} & \longrightarrow & Y^{\log} & & \\
 \downarrow & & \downarrow & & \\
 \underline{\underline{X}}^{\log} & \longrightarrow & \underline{X}^{\log} & \longrightarrow & X^{\log} \\
 & & \downarrow & & \downarrow \\
 & & \underline{C}^{\log} & \longrightarrow & C^{\log}
 \end{array}$$

Diag.1

Theorem ($\Pi \Rightarrow$ étale theta functions — cf. [EtTh], Thm 1.6)

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$$\Pi \longmapsto G(\Pi),$$

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$$\Pi \longmapsto G(\Pi),$$

$$\Pi \longmapsto (l \cdot \Delta_{\Theta})(\Pi)$$

Theorem ($\Pi \Rightarrow$ étale theta functions — cf. [EtTh], Thm 1.6)

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Then \exists functorial group-theoretic algorithms

$$\Pi \longmapsto G(\Pi),$$

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for constructing top. gps. corresp. to $G_k, l \cdot \Delta_{\Theta}$,

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for constructing top. gps. corresp. to G_k , $l \cdot \Delta_{\Theta}$, as well as

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\exists functorial group-theoretic algorithm

$$\Pi \longmapsto (\mathcal{O}_k^\times \cdot \underline{\underline{\eta}}^{\Theta, l \cdot \mathbb{Z} \times \mu_2})(\Pi) \subseteq H^1(\underline{\underline{\Pi}}_{\underline{\underline{Y}}}(\Pi), (l \cdot \Delta_\Theta)(\Pi))$$

Theorem ($\Pi \Rightarrow$ étale theta functions — cf. [EtTh], Thm 1.6)

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$$\Pi \longmapsto (\mathcal{O}_k^{\times} \cdot \underline{\underline{\ddot{\eta}}}^{\Theta, l \cdot \mathbb{Z} \times \mu_2})(\Pi) \subseteq H^1(\underline{\underline{\Pi}}_{\check{Y}}(\Pi), (l \cdot \Delta_{\Theta})(\Pi))$$

for constructing a subset of H^1 corresp. to $\mathcal{O}_k^{\times} \cdot \underline{\underline{\ddot{\eta}}}^{\Theta, l \cdot \mathbb{Z} \times \mu_2}$, i.e.,

$$(l \cdot \mathbb{Z} \times \mu_2)\text{-orbit of } \mathcal{O}_k^{\times} \cdot \underline{\underline{\ddot{\Theta}}}^{\frac{1}{l}}.$$

Theorem ($\Pi \Rightarrow$ mono-theta env. — cf. [EtTh], Cor 2.18, (ii))

Π : a topological group $\cong \Pi_{\underline{X}}^{\text{tp}}$

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$$\Pi \longmapsto \mathbb{M}^\Theta(\Pi)$$

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Then \exists functorial group-theoretic algorithm

$$\Pi \longmapsto \mathbb{M}^{\Theta}(\Pi)$$

for constructing a mod N mono-theta environment.

Theorem (mono-theta env. $\Rightarrow \Pi$ — cf. [EtTh], Cor 2.18, (iii))

\exists functorial algorithm

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$$\mathbb{M}^{\Theta} \longmapsto \Pi_{\underline{X}}(\mathbb{M}^{\Theta})$$

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Theorem (mono-theta env. $\Rightarrow \Pi$ — cf. [EtTh], Cor 2.18, (iii))

\exists functorial algorithm

$$M^\Theta \longmapsto \Pi_{\underline{X}}(M^\Theta)$$

for constructing a top. gp corresp. to $\Pi_{\underline{X}}^{\text{tp}}$.